

Midterm 3 Study Guide (Solutions)

MATH2300 - Calculus II

Spring 2026

Contents

Test for Divergence	1
Geometric Series	3
p -Series	6
Integral Test	7
Direct Comparison Test	14
Limit Comparison Test	19
Alternating Series Test	25
Absolute Convergence	27
Ratio Test	33
Mixed Series Practice	36
Remainder Estimates	42
Taylor Polynomials	46
Power Series	49

Test for Divergence

1. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Let $a_n = \frac{n}{n+1}$. Then

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1.$$

Since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges by the Test for Divergence.

2. $\sum_{n=1}^{\infty} \frac{n^2 + 3}{2n^2 + 1}$

Let $a_n = \frac{n^2 + 3}{2n^2 + 1}$. Then

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 + 3}{2n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1 + \frac{3}{n^2}}{2 + \frac{1}{n^2}} = \frac{1}{2}.$$

Since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges by the Test for Divergence.

3. $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

Let $a_n = \frac{\sin n}{n}$. Since $-1 \leq \sin n \leq 1$, we have

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Because both bounds approach 0 as $n \rightarrow \infty$, the Squeeze Theorem gives

$$\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0.$$

Since $\lim_{n \rightarrow \infty} a_n = 0$, the Test for Divergence is inconclusive.

4. $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

Let $a_n = \cos\left(\frac{1}{n}\right)$. Then

$$\lim_{n \rightarrow \infty} a_n = \cos\left(\lim_{n \rightarrow \infty} \frac{1}{n}\right) = \cos(0) = 1.$$

Since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges by the Test for Divergence.

5. $\sum_{n=1}^{\infty} \frac{5n - 4}{n}$

Let $a_n = \frac{5n - 4}{n}$. First simplify:

$$a_n = 5 - \frac{4}{n}.$$

Thus,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(5 - \frac{4}{n}\right) = 5.$$

Since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges by the Test for Divergence.

$$6. \sum_{n=1}^{\infty} \frac{3n^2}{n^2 + 1}$$

Let $a_n = \frac{3n^2}{n^2 + 1}$. Then

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3}{1 + \frac{1}{n^2}} = 3.$$

Since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges by the Test for Divergence.

$$7. \sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

Let $a_n = \tan\left(\frac{1}{n}\right)$. Then

$$\lim_{n \rightarrow \infty} a_n = \tan\left(\lim_{n \rightarrow \infty} \frac{1}{n}\right) = \tan(0) = 0.$$

Since $\lim_{n \rightarrow \infty} a_n = 0$, the Test for Divergence is inconclusive.

$$8. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Let $a_n = \frac{1}{\sqrt{n}}$. Then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Since $\lim_{n \rightarrow \infty} a_n = 0$, the Test for Divergence is inconclusive.

$$9. \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

Let $a_n = \frac{1}{n^2 + 1}$. Then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Since $\lim_{n \rightarrow \infty} a_n = 0$, the Test for Divergence is inconclusive.

$$10. \sum_{n=1}^{\infty} \frac{1}{n}$$

Let $a_n = \frac{1}{n}$. Then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Since $\lim_{n \rightarrow \infty} a_n = 0$, the Test for Divergence is inconclusive.

Geometric Series

1. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

This is a geometric series with

$$a = 1, \quad r = \frac{1}{2}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

2. $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

This is a geometric series with

$$a = \frac{3}{4}, \quad r = \frac{3}{4}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{a}{1-r} = \frac{\frac{3}{4}}{1-\frac{3}{4}} = 3.$$

3. $\sum_{n=0}^{\infty} \frac{5}{10^n}$

Rewrite the series as

$$\sum_{n=0}^{\infty} 5 \left(\frac{1}{10}\right)^n.$$

So

$$a = 5, \quad r = \frac{1}{10}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=0}^{\infty} \frac{5}{10^n} = \frac{5}{1-\frac{1}{10}} = \frac{50}{9}.$$

4. $\sum_{n=1}^{\infty} \frac{1}{3^n}$

Rewrite the series as

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

Thus,

$$a = \frac{1}{3}, \quad r = \frac{1}{3}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}.$$

$$5. \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

This is a geometric series with

$$a = \frac{2}{3}, \quad r = \frac{2}{3}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 2.$$

$$6. \sum_{n=0}^{\infty} 7 \left(\frac{2}{3}\right)^n$$

This is a geometric series with

$$a = 7, \quad r = \frac{2}{3}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=0}^{\infty} 7 \left(\frac{2}{3}\right)^n = \frac{7}{1 - \frac{2}{3}} = 21.$$

$$7. \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$$

Rewrite the series as

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{1}{2}\right)^{n-1}.$$

So

$$a = \frac{1}{4}, \quad r = \frac{1}{2}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{2}.$$

$$8. \sum_{n=2}^{\infty} \frac{1}{5^n}$$

Rewrite the series as

$$\sum_{n=2}^{\infty} \frac{1}{5^n} = \sum_{n=2}^{\infty} \left(\frac{1}{5}\right)^n.$$

The first term is

$$a = \left(\frac{1}{5}\right)^2 = \frac{1}{25}, \quad r = \frac{1}{5}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=2}^{\infty} \frac{1}{5^n} = \frac{\frac{1}{25}}{1 - \frac{1}{5}} = \frac{1}{20}.$$

$$9. \sum_{n=0}^{\infty} \frac{4}{3^{n+2}}$$

Rewrite the series as

$$\sum_{n=0}^{\infty} \frac{4}{3^{n+2}} = \sum_{n=0}^{\infty} \frac{4}{9} \left(\frac{1}{3}\right)^n.$$

So

$$a = \frac{4}{9}, \quad r = \frac{1}{3}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=0}^{\infty} \frac{4}{3^{n+2}} = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

$$10. \sum_{n=2}^{\infty} \frac{4^n}{7^{n-2}}$$

First rewrite the general term:

$$\frac{4^n}{7^{n-2}} = \frac{4^n}{7^n} \cdot 7^2 = 49 \left(\frac{4}{7}\right)^n.$$

Thus the series becomes

$$\sum_{n=2}^{\infty} 49 \left(\frac{4}{7}\right)^n,$$

which is geometric with

$$a = 49 \left(\frac{4}{7}\right)^2 = 16, \quad r = \frac{4}{7}.$$

Since $|r| < 1$, the series converges, and

$$\sum_{n=2}^{\infty} \frac{4^n}{7^{n-2}} = \frac{16}{1 - \frac{4}{7}} = \frac{112}{3}.$$

p -Series

1. $\sum_{n=1}^{\infty} \frac{1}{n}$

This is a p -series with $p = 1$. Since $p \leq 1$, the series diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

This is a p -series with $p = 2$. Since $p > 1$, the series converges.

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Rewrite $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$. This is a p -series with $p = \frac{1}{2}$. Since $p \leq 1$, the series diverges.

4. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

This is a p -series with $p = 3$. Since $p > 1$, the series converges.

5. $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$

This is a p -series with $p = 0.9$. Since $p \leq 1$, the series diverges.

6. $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

This is a p -series with $p = \frac{5}{4}$. Since $p > 1$, the series converges.

7. $\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$

This is a p -series with $p = 1.0001$. Since $p > 1$, the series converges.

8. $\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$

This is a p -series with $p = 0.99$. Since $p \leq 1$, the series diverges.

9. $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$

This is a p -series with $p = \frac{4}{3}$. Since $p > 1$, the series converges.

10. $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^9}}$

Rewrite $\frac{n^2}{\sqrt{n^9}} = \frac{n^2}{n^{9/2}} = \frac{1}{n^{5/2}}$. This is a p -series with $p = \frac{5}{2}$. Since $p > 1$, the series converges.

Integral Test

1. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Let

$$f(x) = \frac{1}{x \ln x}.$$

For $x \geq 2$, f is continuous and positive. Also,

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2}.$$

Since $x \geq 2$ implies $\ln x > 0$, we have $\ln x + 1 > 0$, so

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus f is decreasing on $[2, \infty)$, so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln x} dx.$$

Let $u = \ln x$, so $du = \frac{1}{x} dx$. Then

$$\int_2^t \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln t} \frac{1}{u} du = \ln |u| \Big|_{\ln 2}^{\ln t} = \ln(\ln t) - \ln(\ln 2).$$

As $t \rightarrow \infty$, we have $\ln t \rightarrow \infty$, so

$$\ln(\ln t) - \ln(\ln 2) \rightarrow \infty.$$

Therefore,

$$\int_2^{\infty} \frac{1}{x \ln x} dx$$

diverges, so

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges by the Integral Test.

2. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Let

$$f(x) = \frac{1}{x(\ln x)^2}.$$

For $x \geq 2$, f is continuous and positive. Also,

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3}.$$

Since $x \geq 2$ implies $\ln x > 0$, we have $\ln x + 2 > 0$, so

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus f is decreasing on $[2, \infty)$, so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx.$$

Let $u = \ln x$, so $du = \frac{1}{x} dx$. Then

$$\int_2^t \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\ln t} u^{-2} du = \left[-\frac{1}{u} \right]_{\ln 2}^{\ln t} = -\frac{1}{\ln t} + \frac{1}{\ln 2}.$$

As $t \rightarrow \infty$, we have $\frac{1}{\ln t} \rightarrow 0$, so

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 2}.$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the Integral Test.

3.
$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$$

Let

$$f(x) = \frac{1}{x(\ln x)(\ln \ln x)}.$$

For $x \geq 3$, f is continuous and positive. To verify that f is decreasing, write

$$g(x) = x(\ln x)(\ln \ln x), \quad \text{so} \quad f(x) = \frac{1}{g(x)}.$$

Then

$$g'(x) = (\ln x)(\ln \ln x) + (\ln \ln x) + 1.$$

For $x \geq 3$, we have $\ln x > 0$ and $\ln \ln x > 0$, so every term on the right is positive. Hence

$$g'(x) > 0 \quad \text{for } x \geq 3.$$

Thus g is increasing, which means $f(x) = 1/g(x)$ is decreasing on $[3, \infty)$. So the Integral Test applies.

Now compute:

$$\int_3^{\infty} \frac{1}{x(\ln x)(\ln \ln x)} dx = \lim_{t \rightarrow \infty} \int_3^t \frac{1}{x(\ln x)(\ln \ln x)} dx.$$

Let

$$u = \ln \ln x.$$

Then

$$du = \frac{1}{x \ln x} dx.$$

So

$$\int_3^t \frac{1}{x(\ln x)(\ln \ln x)} dx = \int_{\ln \ln 3}^{\ln \ln t} \frac{1}{u} du = \ln |u| \Big|_{\ln \ln 3}^{\ln \ln t} = \ln(\ln \ln t) - \ln(\ln \ln 3).$$

As $t \rightarrow \infty$, we have $\ln \ln t \rightarrow \infty$, so

$$\ln(\ln \ln t) - \ln(\ln \ln 3) \rightarrow \infty.$$

Therefore,

$$\int_3^{\infty} \frac{1}{x(\ln x)(\ln \ln x)} dx$$

diverges, so

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$$

diverges by the Integral Test.

$$4. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

Let

$$f(x) = \frac{1}{x\sqrt{\ln x}}.$$

For $x \geq 2$, f is continuous and positive. Also,

$$f'(x) = -\frac{2\ln x + 1}{2x^2(\ln x)^{3/2}}.$$

Since $x \geq 2$ implies $\ln x > 0$, we have $2\ln x + 1 > 0$, so

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus f is decreasing on $[2, \infty)$, so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x\sqrt{\ln x}} dx.$$

Let

$$u = \sqrt{\ln x}.$$

Then

$$du = \frac{1}{2x\sqrt{\ln x}} dx, \quad \text{so} \quad \frac{1}{x\sqrt{\ln x}} dx = 2 du.$$

Therefore,

$$\int_2^t \frac{1}{x\sqrt{\ln x}} dx = \int_{\sqrt{\ln 2}}^{\sqrt{\ln t}} 2 du = 2u \Big|_{\sqrt{\ln 2}}^{\sqrt{\ln t}} = 2\sqrt{\ln t} - 2\sqrt{\ln 2}.$$

As $t \rightarrow \infty$, we have $\sqrt{\ln t} \rightarrow \infty$, so

$$2\sqrt{\ln t} - 2\sqrt{\ln 2} \rightarrow \infty.$$

Therefore,

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx$$

diverges, so

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

diverges by the Integral Test.

$$5. \sum_{n=1}^{\infty} \frac{1}{n^p} \quad (\text{general case})$$

Let

$$f(x) = \frac{1}{x^p} = x^{-p}.$$

Assume $p > 0$. Then for $x \geq 1$, f is continuous and positive. Also,

$$f'(x) = -px^{-p-1}.$$

Since $p > 0$ and $x \geq 1$, we have

$$f'(x) < 0 \quad \text{for } x \geq 1.$$

Thus f is decreasing on $[1, \infty)$, so the Integral Test applies.

We now compute

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx.$$

Case 1: $p \neq 1$.

$$\int_1^t x^{-p} dx = \left[\frac{x^{1-p}}{1-p} \right]_1^t = \frac{t^{1-p} - 1}{1-p}.$$

If $p > 1$, then $1 - p < 0$, so $t^{1-p} \rightarrow 0$. Hence

$$\lim_{t \rightarrow \infty} \frac{t^{1-p} - 1}{1-p} = \frac{0 - 1}{1-p} = \frac{1}{p-1}.$$

So the integral converges.

If $0 < p < 1$, then $1 - p > 0$, so $t^{1-p} \rightarrow \infty$. Hence

$$\frac{t^{1-p} - 1}{1-p} \rightarrow \infty,$$

so the integral diverges.

Case 2: $p = 1$.

$$\int_1^t \frac{1}{x} dx = \ln x \Big|_1^t = \ln t.$$

Since $\ln t \rightarrow \infty$, the integral diverges.

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if

$$p > 1.$$

6. $\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n}$

Let

$$f(x) = \frac{\sqrt{\ln x}}{x}.$$

For $x \geq 2$, f is continuous and positive. Also,

$$f'(x) = \frac{1 - 2 \ln x}{2x^2 \sqrt{\ln x}}.$$

Since $x \geq 2$ implies $\ln 2 > \frac{1}{2}$, we have

$$\ln x \geq \ln 2 > \frac{1}{2},$$

so

$$1 - 2 \ln x < 0.$$

Therefore,

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

Thus f is decreasing on $[2, \infty)$, so the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{\sqrt{\ln x}}{x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{\sqrt{\ln x}}{x} dx.$$

Let

$$u = \ln x, \quad du = \frac{1}{x} dx.$$

Then

$$\int_2^t \frac{\sqrt{\ln x}}{x} dx = \int_{\ln 2}^{\ln t} \sqrt{u} du = \int_{\ln 2}^{\ln t} u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_{\ln 2}^{\ln t}.$$

So

$$\int_2^t \frac{\sqrt{\ln x}}{x} dx = \frac{2}{3} (\ln t)^{3/2} - \frac{2}{3} (\ln 2)^{3/2}.$$

As $t \rightarrow \infty$, this tends to ∞ . Therefore,

$$\int_2^{\infty} \frac{\sqrt{\ln x}}{x} dx$$

diverges, so

$$\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n}$$

diverges by the Integral Test.

7. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

Let

$$f(x) = \frac{1}{x^2 + 1}.$$

For $x \geq 1$, f is continuous and positive. Also,

$$f'(x) = -\frac{2x}{(x^2 + 1)^2}.$$

Since $x \geq 1$, we have $f'(x) < 0$. Thus f is decreasing on $[1, \infty)$, so the Integral Test applies.

Now compute:

$$\int_1^{\infty} \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \tan^{-1}(x) \Big|_1^t.$$

Thus

$$\int_1^{\infty} \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \left(\tan^{-1}(t) - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by the Integral Test.

8. $\sum_{n=1}^{\infty} \frac{2n}{n^4 + 25}$

Let

$$f(x) = \frac{2x}{x^4 + 25}.$$

For $x \geq 1$, f is continuous and positive. Also,

$$f'(x) = \frac{2(x^4 + 25) - 8x^4}{(x^4 + 25)^2} = \frac{50 - 6x^4}{(x^4 + 25)^2}.$$

If $x \geq 2$, then $x^4 \geq 16$, so

$$50 - 6x^4 \leq 50 - 96 < 0.$$

Thus

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

So f is eventually decreasing, and the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{2x}{x^4 + 25} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{2x}{x^4 + 25} dx.$$

Let

$$u = x^2, \quad du = 2x dx.$$

Then

$$\int_2^t \frac{2x}{x^4 + 25} dx = \int_4^{t^2} \frac{1}{u^2 + 25} du = \frac{1}{5} \tan^{-1}\left(\frac{u}{5}\right) \Big|_4^{t^2}.$$

So

$$\int_2^t \frac{2x}{x^4 + 25} dx = \frac{1}{5} \tan^{-1}\left(\frac{t^2}{5}\right) - \frac{1}{5} \tan^{-1}\left(\frac{4}{5}\right).$$

As $t \rightarrow \infty$,

$$\tan^{-1}\left(\frac{t^2}{5}\right) \rightarrow \frac{\pi}{2}.$$

Hence

$$\int_2^{\infty} \frac{2x}{x^4 + 25} dx = \frac{1}{5} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{4}{5}\right) \right).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n}{n^4 + 25}$$

converges by the Integral Test.

9.
$$\sum_{n=1}^{\infty} \frac{3n^2}{n^6 + 36}$$

Let

$$f(x) = \frac{3x^2}{x^6 + 36}.$$

For $x \geq 1$, f is continuous and positive. Also,

$$f'(x) = \frac{6x(x^6 + 36) - 18x^7}{(x^6 + 36)^2} = \frac{12x(18 - x^6)}{(x^6 + 36)^2}.$$

If $x \geq 2$, then $x^6 \geq 64$, so

$$18 - x^6 < 0.$$

Thus

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

So f is eventually decreasing, and the Integral Test applies.

Now compute:

$$\int_2^{\infty} \frac{3x^2}{x^6 + 36} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{3x^2}{x^6 + 36} dx.$$

Let

$$u = x^3, \quad du = 3x^2 dx.$$

Then

$$\int_2^t \frac{3x^2}{x^6 + 36} dx = \int_8^{t^3} \frac{1}{u^2 + 36} du = \frac{1}{6} \tan^{-1}\left(\frac{u}{6}\right) \Big|_8^{t^3}.$$

So

$$\int_2^t \frac{3x^2}{x^6 + 36} dx = \frac{1}{6} \tan^{-1}\left(\frac{t^3}{6}\right) - \frac{1}{6} \tan^{-1}\left(\frac{8}{6}\right).$$

As $t \rightarrow \infty$,

$$\tan^{-1}\left(\frac{t^3}{6}\right) \rightarrow \frac{\pi}{2}.$$

Hence

$$\int_2^\infty \frac{3x^2}{x^6 + 36} dx = \frac{1}{6} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right) \right).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{3n^2}{n^6 + 36}$$

converges by the Integral Test.

10. $\sum_{n=1}^{\infty} \frac{4n^3}{n^8 + 64}$

Let

$$f(x) = \frac{4x^3}{x^8 + 64}.$$

For $x \geq 1$, f is continuous and positive. Also,

$$f'(x) = \frac{12x^2(x^8 + 64) - 32x^{10}}{(x^8 + 64)^2} = \frac{4x^2(192 - 5x^8)}{(x^8 + 64)^2}.$$

If $x \geq 2$, then $x^8 \geq 256$, so

$$192 - 5x^8 \leq 192 - 1280 < 0.$$

Thus

$$f'(x) < 0 \quad \text{for } x \geq 2.$$

So f is eventually decreasing, and the Integral Test applies.

Now compute:

$$\int_2^\infty \frac{4x^3}{x^8 + 64} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{4x^3}{x^8 + 64} dx.$$

Let

$$u = x^4, \quad du = 4x^3 dx.$$

Then

$$\int_2^t \frac{4x^3}{x^8 + 64} dx = \int_{16}^{t^4} \frac{1}{u^2 + 64} du = \frac{1}{8} \tan^{-1}\left(\frac{u}{8}\right) \Big|_{16}^{t^4}.$$

So

$$\int_2^t \frac{4x^3}{x^8 + 64} dx = \frac{1}{8} \tan^{-1}\left(\frac{t^4}{8}\right) - \frac{1}{8} \tan^{-1}(2).$$

As $t \rightarrow \infty$,

$$\tan^{-1}\left(\frac{t^4}{8}\right) \rightarrow \frac{\pi}{2}.$$

Hence

$$\int_2^\infty \frac{4x^3}{x^8 + 64} dx = \frac{1}{8} \left(\frac{\pi}{2} - \tan^{-1}(2) \right).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4n^3}{n^8 + 64}$$

converges by the Integral Test.

Direct Comparison Test

1. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$
Let

$$a_n = \frac{1}{n^2 + 1}.$$

For all $n \geq 1$,

$$0 < a_n = \frac{1}{n^2 + 1} \leq \frac{1}{n^2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, so it converges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges.

Conclusion: The series **converges**.

2. $\sum_{n=1}^{\infty} \frac{2^n}{3^n + 5}$
Let

$$a_n = \frac{2^n}{3^n + 5}.$$

For all $n \geq 1$,

$$0 < a_n = \frac{2^n}{3^n + 5} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

The series

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

is geometric with common ratio $r = \frac{2}{3}$, and since $|r| < 1$, it converges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n + 5}$$

converges.

Conclusion: The series **converges**.

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + n}$
Let

$$a_n = \frac{1}{\sqrt{n} + n}.$$

For $n \geq 1$, we have $\sqrt{n} \leq n$, so

$$\sqrt{n} + n \leq 2n.$$

Taking reciprocals gives

$$a_n = \frac{1}{\sqrt{n} + n} \geq \frac{1}{2n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges because the harmonic series diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + n}$$

diverges.

Conclusion: The series **diverges**.

4.
$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$$

Let

$$a_n = \frac{n}{n^3 + 1}.$$

For all $n \geq 1$,

$$0 < a_n = \frac{n}{n^3 + 1} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, so it converges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$$

converges.

Conclusion: The series **converges**.

5.
$$\sum_{n=1}^{\infty} \frac{3n^2 + 2}{n^4 + n^2 + 1}$$

Let

$$a_n = \frac{3n^2 + 2}{n^4 + n^2 + 1}.$$

For $n \geq 1$, we have $2 \leq 2n^2$, so $3n^2 + 2 \leq 5n^2$. Also,

$$n^4 + n^2 + 1 \geq n^4.$$

Hence, for $n \geq 1$,

$$0 < a_n = \frac{3n^2 + 2}{n^4 + n^2 + 1} \leq \frac{5n^2}{n^4} = \frac{5}{n^2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{5}{n^2} = 5 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges because $\sum \frac{1}{n^2}$ is a p -series with $p = 2 > 1$. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{3n^2 + 2}{n^4 + n^2 + 1}$$

converges.

Conclusion: The series **converges**.

6. $\sum_{n=1}^{\infty} \frac{2^n}{4^n + n}$

Let

$$a_n = \frac{2^n}{4^n + n}.$$

For all $n \geq 1$,

$$0 < a_n = \frac{2^n}{4^n + n} \leq \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n.$$

The series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is geometric with common ratio $r = \frac{1}{2}$, and since $|r| < 1$, it converges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{2^n}{4^n + n}$$

converges.

Conclusion: The series **converges**.

7. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 2}$

Let

$$a_n = \frac{n}{n^2 + 2}.$$

For $n \geq 2$, we have $2 \leq n^2$, so

$$n^2 + 2 \leq 2n^2.$$

Therefore, for $n \geq 2$,

$$a_n = \frac{n}{n^2 + 2} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

The series

$$\sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n}$$

diverges because the harmonic series diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 2}$$

diverges.

Conclusion: The series **diverges**.

$$8. \sum_{n=1}^{\infty} \frac{5^n}{3^n + 2^n}$$

Let

$$a_n = \frac{5^n}{3^n + 2^n}.$$

Since $2^n \leq 3^n$ for all $n \geq 1$, we have

$$3^n + 2^n \leq 2 \cdot 3^n.$$

Thus, for all $n \geq 1$,

$$a_n = \frac{5^n}{3^n + 2^n} \geq \frac{5^n}{2 \cdot 3^n} = \frac{1}{2} \left(\frac{5}{3}\right)^n.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{5}{3}\right)^n$$

is geometric with common ratio $r = \frac{5}{3}$, and since $r > 1$, it diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{5^n}{3^n + 2^n}$$

diverges.

Conclusion: The series **diverges**.

$$9. \sum_{n=1}^{\infty} \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n}$$

Let

$$a_n = \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n}.$$

Since $\sin^2\left(\frac{1}{n}\right) \geq 0$ for all $n \geq 1$, we have

$$a_n = \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n} \geq \frac{1}{n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1 + \sin^2\left(\frac{1}{n}\right)}{n}$$

diverges.

Conclusion: The series **diverges**.

$$10. \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Let

$$a_n = \frac{n!}{n^n}.$$

For $n \geq 2$,

$$a_n = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n}.$$

Each factor is at most 1, so

$$\begin{aligned} a_n &\leq \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots 1 \\ &= \frac{2}{n^2}. \end{aligned}$$

Thus, for $n \geq 2$,

$$0 < a_n \leq \frac{2}{n^2}.$$

The series

$$\sum_{n=2}^{\infty} \frac{2}{n^2} = 2 \sum_{n=2}^{\infty} \frac{1}{n^2}$$

converges because $\sum \frac{1}{n^2}$ is a p -series with $p = 2 > 1$. Therefore, by the Direct Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges.

Conclusion: The series **converges**.

Limit Comparison Test

1. $\sum_{n=1}^{\infty} \frac{n^2 + 3n}{n^3 - 4}$

Let

$$a_n = \frac{n^2 + 3n}{n^3 - 4} \quad \text{and} \quad b_n = \frac{1}{n}.$$

For $n \geq 2$, both a_n and b_n are positive. Compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n^2 + 3n}{n^3 - 4}}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2}{n^3 - 4}. \end{aligned}$$

Divide numerator and denominator by n^3 :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2}{n^3 - 4} &= \lim_{n \rightarrow \infty} \frac{1 + \frac{3}{n}}{1 - \frac{4}{n^3}} \\ &= 1. \end{aligned}$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series and diverges, it follows that

$$\sum_{n=1}^{\infty} \frac{n^2 + 3n}{n^3 - 4}$$

also diverges.

Conclusion: The series **diverges**.

2. $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$

Let

$$a_n = \frac{n^2}{n^4 - 1} \quad \text{and} \quad b_n = \frac{1}{n^2}.$$

For $n \geq 2$, both a_n and b_n are positive. Compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^4 - 1}}{\frac{1}{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{n^4}{n^4 - 1}. \end{aligned}$$

Divide numerator and denominator by n^4 :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^4}{n^4 - 1} &= \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{n^4}} \\ &= 1. \end{aligned}$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$$

also converges.

Conclusion: The series **converges**.

3.
$$\sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{n^2 + 5}$$

Let

$$a_n = \frac{\sqrt{n} + 1}{n^2 + 5} \quad \text{and} \quad b_n = \frac{1}{n^{3/2}}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n} + 1}{n^2 + 5}}{\frac{1}{n^{3/2}}} \\ &= \lim_{n \rightarrow \infty} \frac{n^{3/2}(\sqrt{n} + 1)}{n^2 + 5} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + n^{3/2}}{n^2 + 5}. \end{aligned}$$

Divide numerator and denominator by n^2 :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^2 + n^{3/2}}{n^2 + 5} &= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{\sqrt{n}}}{1 + \frac{5}{n^2}} \\ &= 1. \end{aligned}$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a p -series with $p = \frac{3}{2} > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{n^2 + 5}$$

also converges.

Conclusion: The series **converges**.

4.
$$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$$

Let

$$a_n = \frac{1}{n + \sqrt{n}} \quad \text{and} \quad b_n = \frac{1}{n}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n + \sqrt{n}}}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n}{n + \sqrt{n}}. \end{aligned}$$

Divide numerator and denominator by n :

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n}{n + \sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{\sqrt{n}}} \\ &= 1.\end{aligned}$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series and diverges, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$$

also diverges.

Conclusion: The series **diverges**.

5.
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

Let

$$a_n = \frac{1}{\sqrt{n} + \sqrt{n+1}} \quad \text{and} \quad b_n = \frac{1}{\sqrt{n}}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n} + \sqrt{n+1}}}{\frac{1}{\sqrt{n}}} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{1 + \frac{1}{n}}} \\ &= \frac{1}{2}.\end{aligned}$$

Since $0 < \frac{1}{2} < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a p -series with $p = \frac{1}{2} < 1$, it diverges. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

also diverges.

Conclusion: The series **diverges**.

6.
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4 + 3n}}$$

Let

$$a_n = \frac{1}{\sqrt{n^4 + 3n}} \quad \text{and} \quad b_n = \frac{1}{n^2}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n^4+3n}}}{\frac{1}{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt{n^4+3n}}.\end{aligned}$$

Factor n^4 from inside the square root:

$$\begin{aligned}\frac{n^2}{\sqrt{n^4+3n}} &= \frac{n^2}{\sqrt{n^4\left(1+\frac{3}{n^3}\right)}} \\ &= \frac{n^2}{n^2\sqrt{1+\frac{3}{n^3}}} \\ &= \frac{1}{\sqrt{1+\frac{3}{n^3}}}.\end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{3}{n^3}}} = 1.$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+3n}}$$

also converges.

Conclusion: The series **converges**.

7.
$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{4^n + 1}$$

Let

$$a_n = \frac{3^n + 2^n}{4^n + 1} \quad \text{and} \quad b_n = \left(\frac{3}{4}\right)^n.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{3^n+2^n}{4^n+1}}{\left(\frac{3}{4}\right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{4^n + 1} \cdot \frac{4^n}{3^n} \\ &= \lim_{n \rightarrow \infty} \frac{4^n(3^n + 2^n)}{3^n(4^n + 1)} \\ &= \lim_{n \rightarrow \infty} \frac{1 + \left(\frac{2}{3}\right)^n}{1 + \frac{1}{4^n}} \\ &= 1.\end{aligned}$$

Since $0 < 1 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

is geometric with common ratio $r = \frac{3}{4}$, and since $|r| < 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{4^n + 1}$$

also converges.

Conclusion: The series **converges**.

8.
$$\sum_{n=1}^{\infty} \frac{5}{n^2 + (-1)^n}$$

Let

$$a_n = \frac{5}{n^2 + (-1)^n} \quad \text{and} \quad b_n = \frac{1}{n^2}.$$

For $n \geq 2$, both a_n and b_n are positive. Compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{5}{n^2 + (-1)^n}}{\frac{1}{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{5n^2}{n^2 + (-1)^n}. \end{aligned}$$

Divide numerator and denominator by n^2 :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{5n^2}{n^2 + (-1)^n} &= \lim_{n \rightarrow \infty} \frac{5}{1 + \frac{(-1)^n}{n^2}} \\ &= 5. \end{aligned}$$

Since $0 < 5 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{n^2 + (-1)^n}$$

also converges.

Conclusion: The series **converges**.

9.
$$\sum_{n=1}^{\infty} \frac{4}{n^2 + \tan^{-1}(n)}$$

Let

$$a_n = \frac{4}{n^2 + \tan^{-1}(n)} \quad \text{and} \quad b_n = \frac{1}{n^2}.$$

Both a_n and b_n are positive for all $n \geq 1$. Compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{4}{n^2 + \tan^{-1}(n)}}{\frac{1}{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{4n^2}{n^2 + \tan^{-1}(n)}. \end{aligned}$$

Divide numerator and denominator by n^2 :

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{4n^2}{n^2 + \tan^{-1}(n)} &= \lim_{n \rightarrow \infty} \frac{4}{1 + \frac{\tan^{-1}(n)}{n^2}} \\ &= 4.\end{aligned}$$

Since $0 < 4 < \infty$, the Limit Comparison Test applies. Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, it converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{4}{n^2 + \tan^{-1}(n)}$$

also converges.

Conclusion: The series **converges**.

Alternating Series Test

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Let $b_n = \frac{1}{n}$. Then:

- $b_n > 0$, decreasing
- $\lim_{n \rightarrow \infty} b_n = 0$

The series converges by the Alternating Series Test.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Let $b_n = \frac{1}{n^2}$. Then:

- $b_n > 0$, decreasing
- $\lim_{n \rightarrow \infty} b_n = 0$

The series converges by the Alternating Series Test.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Let $b_n = \frac{1}{\sqrt{n}}$. Then:

- $b_n > 0$, decreasing
- $\lim_{n \rightarrow \infty} b_n = 0$

The series converges by the Alternating Series Test.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

Let $b_n = \frac{1}{\ln(n+1)}$. Then:

- $b_n > 0$, decreasing
- $\lim_{n \rightarrow \infty} b_n = 0$

The series converges by the Alternating Series Test.

5. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

Let $b_n = \frac{n}{n+1}$. Then:

- $\lim_{n \rightarrow \infty} b_n = 1 \neq 0$

The Alternating Series Test does not apply. The series diverges by the Test for Divergence.

6. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n + (-1)^n}$

Let $b_n = \frac{1}{n + (-1)^n}$. Then:

- b_n is not decreasing, because the even and odd terms alternate:

$$b_{2k} = \frac{1}{2k+1}, \quad b_{2k+1} = \frac{1}{2k}$$

so, for example, $b_{2k} > b_{2k+1}$.

The Alternating Series Test does not apply.

7.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$$

Let $b_n = \frac{1}{n^{1/3}}$. Then:

- $b_n > 0$, decreasing
- $\lim_{n \rightarrow \infty} b_n = 0$

The series converges by the Alternating Series Test.

8.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}$$

Let $b_n = \frac{1}{n^2 + \ln n}$. Then:

- $b_n > 0$, decreasing for $n \geq 2$
- $\lim_{n \rightarrow \infty} b_n = 0$

The series converges by the Alternating Series Test.

9.
$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sin(1/n)}{n}$$

Let $b_n = \frac{\sin(1/n)}{n}$. Then:

- $b_n > 0$
- $\lim_{n \rightarrow \infty} b_n = 0$
- To show b_n is decreasing for $n \geq 1$, define $f(x) = \frac{\sin(1/x)}{x}$. Then

$$f'(x) = -\frac{\cos(1/x)}{x^3} - \frac{\sin(1/x)}{x^2} < 0 \text{ for } x \geq 1,$$

so $b_n = f(n)$ is decreasing.

The series converges by the Alternating Series Test.

10.
$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{0.9}}$$

Let $b_n = \frac{1}{n^{0.9}}$. Then:

- $b_n > 0$, decreasing
- $\lim_{n \rightarrow \infty} b_n = 0$

The series converges by the Alternating Series Test.

Absolute Convergence

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Let

$$b_n = \frac{1}{n}.$$

Then

$$b_n > 0, \quad \lim_{n \rightarrow \infty} b_n = 0,$$

and b_n is decreasing since $\frac{1}{n+1} < \frac{1}{n}$ for all $n \geq 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by the Alternating Series Test.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series, which diverges.

Conclusion: The series **converges conditionally**.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a p -series with $p = 2$, and since $p > 1$, it converges.

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges absolutely.

Conclusion: The series **converges absolutely**.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Let

$$b_n = \frac{1}{\sqrt{n}}.$$

Then

$$b_n > 0, \quad \lim_{n \rightarrow \infty} b_n = 0,$$

and b_n is decreasing since $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all $n \geq 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by the Alternating Series Test.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a p -series with $p = \frac{1}{2}$, and since $p \leq 1$, it diverges.

Conclusion: The series **converges conditionally**.

4.
$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

Let

$$b_n = \frac{\ln n}{n}.$$

For $n \geq 2$, we have $b_n \geq 0$, and

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$$

Also,

$$f(x) = \frac{\ln x}{x} \quad \Rightarrow \quad f'(x) = \frac{1 - \ln x}{x^2}.$$

Thus $f'(x) < 0$ when $x > e$, so b_n is decreasing for $n \geq 3$. Therefore,

$$\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$$

converges by the Alternating Series Test, and hence

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

also converges.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}.$$

Using the Integral Test with $f(x) = \frac{\ln x}{x}$,

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^b \\ &= \infty. \end{aligned}$$

So

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

diverges.

Conclusion: The series **converges conditionally**.

5.
$$\sum_{n=1}^{\infty} \frac{\sin(n^2 + 1)}{n^3}$$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n^2 + 1)}{n^3} \right|.$$

Since

$$|\sin(n^2 + 1)| \leq 1$$

for all n , we have

$$\left| \frac{\sin(n^2 + 1)}{n^3} \right| \leq \frac{1}{n^3}.$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is a p -series with $p = 3 > 1$, it converges. Therefore, by the Comparison Test,

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n^2 + 1)}{n^3} \right|$$

converges.

Conclusion: The series **converges absolutely**.

6.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$$

Let

$$b_n = \frac{1}{(n+1)\ln(n+1)}.$$

Then $b_n > 0$, $b_n \rightarrow 0$, and b_n is decreasing for $n \geq 1$. Therefore, by the Alternating Series Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$$

converges.

To test for absolute convergence, consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(n+1)\ln(n+1)} \right| = \sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}.$$

Using the Integral Test with $f(x) = \frac{1}{(x+1)\ln(x+1)}$,

$$\begin{aligned} \int_1^{\infty} \frac{1}{(x+1)\ln(x+1)} dx &= \int_{\ln 2}^{\infty} \frac{1}{u} du \\ &= \infty. \end{aligned}$$

So

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$$

diverges.

Conclusion: The series converges conditionally.

$$7. \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n!}.$$

Apply the Ratio Test:

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \\ &= \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 0. \end{aligned}$$

Since $L < 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

converges absolutely.

Conclusion: The series **converges absolutely**.

$$8. \sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}}$$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^{3/2}} \right|.$$

Since

$$|\cos(n)| \leq 1$$

for all n , we have

$$\left| \frac{\cos(n)}{n^{3/2}} \right| \leq \frac{1}{n^{3/2}}.$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a p -series with $p = \frac{3}{2} > 1$, it converges. Therefore, by the Comparison Test,

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^{3/2}} \right|$$

converges.

Conclusion: The series **converges absolutely**.

9. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

Consider the series of absolute values:

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^2} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

Use the Integral Test with

$$f(x) = \frac{1}{x(\ln x)^2}.$$

For $x \geq 2$, $f(x)$ is positive and continuous. Also,

$$\begin{aligned} f'(x) &= -\frac{(\ln x)^2 + 2 \ln x}{x^2(\ln x)^4} \\ &= -\frac{\ln x + 2}{x^2(\ln x)^3}. \end{aligned}$$

Since $x \geq 2$ implies $\ln x > 0$, we have $f'(x) < 0$, so $f(x)$ is decreasing on $[2, \infty)$.

Now compute:

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^2} dx &= \int_{\ln 2}^{\infty} \frac{1}{u^2} du \\ &= \left[-\frac{1}{u} \right]_{\ln 2}^{\infty} \\ &= \frac{1}{\ln 2}. \end{aligned}$$

Since this improper integral converges, the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the Integral Test. Therefore,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

converges absolutely.

Conclusion: The series **converges absolutely**.

10. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|.$$

Since

$$|\sin(n)| \leq 1$$

for all n , we have

$$\left| \frac{\sin(n)}{n^2} \right| \leq \frac{1}{n^2}.$$

Because

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, it converges. Therefore, by the Comparison Test,

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|$$

converges.

Conclusion: The series **converges absolutely**.

Ratio Test

1. $\sum_{n=1}^{\infty} \frac{5^n}{n^2}$

Let $a_n = \frac{5^n}{n^2}$. Then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{5^{n+1}}{(n+1)^2} \cdot \frac{n^2}{5^n} \\ &= 5 \cdot \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 = 5.\end{aligned}$$

Since the limit is greater than 1, the series diverges.

2. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Let $a_n = \frac{n^2}{2^n}$. Then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\ &= \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^2 = \frac{1}{2}.\end{aligned}$$

The series converges absolutely.

3. $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

Let $a_n = \frac{2^n}{n^n}$. Then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n} \\ &= 2 \cdot \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \cdot \frac{1}{n+1} = 0.\end{aligned}$$

The series converges absolutely.

4. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Let $a_n = \frac{2^n}{n!}$. Then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0.\end{aligned}$$

The series converges absolutely.

5. $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

Let $a_n = \frac{n!}{3^n}$. Then

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{3} = \infty.\end{aligned}$$

The series diverges.

$$6. \sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$$

Let $a_n = \frac{3^n n!}{n^n}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} \\ &= 3 \cdot \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = 3 \cdot \frac{1}{e}. \end{aligned}$$

Since $3/e > 1$, the series diverges.

$$7. \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

Let $a_n = \frac{n!}{(2n)!}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)}{(2n+2)(2n+1)} = 0. \end{aligned}$$

The series converges absolutely.

$$8. \sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n^n}$$

Let $a_n = \frac{(2n)!}{n! \cdot n^n}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(2n+2)!}{(n+1)! \cdot (n+1)^{n+1}} \cdot \frac{n! \cdot n^n}{(2n)!} \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1) \cdot (n+1)^{n+1}} \cdot n^n \\ &= 0. \end{aligned}$$

The series converges absolutely.

$$9. \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{2^n}$$

Let $a_n = \left| (-1)^n \cdot \frac{n}{2^n} \right| = \frac{n}{2^n}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \\ &= \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \frac{1}{2}. \end{aligned}$$

The series converges absolutely.

$$10. \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2}{n!}$$

Let $a_n = \left| (-1)^n \cdot \frac{n^2}{n!} \right| = \frac{n^2}{n!}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n+1} \cdot \frac{1}{n^2} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0. \end{aligned}$$

The series converges absolutely.

Mixed Series Practice

1. Evaluate $\lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + \cos n}}{n}$.

Since $n > 0$,

$$\frac{\sqrt{n^2 + \cos n}}{n} = \sqrt{1 + \frac{\cos n}{n^2}}.$$

Because $-1 \leq \cos n \leq 1$, we have

$$-\frac{1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2},$$

so $\frac{\cos n}{n^2} \rightarrow 0$. Therefore,

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + \cos n}}{n} = \sqrt{1 + 0} = 1.$$

2. Evaluate $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n}\right)^{3n}$.

We use the fact that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Now observe that

$$\left(1 - \frac{1}{2n}\right)^{3n} = \left[\left(1 - \frac{1}{2n}\right)^{2n}\right]^{3/2}.$$

The expression inside the brackets is in the correct form:

$$\left(1 - \frac{1}{2n}\right)^{2n} = \left(1 + \frac{-1}{2n}\right)^{2n}.$$

So if we let $m = 2n$, then as $n \rightarrow \infty$, we also have $m \rightarrow \infty$, and

$$\left(1 - \frac{1}{2n}\right)^{2n} = \left(1 - \frac{1}{m}\right)^m \rightarrow e^{-1}.$$

Therefore,

$$\left(1 - \frac{1}{2n}\right)^{3n} = \left[\left(1 - \frac{1}{2n}\right)^{2n}\right]^{3/2} \rightarrow (e^{-1})^{3/2} = e^{-3/2}.$$

Thus,

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n}\right)^{3n} = e^{-3/2}.$$

3. Suppose $S_n = \sum_{k=1}^n a_k = \frac{n^2}{\ln(n+1) + 3n}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

A series converges if and only if its sequence of partial sums converges to a finite limit. Here,

$$S_n = \frac{n^2}{\ln(n+1) + 3n} = \frac{n}{\frac{\ln(n+1)}{n} + 3}.$$

Since $\frac{\ln(n+1)}{n} \rightarrow 0$, it follows that

$$S_n \sim \frac{n}{3} \rightarrow \infty.$$

Thus the sequence of partial sums does not approach a finite limit, so

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

4. Suppose $S_n = \sum_{k=1}^n a_k = \frac{4n^2 + 1}{2n^2 + 5n + 1}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

Since the sum of an infinite series is the limit of its partial sums, we compute

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{4n^2 + 1}{2n^2 + 5n + 1} = \frac{4}{2} = 2.$$

Therefore, the series converges, and

$$\sum_{n=1}^{\infty} a_n = 2.$$

5. If $\sum a_n = 5$ and $\sum b_n = 4$, find $\sum (2a_n - 3b_n)$.

By linearity of convergent series,

$$\sum (2a_n - 3b_n) = 2 \sum a_n - 3 \sum b_n = 2(5) - 3(4) = 10 - 12 = -2.$$

6. Find the sum of the geometric series: $\sum_{n=2}^{\infty} \frac{6}{4^n}$.

Rewrite the series as

$$\sum_{n=2}^{\infty} \frac{6}{4^n} = \sum_{n=2}^{\infty} 6 \left(\frac{1}{4}\right)^n.$$

This is geometric with first term

$$a = \frac{6}{4^2} = \frac{3}{8}$$

and common ratio

$$r = \frac{1}{4}.$$

Since $|r| < 1$, the series converges, and its sum is

$$\sum_{n=2}^{\infty} \frac{6}{4^n} = \frac{a}{1-r} = \frac{\frac{3}{8}}{1-\frac{1}{4}} = \frac{\frac{3}{8}}{\frac{3}{4}} = \frac{1}{2}.$$

7. Determine whether the series $\sum_{n=2}^{\infty} \frac{n^4 + 3}{n(n+1)^2}$ converges. If it converges, find its sum.

Let

$$a_n = \frac{n^4 + 3}{n(n+1)^2} = \frac{n^4 + 3}{n^3 + 2n^2 + n} = n \cdot \frac{1 + \frac{3}{n^4}}{1 + \frac{2}{n} + \frac{1}{n^2}}.$$

As $n \rightarrow \infty$, the fraction approaches 1, so $a_n \sim n$, and in particular

$$a_n \rightarrow \infty.$$

Since the terms of the series do not approach 0, the series diverges by the Test for Divergence.

8. Determine whether the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n - 1}$ converges.

Let

$$a_n = \frac{4^n}{5^n - 1}, \quad b_n = \left(\frac{4}{5}\right)^n.$$

Then

$$\frac{a_n}{b_n} = \frac{4^n}{5^n - 1} \cdot \frac{5^n}{4^n} = \frac{5^n}{5^n - 1} = \frac{1}{1 - 5^{-n}} \rightarrow 1.$$

Since

$$\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$$

is a convergent geometric series, the Limit Comparison Test implies that

$$\sum_{n=1}^{\infty} \frac{4^n}{5^n - 1}$$

also converges.

9. Let $a_n = \frac{n^2}{2^n}$. Use the Ratio Test to determine whether $\sum a_n$ converges.

We compute

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \left(\frac{n+1}{n} \right)^2.$$

Therefore,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 = \frac{1}{2}.$$

Since this limit is less than 1, the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges by the Ratio Test.

10. Use the Ratio Test to determine for which values of $c > 0$ the series $\sum_{n=1}^{\infty} \frac{c^n}{n}$ converges.

Let

$$a_n = \frac{c^n}{n}.$$

Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{c^{n+1}}{n+1} \cdot \frac{n}{c^n} = c \cdot \frac{n}{n+1}.$$

Thus,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = c.$$

By the Ratio Test:

- if $0 < c < 1$, then the series converges;
- if $c > 1$, then the series diverges;
- if $c = 1$, the Ratio Test is inconclusive.

When $c = 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

the harmonic series, which diverges. Therefore,

$$\sum_{n=1}^{\infty} \frac{c^n}{n} \text{ converges precisely for } 0 < c < 1.$$

11. Determine whether the series $\sum_{n=1}^{\infty} \frac{2n}{n^3 + \cos n}$ converges.

Let

$$a_n = \frac{2n}{n^3 + \cos n}.$$

Since $-1 \leq \cos n \leq 1$, we have

$$n^3 + \cos n \geq n^3 - 1.$$

Hence, for $n \geq 2$,

$$0 < a_n \leq \frac{2n}{n^3 - 1}.$$

Also, for $n \geq 2$,

$$n^3 - 1 \geq \frac{1}{2}n^3,$$

so

$$\frac{2n}{n^3 - 1} \leq \frac{2n}{\frac{1}{2}n^3} = \frac{4}{n^2}.$$

Therefore, for $n \geq 2$,

$$0 < a_n \leq \frac{4}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{4}{n^2}$$

is a convergent p -series, the given series converges by the Direct Comparison Test.

12. Determine whether $\sum_{n=1}^{\infty} \frac{\sin(n^3)}{n^3 + 1}$ converges absolutely, conditionally, or diverges.

Consider the absolute value:

$$\left| \frac{\sin(n^3)}{n^3 + 1} \right| \leq \frac{1}{n^3 + 1} < \frac{1}{n^3}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges, the Direct Comparison Test shows that

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n^3)}{n^3 + 1} \right|$$

converges. Therefore, the original series converges absolutely.

13. Determine whether $\sum_{n=3}^{\infty} \frac{(-1)^n}{n^2 - 4}$ converges absolutely, conditionally, or diverges.

To test for absolute convergence, consider

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n^2 - 4} \right| = \sum_{n=3}^{\infty} \frac{1}{n^2 - 4}.$$

Compare this with $\sum \frac{1}{n^2}$. We have

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2 - 4}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 - 4} = 1.$$

Since

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

converges, the Limit Comparison Test implies that

$$\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$$

also converges. Thus the original series converges absolutely.

14. Use the Alternating Series Remainder Theorem to estimate the error when approximating $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3}$ using the first 4 terms.

Let

$$b_n = \frac{1}{n^3}.$$

Then $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$, so the Alternating Series Remainder Theorem applies. If R_4 denotes the error after using the first four terms, then

$$|R_4| \leq b_5 = \frac{1}{5^3} = \frac{1}{125}.$$

Therefore, the error is at most

$$\boxed{\frac{1}{125}}.$$

15. Use the Integral Test to show that $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^2}$ converges. Estimate the error when approximating the series with the first 5 terms.

Let

$$f(x) = \frac{1}{x(\ln x)^2}, \quad x \geq 2.$$

For $x \geq 2$, the function f is positive and continuous. Also,

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3} < 0 \quad (x > 1),$$

so f is decreasing. Therefore, the Integral Test applies.

Now compute

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx.$$

Let $u = \ln x$, so $du = \frac{1}{x} dx$. Then

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\infty} \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_{\ln 2}^{\infty} = \frac{1}{\ln 2}.$$

Since the improper integral converges, the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the Integral Test.

The first five terms are the terms with $n = 2, 3, 4, 5, 6$, so the corresponding partial sum is S_6 . Let

$$R_6 = S - S_6.$$

By the Integral Test remainder estimate,

$$\int_7^{\infty} \frac{1}{x(\ln x)^2} dx < R_6 < \int_6^{\infty} \frac{1}{x(\ln x)^2} dx.$$

Using the antiderivative found above,

$$\frac{1}{\ln 7} < R_6 < \frac{1}{\ln 6}.$$

Thus the error satisfies

$$\frac{1}{\ln 7} < R_6 < \frac{1}{\ln 6},$$

so in particular,

$$R_6 < \frac{1}{\ln 6} \approx 0.558.$$

Remainder Estimates

1. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ using the first 4 terms.

$$\begin{aligned} S_4 &= 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \\ &= \frac{115}{144} \\ &\approx 0.7986. \end{aligned}$$

By the Alternating Series Estimation Theorem,

$$|R_4| \leq b_5 = \frac{1}{5^2} = \frac{1}{25} = 0.04.$$

2. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 1}$ using the first 3 terms.

$$\begin{aligned} S_3 &= -\frac{1}{2} + \frac{1}{9} - \frac{1}{28} \\ &= -\frac{107}{252} \\ &\approx -0.4246. \end{aligned}$$

Thus,

$$|R_3| \leq b_4 = \frac{1}{4^3 + 1} = \frac{1}{65} \approx 0.0154.$$

3. How many terms are needed to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ to within 0.0001?

We want

$$|R_n| \leq b_{n+1} = \frac{1}{(n+1)^5} < 0.0001.$$

So

$$\begin{aligned} \frac{1}{(n+1)^5} < 0.0001 &\iff (n+1)^5 > 10000 \\ &\iff n+1 > \sqrt[5]{10000} \approx 6.31. \end{aligned}$$

Therefore, the smallest such n is $n = 6$, so 6 terms are needed.

4. Use the first 5 terms of $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n+1)}{n^2}$ to approximate the sum.

$$\begin{aligned} S_5 &= -\frac{\ln 2}{1^2} + \frac{\ln 3}{2^2} - \frac{\ln 4}{3^2} + \frac{\ln 5}{4^2} - \frac{\ln 6}{5^2} \\ &\approx -0.6931 + 0.2747 - 0.1540 + 0.1006 - 0.0717 \\ &\approx -0.5436. \end{aligned}$$

Thus,

$$|R_5| \leq b_6 = \frac{\ln 7}{6^2} \approx 0.0541.$$

5. Determine the minimum number of terms needed to estimate $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ with error less than 0.01.

We need

$$|R_n| \leq b_{n+1} = \frac{1}{(n+1) \ln(n+1)} < 0.01.$$

Checking nearby values,

$$\frac{1}{29 \ln 29} > 0.01, \quad \frac{1}{30 \ln 30} < 0.01.$$

So the smallest n is $n = 29$.

Since the series starts at $n = 2$, this means using terms $n = 2$ through $n = 29$, i.e. 28 terms.

6. Approximate $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ using the first 5 terms.

$$\begin{aligned} S_5 &= \sum_{n=1}^5 \frac{1}{n^2 + 1} \\ &= \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} \\ &\approx 0.8973. \end{aligned}$$

Using the Integral Test Remainder Estimate,

$$\begin{aligned} R_5 &\leq \int_5^{\infty} \frac{1}{x^2 + 1} dx \\ &= \tan^{-1}(x) \Big|_5^{\infty} \\ &= \frac{\pi}{2} - \tan^{-1}(5) \\ &\approx 0.1974. \end{aligned}$$

So the error is less than 0.1974.

7. Estimate the remainder when approximating

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

by its partial sum through $n = 10$.

Let $f(x) = \frac{1}{x(\ln x)^2}$. Then

$$\int_{11}^{\infty} f(x) dx \leq R_{10} \leq \int_{10}^{\infty} f(x) dx.$$

Using $u = \ln x$, $du = \frac{1}{x} dx$,

$$\begin{aligned} \int \frac{1}{x(\ln x)^2} dx &= \int u^{-2} du \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\ln x} + C. \end{aligned}$$

Thus,

$$\int_{10}^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 10},$$
$$\int_{11}^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 11}.$$

Therefore,

$$\boxed{\frac{1}{\ln 11} \leq R_{10} \leq \frac{1}{\ln 10}}.$$

8. Estimate $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ using the first 8 terms.

$$S_8 = \sum_{n=1}^8 \frac{1}{n^{3/2}}$$
$$\approx 1 + 0.3536 + 0.1925 + 0.1250 + 0.0894 + 0.0680 + 0.0540 + 0.0442$$
$$\approx 1.9267.$$

Also,

$$R_8 \leq \int_8^{\infty} \frac{1}{x^{3/2}} dx$$
$$= -2x^{-1/2} \Big|_8^{\infty}$$
$$= \frac{2}{\sqrt{8}}$$
$$\approx 0.7071.$$

9. Find the smallest n such that $\sum_{k=2}^n \frac{1}{k(\ln k)^2}$ approximates the full sum within 0.05.

We want

$$R_n \leq \int_n^{\infty} \frac{1}{x(\ln x)^2} dx < 0.05.$$

Using $u = \ln x$, $du = \frac{1}{x} dx$,

$$\int_n^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{\ln n}^{\infty} \frac{1}{u^2} du$$
$$= \frac{1}{\ln n}.$$

So

$$\frac{1}{\ln n} < 0.05 \iff \ln n > 20$$
$$\iff n > e^{20}.$$

Therefore, the smallest integer n is

$$\boxed{n = 485,165,196}.$$

10. Approximate $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ using the first 6 terms.

$$\begin{aligned} S_6 &= \sum_{n=2}^7 \frac{\ln n}{n^3} \\ &\approx 0.0866 + 0.0407 + 0.0217 + 0.0129 + 0.0083 + 0.0057 \\ &\approx 0.1758. \end{aligned}$$

For the error,

$$R_6 \leq \int_7^{\infty} \frac{\ln x}{x^3} dx.$$

Using integration by parts with

$$u = \ln x, \quad dv = x^{-3} dx, \quad du = \frac{1}{x} dx, \quad v = -\frac{1}{2x^2},$$

we get

$$\begin{aligned} \int \frac{\ln x}{x^3} dx &= -\frac{\ln x}{2x^2} + \frac{1}{2} \int x^{-3} dx \\ &= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} R_6 &\leq -\frac{\ln x}{2x^2} - \frac{1}{4x^2} \Big|_7^{\infty} \\ &= \frac{\ln 7}{98} + \frac{1}{196} \\ &\approx 0.0250. \end{aligned}$$

So the error is less than 0.0250.

Taylor Polynomials

For each function,

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4.$$

1. $f(x) = x^2e^x$, $a = 0$

$$\begin{aligned} f(x) &= x^2e^x & f(0) &= 0 \\ f'(x) &= (2x + x^2)e^x & f'(0) &= 0 \\ f''(x) &= (2 + 4x + x^2)e^x & f''(0) &= 2 \\ f'''(x) &= (6 + 6x + x^2)e^x & f'''(0) &= 6 \\ f^{(4)}(x) &= (12 + 8x + x^2)e^x & f^{(4)}(0) &= 12 \end{aligned}$$

$$T_4(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{12}{4!}x^4 = x^2 + x^3 + \frac{x^4}{2}.$$

2. $f(x) = \frac{1}{1+x^2}$, $a = 1$

$$\begin{aligned} f(x) &= (1+x^2)^{-1} & f(1) &= \frac{1}{2} \\ f'(x) &= -\frac{2x}{(1+x^2)^2} & f'(1) &= -\frac{1}{2} \\ f''(x) &= \frac{6x^2-2}{(1+x^2)^3} & f''(1) &= \frac{1}{2} \\ f'''(x) &= \frac{24x(1-x^2)}{(1+x^2)^4} & f'''(1) &= 0 \\ f^{(4)}(x) &= \frac{24(1-10x^2+5x^4)}{(1+x^2)^5} & f^{(4)}(1) &= -3 \end{aligned}$$

$$T_4(x) = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1/2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4.$$

$$T_4(x) = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4.$$

3. $f(x) = \cos(x^2)$, $a = 0$

$$\begin{aligned} f(x) &= \cos(x^2) & f(0) &= 1 \\ f'(x) &= -2x \sin(x^2) & f'(0) &= 0 \\ f''(x) &= -2 \sin(x^2) - 4x^2 \cos(x^2) & f''(0) &= 0 \\ f'''(x) &= -12x \cos(x^2) + 8x^3 \sin(x^2) & f'''(0) &= 0 \\ f^{(4)}(x) &= -12 \cos(x^2) + 48x^2 \sin(x^2) + 16x^4 \cos(x^2) & f^{(4)}(0) &= -12 \end{aligned}$$

$$T_4(x) = 1 + 0x + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-12}{4!}x^4 = 1 - \frac{x^4}{2}.$$

4. $f(x) = \sin(2x)$, $a = \frac{\pi}{4}$

$$\begin{array}{ll} f(x) = \sin(2x) & f\left(\frac{\pi}{4}\right) = 1 \\ f'(x) = 2\cos(2x) & f'\left(\frac{\pi}{4}\right) = 0 \\ f''(x) = -4\sin(2x) & f''\left(\frac{\pi}{4}\right) = -4 \\ f'''(x) = -8\cos(2x) & f'''\left(\frac{\pi}{4}\right) = 0 \\ f^{(4)}(x) = 16\sin(2x) & f^{(4)}\left(\frac{\pi}{4}\right) = 16 \end{array}$$

$$T_4(x) = 1 + 0\left(x - \frac{\pi}{4}\right) + \frac{-4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{0}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{16}{4!}\left(x - \frac{\pi}{4}\right)^4.$$

$$T_4(x) = 1 - 2\left(x - \frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4.$$

5. $f(x) = \ln(1 + 2x)$, $a = 0$

$$\begin{array}{ll} f(x) = \ln(1 + 2x) & \\ f'(x) = \frac{2}{1 + 2x} & f(0) = 0 \\ f''(x) = -\frac{4}{(1 + 2x)^2} & f'(0) = 2 \\ f'''(x) = \frac{16}{(1 + 2x)^3} & f''(0) = -4 \\ f^{(4)}(x) = -\frac{96}{(1 + 2x)^4} & f'''(0) = 16 \\ & f^{(4)}(0) = -96 \end{array}$$

$$T_4(x) = 0 + 2x + \frac{-4}{2!}x^2 + \frac{16}{3!}x^3 + \frac{-96}{4!}x^4 = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4.$$

6. $f(x) = \tan^{-1}(x)$, $a = 0$

$$\begin{array}{ll} f(x) = \tan^{-1}(x) & \\ f'(x) = \frac{1}{1 + x^2} & f(0) = 0 \\ f''(x) = -\frac{2x}{(1 + x^2)^2} & f'(0) = 1 \\ f'''(x) = \frac{6x^2 - 2}{(1 + x^2)^3} & f''(0) = 0 \\ f^{(4)}(x) = \frac{24x(1 - x^2)}{(1 + x^2)^4} & f'''(0) = -2 \\ & f^{(4)}(0) = 0 \end{array}$$

$$T_4(x) = 0 + x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3 + \frac{0}{4!}x^4 = x - \frac{x^3}{3}.$$

$$7. f(x) = \frac{x}{1-x}, \quad a = 0$$

$$\begin{aligned} f(x) &= \frac{x}{1-x} & f(0) &= 0 \\ f'(x) &= \frac{1}{(1-x)^2} & f'(0) &= 1 \\ f''(x) &= \frac{2}{(1-x)^3} & f''(0) &= 2 \\ f'''(x) &= \frac{6}{(1-x)^4} & f'''(0) &= 6 \\ f^{(4)}(x) &= \frac{24}{(1-x)^5} & f^{(4)}(0) &= 24 \end{aligned}$$

$$T_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4 = x + x^2 + x^3 + x^4.$$

$$8. f(x) = e^{x^2}, \quad a = 0$$

$$\begin{aligned} f(x) &= e^{x^2} & f(0) &= 1 \\ f'(x) &= 2xe^{x^2} & f'(0) &= 0 \\ f''(x) &= (2 + 4x^2)e^{x^2} & f''(0) &= 2 \\ f'''(x) &= (12x + 8x^3)e^{x^2} & f'''(0) &= 0 \\ f^{(4)}(x) &= (12 + 48x^2 + 16x^4)e^{x^2} & f^{(4)}(0) &= 12 \end{aligned}$$

$$T_4(x) = 1 + 0x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 = 1 + x^2 + \frac{x^4}{2}.$$

$$9. f(x) = \frac{1}{\sqrt{1-x}}, \quad a = 0$$

$$\begin{aligned} f(x) &= (1-x)^{-1/2} & f(0) &= 1 \\ f'(x) &= \frac{1}{2}(1-x)^{-3/2} & f'(0) &= \frac{1}{2} \\ f''(x) &= \frac{3}{4}(1-x)^{-5/2} & f''(0) &= \frac{3}{4} \\ f'''(x) &= \frac{15}{8}(1-x)^{-7/2} & f'''(0) &= \frac{15}{8} \\ f^{(4)}(x) &= \frac{105}{16}(1-x)^{-9/2} & f^{(4)}(0) &= \frac{105}{16} \end{aligned}$$

$$T_4(x) = 1 + \frac{1}{2}x + \frac{3/4}{2!}x^2 + \frac{15/8}{3!}x^3 + \frac{105/16}{4!}x^4.$$

$$T_4(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4.$$

Power Series

1.
$$\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n \cdot 3^n}$$

Let

$$a_n = \frac{(2x-5)^n}{n \cdot 3^n}.$$

To find the radius and interval of convergence, we apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2x-5)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(2x-5)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{2x-5}{3} \cdot \frac{n}{n+1} \right| \\ &= \left| \frac{2x-5}{3} \right|. \end{aligned}$$

The series converges when

$$\left| \frac{2x-5}{3} \right| < 1.$$

This inequality is equivalent to

$$|2x-5| < 3,$$

so

$$-3 < 2x-5 < 3.$$

Adding 5 throughout gives

$$2 < 2x < 8,$$

and dividing by 2 gives

$$1 < x < 4.$$

Thus the open interval of convergence is $(1, 4)$.

Now we test the endpoints.

At $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(2(1)-5)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is an alternating harmonic series. It converges by the Alternating Series Test since

$$b_n = \frac{1}{n} > 0, \quad b_{n+1} < b_n, \quad \lim_{n \rightarrow \infty} b_n = 0.$$

At $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(2(4)-5)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series, so it diverges.

Therefore,

$$\boxed{R = \frac{3}{2}, \quad I = [1, 4].}$$

$$2. \sum_{n=1}^{\infty} \frac{n^2(x+4)^n}{7^n}$$

Let

$$a_n = \frac{n^2(x+4)^n}{7^n}.$$

Apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2(x+4)^{n+1}}{7^{n+1}} \cdot \frac{7^n}{n^2(x+4)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{n^2} \cdot \left| \frac{x+4}{7} \right| \right) \\ &= \left| \frac{x+4}{7} \right|. \end{aligned}$$

The series converges when

$$\left| \frac{x+4}{7} \right| < 1.$$

This gives

$$|x+4| < 7,$$

so

$$-7 < x+4 < 7.$$

Subtracting 4 yields

$$-11 < x < 3.$$

Thus the open interval of convergence is $(-11, 3)$.

Now test the endpoints.

At $x = -11$:

$$\sum_{n=1}^{\infty} \frac{n^2(-11+4)^n}{7^n} = \sum_{n=1}^{\infty} \frac{n^2(-7)^n}{7^n} = \sum_{n=1}^{\infty} n^2(-1)^n.$$

Here the terms are

$$a_n = n^2(-1)^n.$$

Since

$$\lim_{n \rightarrow \infty} a_n$$

does not exist, and in particular the terms do *not* approach 0, the series diverges by the Test for Divergence.

At $x = 3$:

$$\sum_{n=1}^{\infty} \frac{n^2(3+4)^n}{7^n} = \sum_{n=1}^{\infty} \frac{n^2 7^n}{7^n} = \sum_{n=1}^{\infty} n^2.$$

Again, the terms do not approach 0, so the series diverges by the Test for Divergence.

Therefore,

$$\boxed{R = 7, \quad I = (-11, 3).}$$

$$3. \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

Let

$$a_n = \frac{(x-1)^n}{n!}.$$

Apply the Ratio Test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-1)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x-1}{n+1} \right| \\ &= 0.\end{aligned}$$

Since $0 < 1$ for every real number x , the series converges for all x .

Therefore, the radius of convergence is infinite, and the interval of convergence is all real numbers:

$$\boxed{R = \infty, \quad I = (-\infty, \infty).}$$

4.
$$\sum_{n=1}^{\infty} \frac{(x+2)^{2n}}{5^n}$$

Rewrite the series as

$$\sum_{n=1}^{\infty} \left(\frac{(x+2)^2}{5} \right)^n.$$

This is a geometric series with common ratio

$$r = \frac{(x+2)^2}{5}.$$

A geometric series converges exactly when $|r| < 1$. Therefore, we require

$$\left| \frac{(x+2)^2}{5} \right| < 1.$$

Since $(x+2)^2 \geq 0$, this becomes

$$\frac{(x+2)^2}{5} < 1,$$

so

$$(x+2)^2 < 5.$$

Taking square roots gives

$$-\sqrt{5} < x+2 < \sqrt{5},$$

and subtracting 2 gives

$$-2 - \sqrt{5} < x < -2 + \sqrt{5}.$$

Now test the endpoints.

At $x = -2 - \sqrt{5}$:

$$x+2 = -\sqrt{5}, \quad \frac{(x+2)^2}{5} = \frac{5}{5} = 1.$$

So the series becomes

$$\sum_{n=1}^{\infty} 1,$$

which diverges.

At $x = -2 + \sqrt{5}$:

$$x+2 = \sqrt{5}, \quad \frac{(x+2)^2}{5} = \frac{5}{5} = 1.$$

Again the series becomes

$$\sum_{n=1}^{\infty} 1,$$

which diverges.

Therefore,

$$\boxed{R = \sqrt{5}, \quad I = (-2 - \sqrt{5}, -2 + \sqrt{5}).}$$

5.
$$\sum_{n=1}^{\infty} \frac{(-1)^n (3x)^n}{n^2}$$

We may rewrite the series as

$$\sum_{n=1}^{\infty} \frac{(-3x)^n}{n^2}.$$

Let

$$a_n = \frac{(-3x)^n}{n^2}.$$

Apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-3x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-3x)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left(3|x| \cdot \frac{n^2}{(n+1)^2} \right) \\ &= 3|x|. \end{aligned}$$

The series converges when

$$3|x| < 1,$$

that is,

$$|x| < \frac{1}{3}.$$

So the open interval of convergence is

$$\left(-\frac{1}{3}, \frac{1}{3} \right).$$

Now test the endpoints.

At $x = \frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (3 \cdot \frac{1}{3})^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p -series with $p = 2 > 1$, the series converges absolutely.

At $x = -\frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (3 \cdot (-\frac{1}{3}))^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is also a p -series with $p = 2 > 1$. Hence it converges.

Therefore,

$$\boxed{R = \frac{1}{3}, \quad I = \left[-\frac{1}{3}, \frac{1}{3} \right].}$$

$$6. \sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$$

Let

$$a_n = \frac{x^n}{(n+2)!}.$$

To determine the radius and interval of convergence, we apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+3} \right| \\ &= 0. \end{aligned}$$

Since $0 < 1$ for every real number x , the series converges for all real x .

Therefore,

$$\boxed{R = \infty, \quad I = (-\infty, \infty).}$$

$$7. \sum_{n=2}^{\infty} \frac{(x-7)^n}{n \ln n}$$

Let

$$a_n = \frac{(x-7)^n}{n \ln n}.$$

Apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-7)^{n+1}}{(n+1) \ln(n+1)} \cdot \frac{n \ln n}{(x-7)^n} \right| \\ &= |x-7| \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} \\ &= |x-7|. \end{aligned}$$

Thus the series converges when

$$|x-7| < 1,$$

which gives

$$6 < x < 8.$$

So the open interval of convergence is $(6, 8)$, and the center is 7. Hence

$$R = 1.$$

Now test the endpoints.

At $x = 6$:

$$\sum_{n=2}^{\infty} \frac{(6-7)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}.$$

This is an alternating series of the form

$$\sum_{n=2}^{\infty} (-1)^n b_n \quad \text{with} \quad b_n = \frac{1}{n \ln n}.$$

To apply the Alternating Series Test, we verify the required conditions:

$$b_n > 0 \quad \text{for } n \geq 2,$$

since $n > 0$ and $\ln n > 0$ for $n \geq 2$.

Also, $b_n \rightarrow 0$ as $n \rightarrow \infty$ because $n \ln n \rightarrow \infty$.

Finally, b_n is decreasing for $x \geq 2$. Indeed, let

$$f(x) = \frac{1}{x \ln x}.$$

Then

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0 \quad \text{for } x \geq 2.$$

So f is decreasing, which means $b_n = \frac{1}{n \ln n}$ is decreasing.

Therefore,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

converges by the Alternating Series Test.

At $x = 8$:

$$\sum_{n=2}^{\infty} \frac{(8 - 7)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

This is a positive-term series. Using the Integral Test with

$$f(x) = \frac{1}{x \ln x},$$

we know f is continuous, positive, and decreasing for $x \geq 2$, and

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln x} dx.$$

Let $u = \ln x$, so $du = \frac{1}{x} dx$. Then

$$\int_2^t \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln t} \frac{1}{u} du = \ln(\ln t) - \ln(\ln 2).$$

As $t \rightarrow \infty$, this tends to ∞ . Hence

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

Therefore,

$$\boxed{R = 1, \quad I = [6, 8].}$$

8.
$$\sum_{n=0}^{\infty} \frac{n!(x+1)^n}{n^n}$$

Let

$$a_n = \frac{n!(x+1)^n}{n^n}.$$

Apply the Ratio Test:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)!(x+1)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!(x+1)^n}\right| \\ &= |x+1| \cdot \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= |x+1| \left(\frac{n}{n+1}\right)^n.\end{aligned}$$

Now

$$\left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n} \rightarrow \frac{1}{e}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+1|}{e}.$$

The series converges when

$$\frac{|x+1|}{e} < 1,$$

that is,

$$|x+1| < e.$$

So the open interval of convergence is

$$-1 - e < x < -1 + e.$$

Hence the center is -1 and the radius is

$$R = e.$$

Now test the endpoints.

At $x = -1 + e$:

$$x + 1 = e,$$

so the series becomes

$$\sum_{n=0}^{\infty} \frac{n!e^n}{n^n}.$$

Let

$$b_n = \frac{n!e^n}{n^n}.$$

Then

$$\frac{b_{n+1}}{b_n} = e \left(\frac{n}{n+1}\right)^n.$$

Since

$$\left(1 + \frac{1}{n}\right)^n < e,$$

taking reciprocals gives

$$\left(\frac{n}{n+1}\right)^n > \frac{1}{e}.$$

Therefore,

$$\frac{b_{n+1}}{b_n} = e \left(\frac{n}{n+1}\right)^n > 1.$$

So $\{b_n\}$ is increasing. Since $b_1 = e > 0$, it follows that $b_n \geq e$ for all $n \geq 1$. In particular,

$$\lim_{n \rightarrow \infty} b_n \neq 0.$$

Therefore, the series diverges by the Test for Divergence.

At $x = -1 - e$:

$$x + 1 = -e,$$

so the series becomes

$$\sum_{n=0}^{\infty} \frac{n!(-e)^n}{n^n}.$$

Its terms have absolute value

$$\frac{n!e^n}{n^n} = b_n.$$

From the work above, b_n is increasing and does not approach 0. Hence the terms of the series do not approach 0, so the series diverges by the Test for Divergence.

Therefore,

$$\boxed{R = e, \quad I = (-1 - e, -1 + e)}.$$

9.
$$\sum_{n=1}^{\infty} \frac{(2x - 3)^{2n}}{n \cdot 4^n}$$

Let

$$a_n = \frac{(2x - 3)^{2n}}{n \cdot 4^n}.$$

Apply the Ratio Test:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(2x - 3)^{2n+2}}{(n + 1)4^{n+1}} \cdot \frac{n4^n}{(2x - 3)^{2n}} \right| \\ &= \frac{(2x - 3)^2}{4} \cdot \frac{n}{n + 1}. \end{aligned}$$

Taking the limit gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{(2x - 3)^2}{4}.$$

The series converges when

$$\frac{(2x - 3)^2}{4} < 1.$$

This is equivalent to

$$(2x - 3)^2 < 4,$$

so

$$|2x - 3| < 2.$$

Therefore,

$$-2 < 2x - 3 < 2.$$

Adding 3 gives

$$1 < 2x < 5,$$

and dividing by 2 gives

$$\frac{1}{2} < x < \frac{5}{2}.$$

Thus the open interval of convergence is

$$\left(\frac{1}{2}, \frac{5}{2} \right).$$

The center is $x = \frac{3}{2}$, so the radius is

$$R = 1.$$

Now test the endpoints.

At $x = \frac{5}{2}$:

$$2x - 3 = 2,$$

so the series becomes

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n4^n} = \sum_{n=1}^{\infty} \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges.

At $x = \frac{1}{2}$:

$$2x - 3 = -2,$$

so

$$(2x - 3)^{2n} = (-2)^{2n} = 4^n.$$

Thus the series again becomes

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges.

Therefore,

$$\boxed{R = 1, \quad I = \left(\frac{1}{2}, \frac{5}{2}\right)}.$$

10. $\sum_{n=1}^{\infty} \frac{n(x-5)^n}{2^n(n+1)}$

Let

$$a_n = \frac{n(x-5)^n}{2^n(n+1)}.$$

Apply the Ratio Test:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(n+1)(x-5)^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{n(x-5)^n} \right| \\ &= \left| \frac{x-5}{2} \right| \cdot \frac{(n+1)^2}{n(n+2)}. \end{aligned}$$

Taking the limit gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x-5}{2} \right|.$$

The series converges when

$$\left| \frac{x-5}{2} \right| < 1,$$

which is equivalent to

$$|x-5| < 2.$$

So

$$3 < x < 7.$$

Thus the open interval of convergence is $(3, 7)$, centered at $x = 5$, with radius

$$R = 2.$$

Now test the endpoints.

At $x = 7$:

$$\sum_{n=1}^{\infty} \frac{n(7-5)^n}{2^n(n+1)} = \sum_{n=1}^{\infty} \frac{n \cdot 2^n}{2^n(n+1)} = \sum_{n=1}^{\infty} \frac{n}{n+1}.$$

But

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0,$$

so the series diverges by the Test for Divergence.

At $x = 3$:

$$\sum_{n=1}^{\infty} \frac{n(3-5)^n}{2^n(n+1)} = \sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n(n+1)} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}.$$

Its terms are

$$(-1)^n \frac{n}{n+1}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1,$$

the terms do not approach 0. Therefore, the series diverges by the Test for Divergence.

Hence,

$$\boxed{R = 2, \quad I = (3, 7)}.$$

Representing Functions as Power Series

Free Response Practice

1. $f(x) = \frac{x}{1-3x}$

$$\begin{aligned} f(x) &= x \cdot \frac{1}{1-3x} \\ &= x \sum_{n=0}^{\infty} (3x)^n \\ &= \sum_{n=0}^{\infty} 3^n x^{n+1}. \end{aligned}$$

This series converges when

$$|3x| < 1 \iff |x| < \frac{1}{3},$$

so the interval of convergence is

$$\boxed{\left(-\frac{1}{3}, \frac{1}{3}\right)}.$$

2. $f(x) = \frac{4}{7+2x}$

$$\begin{aligned} f(x) &= \frac{4}{7} \cdot \frac{1}{1+\frac{2x}{7}} \\ &= \frac{4}{7} \cdot \frac{1}{1-\left(-\frac{2x}{7}\right)} \\ &= \frac{4}{7} \sum_{n=0}^{\infty} \left(-\frac{2x}{7}\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{4 \cdot 2^n}{7^{n+1}} x^n. \end{aligned}$$

This series converges when

$$\left|-\frac{2x}{7}\right| < 1 \iff |x| < \frac{7}{2},$$

so the interval of convergence is

$$\boxed{\left(-\frac{7}{2}, \frac{7}{2}\right)}.$$

3. $f(x) = \frac{1}{9-x^2}$

$$\begin{aligned} f(x) &= \frac{1}{9} \cdot \frac{1}{1-\frac{x^2}{9}} \\ &= \frac{1}{9} \sum_{n=0}^{\infty} \left(\frac{x^2}{9}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{9^{n+1}}. \end{aligned}$$

This series converges when

$$\left| \frac{x^2}{9} \right| < 1 \iff |x| < 3,$$

so the interval of convergence is

$$\boxed{(-3, 3)}.$$

4. $f(x) = \frac{x^2}{1+x}$

$$\begin{aligned} f(x) &= x^2 \cdot \frac{1}{1+x} \\ &= x^2 \cdot \frac{1}{1-(-x)} \\ &= x^2 \sum_{n=0}^{\infty} (-x)^n \\ &= x^2 \sum_{n=0}^{\infty} (-1)^n x^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{n+2}. \end{aligned}$$

This series converges when

$$|-x| < 1 \iff |x| < 1,$$

so the interval of convergence is

$$\boxed{(-1, 1)}.$$

5. $f(x) = \frac{1}{5-x}$

$$\begin{aligned} f(x) &= \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}} \\ &= \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}. \end{aligned}$$

This series converges when

$$\left| \frac{x}{5} \right| < 1 \iff |x| < 5,$$

so the interval of convergence is

$$\boxed{(-5, 5)}.$$

Multiple Choice Practice

1. (A) $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$.
2. (C) $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$.
3. (A) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ has radius 1.
4. (B) $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ for $|2x| < 1$.
5. (A) Use the geometric series for $\frac{1}{1+t^2}$:

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \quad |t| < 1.$$

Multiply by t and integrate term-by-term:

$$\begin{aligned} \frac{t}{1+t^2} &= \sum_{n=0}^{\infty} (-1)^n t^{2n+1}, \\ \int_0^x \frac{t}{1+t^2} dt &= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n+1} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+2}. \end{aligned}$$

6. (B) If $\frac{1}{1-x}$ has $R = 1$, then $\frac{1}{1-5x}$ has $R = \frac{1}{5}$.