Midterm 3 Study Guide

MATH2300 - Calculus II

Spring 2026

Contents

Overview

Exam information:

- **Date:** Monday, April 13
- **Time:** 5:45pm – 7:15pm
- **Location:** CHEM 140

Topics covered:

- §11.2: Test for Divergence
- §11.2: Geometric Series
- §11.3: p-series
- §11.3: Integral Test
- §11.4: Direct Comparison Test
- §11.4: Limit Comparison Test
- §11.5: Alternating Series
- §11.5: Absolute Convergence
- §11.6: Ratio Test
- §11.5: Alternating Series Test Remainder
- §11.3: Integral Test Remainder
- §11.10: Taylor Polynomials
- §11.8: Power Series
- §11.9: Representing Functions with Power Series

Suggested things to study:

- Fall 2022 practice exam on Canvas; Spring 2023 practice exam on Canvas
- Quizzes (Quiz 9, Quiz 10, Quiz 11, Quiz 12)
- WebAssign
- Written homeworks
- The problems in this study guide

A good way to prepare is to practice actively: try problems without notes first, check your work, and then redo similar problems until each method feels routine. Focus on choosing the right method, writing clean setups, and finishing with a correct final answer.

The Story of Infinite Series

We begin with a sequence of numbers $a_1, a_2, a_3, \ldots$ and ask a natural question: What happens if we add them all together? This leads us to the concept of an infinite series, written as

$$\sum a_n = a_1 + a_2 + a_3 + \cdots .$$

Can we find the exact sum?

In special cases, we can compute the exact value of the infinite series:

- **Telescoping Series:** If many terms cancel out when we write the partial sums, we have a telescoping series. For example:

$$\sum \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

  We compute the partial sum S_n, observe cancellation, and take the limit as $n \to \infty$.

- **Geometric Series:** For a geometric series of the form $\sum ar^{n-1}$, the sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad \text{when } |r| < 1.$$

In these cases, we not only know that the series converges—we also know exactly what it converges to.

Does the series converge or diverge?

Most infinite series are not so convenient. In general, we can't compute the sum directly, so we shift our focus to a different question: *Does the series converge or diverge?*

Positive-Term Series

We begin with series whose terms are all positive. Several tests help us determine whether such a series converges:

- **Test for Divergence:** If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. In fact, this test applies to any series, even if the terms are not all positive!
- **Integral Test:** If $a_n = f(n)$ where f is positive, continuous, and decreasing, then

$$\sum a_n \text{ converges} \quad \Leftrightarrow \quad \int_1^{\infty} f(x)\,dx \text{ converges.}$$

- **Direct Comparison Test:** Compare a_n to a known benchmark series b_n.
  - If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.
  - If $0 \le b_n \le a_n$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

  Benchmark series are often geometric series or p-series, since their convergence behavior is well understood.

- **Limit Comparison Test:** If $a_n, b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{with } 0 < c < \infty,$$

  then either both series converge or both diverge.

Alternating Series and Mixed Signs

Series are not required to have only positive terms. In fact, many important series include both positive and negative terms. We begin with the especially nice case of **alternating series**, where the signs alternate in a regular pattern. If the terms alternate and decrease to zero, we can apply the:

- **Alternating Series Test:** For a series of the form $\sum(-1)^{n+1} b_n$ with $b_n > 0$, if

$$b_{n+1} \le b_n \quad \text{and} \quad \lim_{n\to\infty} b_n = 0,$$

  then the series converges.

When a series has both positive and negative terms, we test for:

- **Absolute Convergence:** If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely (and hence converges).
- **Conditional Convergence:** If $\sum a_n$ converges but $\sum |a_n|$ diverges, then the series converges conditionally.

A powerful tool in these cases is the:

- **Ratio Test:** Let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

  Then:
  - If $L < 1$, the series converges absolutely.
  - If $L > 1$ or $L = \infty$, the series diverges.
  - If $L = 1$, the test is inconclusive.

Estimating the Sum: Remainder Estimates

Even when we can't find the exact sum, we might want to estimate how close our partial sum S_n is to the true value.

- **Alternating Series Remainder:**

$$|R_n| = |S - S_n| \le b_{n+1}$$

  The error is no larger than the next term.

- **Integral Test Remainder:**

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_n^{\infty} f(x)\,dx$$

  This provides upper and lower bounds for the error when using the integral test.

Summary of Series Types

Series Type	General Form	Convergence Behavior
Telescoping Series	Terms cancel in successive pairs, e.g., $\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)$	Partial sums simplify to a finite number of terms; convergence depends on the limit of the remaining terms. Often converges due to cancellation.
Geometric Series	$\sum ar^{n-1}$	Converges if $\lvert r\rvert < 1$ to $\frac{a}{1-r}$. Diverges if $\lvert r\rvert \geq 1$.
p-Series	$\sum \frac{1}{n^p}$, where $p > 0$	Converges if $p > 1$. Diverges if $0 < p \leq 1$.
Alternating Series	$\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$, $b_n > 0$	Converges if b_n decreases and $\lim b_n = 0$. May converge conditionally or absolutely.

Summary of Convergence Tests

Test	Applies To	Conclusion
Test for Divergence	Any series $\sum a_n$	If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. If the limit is 0, the test is inconclusive.
Integral Test	Series with positive, decreasing $f(n) = a_n$	If $\int_1^\infty f(x)\,dx$ converges, so does $\sum a_n$. If the integral diverges, so does the series.
Direct Comparison Test	Positive-term series $\sum a_n$, $\sum b_n$	If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
Limit Comparison Test	Positive-term series $\sum a_n$, $\sum b_n$	If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $0 < c < \infty$, then both series converge or both diverge.
Alternating Series Test	Alternating series $\sum(-1)^n b_n$ with $b_n > 0$	If b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, then the series converges.
Absolute Convergence	Any series $\sum a_n$	If $\sum \lvert a_n\rvert$ converges, then $\sum a_n$ converges absolutely (and hence converges).
Ratio Test	Any series $\sum a_n$	Compute $L = \lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert$: if $L < 1$, series converges absolutely; if $L > 1$ or $L = \infty$, series diverges; if $L = 1$, test is inconclusive.

Summary of Remainder Estimates

Test	Remainder Estimate	Interpretation
Alternating Series Test Remainder	$\lvert R_n\rvert = \lvert S - S_n\rvert \leq b_{n+1}$	For an alternating series satisfying the Alternating Series Test, the error in approximating the sum by the nth partial sum is at most the absolute value of the next term. Error decreases as n increases.
Integral Test Remainder	$\int_{n+1}^\infty f(x)\,dx < R_n < \int_n^\infty f(x)\,dx$	For a positive, decreasing, continuous function $f(n) = a_n$, the true remainder lies between two improper integrals. Useful for estimating or bounding the error in partial sums.

How to Determine Whether a Series Converges or Diverges

1. Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$ or the limit does not exist, then $\sum a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, this test is inconclusive.

2. Known Series Types

Telescoping	Write out several partial sums and look for cancellation. The series converges when the partial sums approach a finite limit.
Geometric	$\sum ar^{n-1}$. Converges if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
p-Series	$\sum \frac{1}{n^p}$. Converges if $p > 1$; diverges if $p \leq 1$.

3. Positive-Term Series $a_n \geq 0$

DCT	If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $0 \leq b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. Use when you can compare a_n by inequality with a known p-series or geometric series.
LCT	If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ have the same behavior. Use when direct comparison is difficult.
Integral Test	Use when $a_n = f(n)$, where f is positive, continuous, and decreasing. Then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ have the same behavior.
Ratio Test	Best for factorials, exponentials, and powers. Let $L = \lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert$. If $L < 1$, the series converges absolutely; if $L > 1$, it diverges; if $L = 1$, the test is inconclusive.

4. Mixed-Sign Series

Absolute convergence	If $\sum \lvert a_n\rvert$ converges, then $\sum a_n$ also converges. In this case, $\sum a_n$ is called absolutely convergent. Use the tests on the left to study $\sum \lvert a_n\rvert$.
Alternating Series Test	If $\sum a_n$ has the form $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$, where $b_n > 0$, and if $b_n \to 0$ and b_n is eventually decreasing, then $\sum a_n$ converges.

Series Classification

Absolutely convergent	$\sum \lvert a_n\rvert$ converges.
Conditionally convergent	$\sum a_n$ converges, but $\sum \lvert a_n\rvert$ diverges.
Divergent	$\sum a_n$ diverges.

Test for Divergence

Let $\sum a_n$ be an infinite series. If

$$\lim_{n\to\infty} a_n \neq 0 \quad \text{or if the limit does not exist,}$$

then the series $\sum a_n$ **diverges**. Equivalently, for a series to have any chance of converging, its terms must approach 0. If $\lim_{n\to\infty} a_n = 0$, this does *not* mean the series converges. It only means the Test for Divergence is inconclusive.

Practice

Apply the Test for Divergence. If the limit is nonzero or does not exist, the series diverges. If the limit is zero, the test is inconclusive.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{2n^2+1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$
4. $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{5n-4}{n}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^2+1}$
7. $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Geometric Series

A **geometric series** has the form

$$\sum_{n=0}^{\infty} ar^n \qquad \text{or} \qquad \sum_{n=1}^{\infty} ar^{\,n-1},$$

where a is the first term and r is the **common ratio**. If needed, first rewrite the series so the repeating factor r^n is clear. A geometric series **converges** exactly when $|r| < 1$. In that case,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Common variations:

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}, \qquad \sum_{n=1}^{\infty} ar^{\,n-1} = \frac{a}{1-r}, \qquad \text{for } |r| < 1.$$

Key idea: In every convergent case, the sum is the *first term* divided by $1 - r$.

Practice

For each of the following series, determine whether the series converges or diverges. If the series converges, find its sum.

1. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$
2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{10^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$
6. $\displaystyle\sum_{n=0}^{\infty} 7 \cdot \left(\frac{2}{3}\right)^n$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$
8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{5^n}$
9. $\displaystyle\sum_{n=0}^{\infty} \frac{4}{3^{n+2}}$
10. $\displaystyle\sum_{n=2}^{\infty} \frac{4^n}{7^{n-2}}$

p-series

A **p-series** is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p > 0.$$

To identify a p-series, rewrite the denominator using exponent notation if needed. For example,

$$\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}},$$

so this is a p-series with $p = \frac{1}{2}$. The behavior of a p-series depends entirely on the value of p:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges}, & p > 1, \\ \text{diverges}, & 0 < p \leq 1. \end{cases}$$

Some series are not written exactly in p-series form at first glance, but can be simplified into that form. Others are not true p-series, yet behave like one for large n; in those cases, comparison tests are often the right tool.

Practice

Determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^9}}$

Integral Test

Integral Test. Suppose $a_n = f(n)$, where f is positive, continuous, and decreasing for all sufficiently large x. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. This test is most useful when the terms of the series come from a function that can be integrated more easily than the series can be summed.

Practice

Use the Integral Test to determine whether each series converges or diverges.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ (general case)
6. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{\ln n}}{n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^4+25}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^6+36}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3}{n^8+64}$

Direct Comparison Test

Direct Comparison Test. Let $a_n \geq 0$ and $b_n \geq 0$ for all sufficiently large n.

- If $a_n \leq b_n$ for all sufficiently large n, and if $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $b_n \leq a_n$ for all sufficiently large n, and if $\sum b_n$ diverges, then $\sum a_n$ also diverges.

This test is most useful when a series can be compared to a simpler series whose behavior is already known.

Practice

Use the Direct Comparison Test to determine whether each series converges or diverges by comparing it to a known p-series or geometric series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2}{n^4+n^2+1}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4^n+n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+2^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin^2\left(\frac{1}{n}\right)}{n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Limit Comparison Test

The **Limit Comparison Test** is used to compare two positive-term series

$$\sum a_n \qquad \text{and} \qquad \sum b_n.$$

Compute

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

If

$$0 < L < \infty,$$

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge. In practice, the Limit Comparison Test is useful when a_n and b_n have the same dominant behavior for large n, even if one is not always larger than the other.

Practice

Use the Limit Comparison Test to determine whether each series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3-4}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4-1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+5}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+3n}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+2^n}{4^n+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2+(-1)^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+\tan^{-1}(n)}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2+(-1)^n}$

Alternating Series Test

The **Alternating Series Test** applies to series of the form

$$\sum(-1)^n b_n \qquad \text{or} \qquad \sum(-1)^{n+1} b_n,$$

where $b_n > 0$. If the terms b_n satisfy

$$\lim_{n\to\infty} b_n = 0 \qquad \text{and} \qquad b_{n+1} \le b_n \text{ for all sufficiently large } n,$$

then the alternating series converges.

Practice

Determine whether each series converges or diverges. If the Alternating Series Test does not apply, state why.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(-1)^n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+\ln n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sin(1/n)}{n}$
10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{0.9}}$

Absolute Convergence

A series $\sum a_n$ is said to **converge absolutely** if the series of absolute values $\sum |a_n|$ converges. Absolute convergence is important because it guarantees that the original series $\sum a_n$ also converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is called **conditionally convergent**.

Practice

For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2+1)}{n^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}}$
9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Ratio Test

The **Ratio Test** is especially useful for series involving factorials, exponentials, or powers of n. For a series $\sum a_n$, compute

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

$$\begin{array}{ll} L < 1 & \text{the series converges absolutely,} \\ L > 1 \text{ or } L = \infty & \text{the series diverges,} \\ L = 1 & \text{the test is inconclusive.} \end{array}$$

In practice, the Ratio Test checks whether the terms are shrinking quickly enough from one term to the next for the series to converge.

Practice

Determine whether each series converges absolutely, converges conditionally, or diverges using the Ratio Test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^2}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n^n}$
9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{2^n}$
10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2}{n!}$

Mixed Series Practice

For each problem below, determine convergence/divergence, evaluate limits or sums, or justify the requested conclusion. Show clear reasoning.

1. Evaluate $\lim_{n\to\infty} \dfrac{\sqrt{n^2+\cos n}}{n}$.

2. Evaluate $\lim_{n\to\infty} \left(1-\dfrac{1}{2n}\right)^{3n}$.

3. Suppose $S_n = \sum_{k=1}^{n} a_k = \dfrac{n^2}{\ln(n+1)+3n}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

4. Suppose $S_n = \sum_{k=1}^{n} a_k = \dfrac{4n^2+1}{2n^2+5n+1}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

5. If $\sum a_n = 5$ and $\sum b_n = 4$, find $\sum (2a_n - 3b_n)$.

6. Find the sum of the geometric series: $\sum_{n=2}^{\infty} \dfrac{6}{4^n}$.

7. Determine whether the series $\sum_{n=2}^{\infty} \dfrac{n^4+3}{n(n+1)^2}$ converges. If it converges, find its sum.

8. Determine whether the series $\sum_{n=1}^{\infty} \dfrac{4^n}{5^n-1}$ converges.

9. Let $a_n = \dfrac{n^2}{2^n}$. Use the Ratio Test to determine whether $\sum a_n$ converges.

10. Use the Ratio Test to determine for which values of $c > 0$ the series $\sum_{n=1}^{\infty} \dfrac{c^n}{n}$ converges.

11. Determine whether the series $\sum_{n=1}^{\infty} \dfrac{2n}{n^3+\cos n}$ converges.

12. Determine whether $\sum_{n=1}^{\infty} \dfrac{\sin(n^3)}{n^3+1}$ converges absolutely, conditionally, or diverges.

13. Determine whether $\sum_{n=3}^{\infty} \dfrac{(-1)^n}{n^2-4}$ converges absolutely, conditionally, or diverges.

14. Use the Alternating Series Remainder Theorem to estimate the error when approximating $\sum_{n=1}^{\infty} (-1)^n \dfrac{1}{n^3}$ using the first 4 terms.

15. Use the Integral Test to show that $\sum_{n=2}^{\infty} \dfrac{1}{n\ln(n)^2}$ converges. Estimate the error when approximating the series with the first 5 terms.

Remainder Estimates

Alternating Series Remainder

- **Applies to:** A convergent alternating series.
- **Remainder estimate:** The error when approximating the sum by the first n terms satisfies:

$$|R_n| = |S - S_n| \leq b_{n+1}$$

Integral Test Remainder

- **Applies to:** A convergent series $\sum a_n$, where $a_n = f(n)$, and $f(x)$ is a positive, continuous, decreasing function.
- **Remainder estimate:** If $S_n = \sum_{k=1}^{n} a_k$, then the remainder $R_n = S - S_n$ satisfies:

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$

Practice

For Problems 1–5, use the Alternating Series Remainder Theorem. For Problems 6–10, use the Integral Test Remainder estimate. Show all reasoning.

Alternating Series Remainder

1. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ using the first 4 terms. Estimate the error.
2. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^3+1}$ using the first 3 terms. How accurate is your approximation?
3. Find how many terms are needed to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ to within 0.0001.
4. Use the first 5 terms of $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n+1)}{n^2}$ to approximate the sum. Estimate the error.
5. Determine the minimum number of terms needed to estimate $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n}}{n \ln n}$ with error less than 0.01.

Integral Test Remainder

6. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ using the first 5 terms. Use the Integral Test to bound the error.
7. Use the Integral Test to estimate the remainder when approximating $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ by its partial sum through $n = 10$.
8. Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ using the first 8 terms, and bound the error using the Integral Test.
9. Find the smallest n such that the partial sum $\displaystyle\sum_{k=2}^{n} \frac{1}{k(\ln k)^2}$ approximates the infinite series to within 0.05.
10. Approximate $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ using the first 6 terms. Estimate how close your approximation is to the true value.

Taylor Polynomials

A **Taylor polynomial** is a polynomial that approximates a function near a chosen center $x = a$. The n-th degree Taylor polynomial uses the value of the function and its first n derivatives at a:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Near $x = a$, the graph of $T_n(x)$ closely matches the graph of $f(x)$. In general, higher-degree Taylor polynomials give better approximations near the center.

Practice

Find the degree 4 Taylor polynomial for $f(x)$ centered at the given value of a.

1. $f(x) = x^2 e^x, \quad a = 0$
2. $f(x) = \dfrac{1}{1+x^2}, \quad a = 1$
3. $f(x) = \cos(x^2), \quad a = 0$
4. $f(x) = \sin(2x), \quad a = \dfrac{\pi}{4}$
5. $f(x) = \ln(1+2x), \quad a = 0$
6. $f(x) = \tan^{-1}(x), \quad a = 0$
7. $f(x) = \dfrac{x}{1-x}, \quad a = 0$
8. $f(x) = e^{x^2}, \quad a = 0$
9. $f(x) = \dfrac{1}{\sqrt{1-x}}, \quad a = 0$

Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

where a is the **center** and the c_n's are constants.

The **radius of convergence**, denoted R, tells how far the series converges away from its center:

$$|x-a| < R \Rightarrow \text{ converges}, \qquad |x-a| > R \Rightarrow \text{ diverges}.$$

If $|x-a| = R$, you must test the endpoint(s) separately.

The **interval of convergence** is the set of all x-values for which the series converges. It is built from the open interval

$$(a-R, a+R)$$

and then adjusted after checking the endpoints.

Practice

For each series below, find the radius of convergence R and the interval of convergence I.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n \cdot 3^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x+4)^n}{7^n}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^{2n}}{5^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3x)^n}{n^2}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$
7. $\displaystyle\sum_{n=2}^{\infty} \frac{(x-7)^n}{n\ln(n)}$
8. $\displaystyle\sum_{n=0}^{\infty} \frac{n!(x+1)^n}{n^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^{2n}}{n4^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-5)^n}{2^n(n+1)}$

Representing Functions as Power Series

The starting point is the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

To find a power series for a new function, rewrite it so that it matches the form

$$\frac{1}{1-u},$$

and then apply the geometric series formula with u in place of x:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \qquad |u| < 1.$$

From this basic formula, new power series can be obtained by **substitution**, **algebraic manipulation**, and by **differentiating** or **integrating** term-by-term. When a power series is differentiated or integrated term-by-term, the resulting series has the same radius of convergence, although the endpoint behavior may change and must be checked separately.

Practice

For each function below, find a power series representation centered at 0, write your answer in summation notation, and state the interval of convergence.

1. $f(x) = \dfrac{x}{1-3x}$
2. $f(x) = \dfrac{4}{7+2x}$
3. $f(x) = \dfrac{1}{9-x^2}$
4. $f(x) = \dfrac{x^2}{1+x}$
5. $f(x) = \dfrac{1}{5-x}$

Multiple Choice Practice

1. Which of the following is the correct power series representation for $\dfrac{1}{1+x}$?

   (A) $\sum_{n=0}^{\infty}(-1)^n x^n$

   (B) $\sum_{n=0}^{\infty} x^n$

   (C) $\sum_{n=0}^{\infty}(-1)^n (x-1)^n$

   (D) $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$

   (E) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^n}{n}$

2. What is the power series representation for $\arctan(x)$?

   (A) $\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$

   (B) $\sum_{n=0}^{\infty} x^{2n+1}$

   (C) $\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$

   (D) $\sum_{n=1}^{\infty}(-1)^n \dfrac{x^{2n+1}}{n}$

   (E) $\sum_{n=1}^{\infty} \dfrac{x^{2n}}{n}$

3. Suppose $\ln(1+x)$ is expanded as a power series centered at 0. What is the radius of convergence?

   (A) 1

   (B) 2

   (C) 1/2

   (D) 3

   (E) Infinite

4. Suppose $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Which series represents $\dfrac{1}{1-2x}$?

   (A) $\sum_{n=0}^{\infty} \left(\dfrac{x}{2}\right)^n$

   (B) $\sum_{n=0}^{\infty} 2^n x^n$

   (C) $\sum_{n=0}^{\infty} 2x^n$

   (D) $\sum_{n=0}^{\infty} 2nx^n$

   (E) $\sum_{n=1}^{\infty} 2^n x^n$

5. What is the power series representation of $\int_0^x \dfrac{t}{1+t^2}\,dt$?

   (A) $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{2n+2}$

   (B) $\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$

   (C) $\sum_{n=0}^{\infty} \dfrac{x^{2n+2}}{2n+2}$

   (D) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

   (E) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^{2n}}{n}$

6. If $\dfrac{1}{1-x}$ has radius of convergence 1, what is the radius of convergence for $\dfrac{1}{1-5x}$?

   (A) 5

   (B) 1/5

   (C) 2

   (D) 1/2

   (E) 1