Midterm 3 Study Guide

MATH2300 - Calculus II

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Contents

The Story of Infinite Series

We begin with a **sequence** of numbers $a_1, a_2, a_3, \ldots$ and ask a natural question: *What happens if we add them all together?* This leads us to the concept of an **infinite series**, written as

$$\sum a_n = a_1 + a_2 + a_3 + \cdots .$$

Can We Find the Exact Sum?

In special cases, we can compute the exact value of the infinite series:

- **Telescoping Series:** If many terms cancel out when we write the partial sums, we have a telescoping series. For example:

$$\sum \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

  We compute the partial sum S_n, observe cancellation, and take the limit as $n \to \infty$.

- **Geometric Series:** For a geometric series of the form $\sum ar^{n-1}$, the sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad \text{when } |r| < 1.$$

In these cases, we not only know that the series converges—we know exactly what it converges to.

Usually, We Can't Find the Sum Exactly

Most infinite series are not so convenient. In general, we can't compute the sum directly, so we shift our focus to a different question: *Does the series converge or diverge?*

Step 1: Positive-Term Series

We begin with series whose terms are all positive. Several tests help us determine whether such a series converges:

- **Test for Divergence:** If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. In fact, this test applies to any series, even if the terms are not all positive!

- **Integral Test:** If $a_n = f(n)$ where f is positive, continuous, and decreasing, then

$$\sum a_n \text{ converges} \quad \Leftrightarrow \quad \int_1^{\infty} f(x)\,dx \text{ converges.}$$

- **Direct Comparison Test:** Compare a_n to a known benchmark series b_n.
  - If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.
  - If $0 \leq b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

  Benchmark series are often geometric series or p-series, since their convergence behavior is well understood.

- **Limit Comparison Test:** If $a_n, b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{with } 0 < c < \infty,$$

  then either both series converge or both diverge.

Step 2: Alternating Series and Mixed Signs

Series are not required to have only positive terms. In fact, many important series include both positive and negative terms. We begin with the especially nice case of **alternating series**, where the signs alternate in a regular pattern. If the terms alternate and decrease to zero, we can apply the:

- **Alternating Series Test:** For a series of the form $\sum(-1)^{n+1}b_n$ with $b_n > 0$, if

$$b_{n+1} \leq b_n \quad \text{and} \quad \lim_{n\to\infty} b_n = 0,$$

  then the series converges.

When a series has both positive and negative terms, we test for:

- **Absolute Convergence:** If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely (and hence converges).
- **Conditional Convergence:** If $\sum a_n$ converges but $\sum |a_n|$ diverges, then the series converges conditionally.

A powerful tool in these cases is the:

- **Ratio Test:** Let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

  Then:
  - If $L < 1$, the series converges absolutely.
  - If $L > 1$ or $L = \infty$, the series diverges.
  - If $L = 1$, the test is inconclusive.

Estimating the Sum: Remainder Estimates

Even when we can't find the exact sum, we might want to estimate how close our partial sum S_n is to the true value.

- **Alternating Series Remainder:**

$$|R_n| = |S - S_n| \leq b_{n+1}$$

  The error is no larger than the next term.

- **Integral Test Remainder:**

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$

  This provides upper and lower bounds for the error when using the integral test.

In Summary

- Some series can be summed exactly (e.g., telescoping, geometric).
- For most series, we focus on convergence rather than the exact value.
- Positive-term series are the starting point, with several comparison-based tests.
- Alternating series are easier to manage due to cancellation.
- Absolute convergence guarantees convergence even for mixed-sign series.
- Remainder estimates help quantify how close a partial sum is to the true sum.

Summary of Series Types

Series Type	General Form	Convergence Behavior
Telescoping Series	Terms cancel in successive pairs, e.g., $\sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$	Partial sums simplify to a finite number of terms; convergence depends on the limit of the remaining terms. Often converges due to cancellation.
Geometric Series	$\sum ar^{n-1}$	Converges if $\lvert r\rvert < 1$ to $\frac{a}{1-r}$. Diverges if $\lvert r\rvert \geq 1$.
p-Series	$\sum \frac{1}{n^p}$, where $p > 0$	Converges if $p > 1$. Diverges if $0 < p \leq 1$.
Alternating Series	$\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$, $b_n > 0$	Converges if b_n decreases and $\lim b_n = 0$. May converge conditionally or absolutely.

Summary of Convergence Tests

Test	Applies To	Conclusion
Test for Divergence	Any series $\sum a_n$	If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. If the limit is 0, the test is inconclusive.
Integral Test	Series with positive, decreasing $f(n) = a_n$	If $\int_1^\infty f(x)\,dx$ converges, so does $\sum a_n$. If the integral diverges, so does the series.
Direct Comparison Test	Positive-term series $\sum a_n$, $\sum b_n$	If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
Limit Comparison Test	Positive-term series $\sum a_n$, $\sum b_n$	If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $0 < c < \infty$, then both series converge or both diverge.
Alternating Series Test	Alternating series $\sum(-1)^n b_n$ with $b_n > 0$	If b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, then the series converges.
Absolute Convergence	Any series $\sum a_n$	If $\sum \lvert a_n\rvert$ converges, then $\sum a_n$ converges absolutely (and hence converges).
Ratio Test	Any series $\sum a_n$	Compute $L = \lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert$: if $L < 1$, series converges absolutely; if $L > 1$ or $L = \infty$, series diverges; if $L = 1$, test is inconclusive.

Summary of Remainder Estimates

Test	Remainder Estimate	Interpretation
Alternating Series Test Remainder	$\lvert R_n\rvert = \lvert S - S_n\rvert \leq b_{n+1}$	For an alternating series satisfying the Alternating Series Test, the error in approximating the sum by the nth partial sum is at most the absolute value of the next term. Error decreases as n increases.
Integral Test Remainder	$\int_{n+1}^\infty f(x)\,dx < R_n < \int_n^\infty f(x)\,dx$	For a positive, decreasing, continuous function $f(n) = a_n$, the true remainder lies between two improper integrals. Useful for estimating or bounding the error in partial sums.

Sequences

Let a_n be a sequence. To evaluate $\lim_{n\to\infty} a_n$, follow these steps:

- **Step 1: Try direct substitution.**
  - If $a_n = \frac{1}{n}, \frac{5n+2}{2n+3}, \frac{n^2}{n^2+1}$, try plugging in $n \to \infty$.
  - Use leading-term analysis for rational expressions.
- **Step 2: Simplify algebraically if needed.**
  - Factor, divide numerator/denominator by highest power of n, or simplify complex expressions.
- **Step 3: Use known limit rules.**
  - $\lim_{n\to\infty} \dfrac{1}{n^p} = 0$ for any $p > 0$
  - $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$
  - $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for any $a > 0$
  - $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$
- **Step 4: Apply L'Hôpital's Rule (only for continuous functions).**
  - Define $f(x)$ from a_n, take $\lim_{x\to\infty} f(x)$ if it's indeterminate.
  - Use only when a_n is expressible as a continuous function of x.
- **Step 5: Use squeeze theorem if appropriate.**
  - Useful for trigonometric sequences or oscillating behavior.
  - Example: $\lim_{n\to\infty} \dfrac{\sin n}{n} = 0$
- **Step 6: Use logarithms for exponential powers.**
  - For $a_n = n^{1/n}$, $a_n = \left(1 + \frac{1}{n}\right)^n$, take ln, find the limit, then exponentiate.
- **Step 7: Consider behavior of factorials.**
  - Use growth comparisons: $n! \gg n^n \gg r^n \gg n^p$
  - Example: $\lim_{n\to\infty} \dfrac{n^2}{n!} = 0$

Find the limit of each of the following sequences as $n \to \infty$. Show your work and indicate which method you are using (e.g., algebraic simplification, squeeze theorem, logarithmic substitution, etc.).

1. $\lim_{n\to\infty} \dfrac{2n^2+1}{3n^2-5}$
2. $\lim_{n\to\infty} \dfrac{n^2+3n}{4n^2+1}$
3. $\lim_{n\to\infty} \dfrac{1}{n^{0.5}}$
4. $\lim_{n\to\infty} \dfrac{\ln(n+1)}{n}$
5. $\lim_{n\to\infty} \dfrac{\sin(n)}{n}$
6. $\lim_{n\to\infty} n^{1/n}$
7. $\lim_{n\to\infty} \dfrac{n^2}{n!}$
8. $\lim_{n\to\infty} \dfrac{\cos(n)+n}{\sqrt{n^2+1}}$
9. $\lim_{n\to\infty} \dfrac{\sin(n)}{\sqrt{n^2+4}-n}$
10. $\lim_{n\to\infty} \dfrac{\sqrt{n^2+\sin(n)}}{n}$

Series & Partial Sums

- Start with a **sequence** $\{a_n\}$, where each a_n is a real number. The associated **series** is the sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

- Define the sequence $\{S_n\}$, where each partial sum is

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

- The series $\sum a_n$ **converges** if the sequence of partial sums $\{S_n\}$ has a finite limit:

$$\lim_{n\to\infty} S_n = S$$

  In this case, we write:

$$\sum_{n=1}^{\infty} a_n = S$$

  If $\lim S_n$ does not exist or is infinite, then the series **diverges**.

- **To find the sum of a series from partial sums:**
  - If you're given a formula for S_n, compute

$$\lim_{n\to\infty} S_n$$

  - That limit is the value of the series:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

Each problem gives the partial sum $S_n = \sum_{k=1}^{n} a_k$. Find the value of the infinite series $\sum_{n=1}^{\infty} a_n$ by computing $\lim_{n\to\infty} S_n$.

1. $S_n = 3 - \dfrac{1}{n}$
2. $S_n = \dfrac{5n}{n+1}$
3. $S_n = \dfrac{2n^2+1}{n^2+2n+1}$
4. $S_n = \dfrac{4n^2}{n^2+1}$
5. $S_n = 1 - \dfrac{2}{n^2+1}$
6. $S_n = \dfrac{6n^2+3n+2}{2n^2+5n+1}$
7. $S_n = \dfrac{5n}{\ln(n+1)+2n}$
8. $S_n = \dfrac{n+4}{\sqrt{n^2+9}}$
9. $S_n = \dfrac{4n+\ln(n)}{3n+1}$
10. $S_n = \dfrac{n^2+1}{n\ln n+1}$

Telescoping Series

- **Step 1: Identify the general term.**
  Examine the structure of the term a_n. If it is a rational expression, it may telescope.

- **Step 2: Decompose using partial fractions (if needed).**
  Rewrite a_n as a difference of simpler terms:
  $$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

- **Step 3: Write out the first few terms.**
  Expand the partial sum $S_N = a_1 + a_2 + \cdots + a_N$ to observe a cancellation pattern.

- **Step 4: Look for cancellation.**
  Most terms will cancel out in a telescoping series, leaving only a few from the beginning and end.

- **Step 5: Take the limit of the partial sum.**
  After cancellation, evaluate the simplified expression:
  $$\lim_{N\to\infty} S_N$$

- **Step 6: Conclude the sum of the series.**
  The value of the infinite series is:
  $$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

Evaluate the following infinite series by identifying the telescoping nature and computing the sum.

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n(n+1)}$
3. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-1}$
4. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$
5. $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$
6. $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n(n+2)}$
7. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$
8. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$
9. $\displaystyle\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$
10. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

Geometric Series

- **Step 1: Recognize or rewrite the series into geometric form.**
  A geometric series has the form:
$$\sum_{n=0}^{\infty} ar^n \quad \text{or} \quad \sum_{n=1}^{\infty} ar^{n-1}.$$
  Use algebraic manipulation if needed:
  - Factor out constants.
  - Reindex the starting value of n.
  - Rewrite powers to match r^n format.
- **Step 2: Identify a and r.**
  Once in standard form, determine:
  - a: the first term of the series.
  - r: the common ratio between terms.
- **Step 3: Check for convergence.**
  A geometric series converges if $|r| < 1$, and diverges otherwise.
- **Step 4: Use the geometric series formula (if convergent).**
$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad \text{for } |r| < 1.$$
  For series starting at $n = 1$:
$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}, \qquad \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$
  Note: in each case, the formula is the first term of the series divided by $1 - r$.

1. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$
2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{10^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$
6. $\displaystyle\sum_{n=0}^{\infty} 7 \cdot \left(\frac{2}{3}\right)^n$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$
8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{5^n}$
9. $\displaystyle\sum_{n=0}^{\infty} \frac{4}{3^{n+2}}$
10. $\displaystyle\sum_{n=2}^{\infty} \frac{4^n}{7^{n-2}}$

p-Series

- **Step 1: Identify the form of the series.**
  A p-series has the form:
  $$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{where } p > 0.$$
- **Step 2: Determine the value of p.**
  Carefully extract the exponent from the denominator and simplify if needed (e.g., square roots, fractional exponents).
- **Step 3: Apply the convergence test for p-series.**
  - If $p > 1$, the series **converges**.
  - If $0 < p \leq 1$, the series **diverges**.
- **Step 4: Recognize disguised p-series.**
  Some series may not look like $\frac{1}{n^p}$ at first, but can be rewritten or compared to one:
  - Example: $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ is a p-series with $p = \frac{1}{2}$.
  - Use comparison tests when the series is close to a p-series but not exactly in that form.

Determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^9}}$

Test For Divergence

- **Step 1: Identify the general term a_n of the series.**
  Start with a series of the form:
  $$\sum a_n$$
- **Step 2: Compute the limit of the general term.**
  Evaluate:
  $$\lim_{n\to\infty} a_n$$
- **Step 3: Apply the conclusion:**
  - If $\lim_{n\to\infty} a_n \neq 0$ or the limit does not exist, then the series **diverges**.
  - If $\lim_{n\to\infty} a_n = 0$, the test is **inconclusive**. The series may converge or diverge.
- **Important Notes:**
  - The Test for Divergence cannot prove convergence.
  - This test works for all types of series (not just positive-term series).

Apply the Test for Divergence. If the limit is nonzero or does not exist, the series diverges. If the limit is zero, the test is inconclusive.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{2n^2+1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$
4. $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{5n-4}{n}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^2+1}$
7. $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Integral Test

- **Step 1: Identify the series.**
  The series should be of the form:
  $$\sum_{n=1}^{\infty} a_n, \quad \text{where } a_n = f(n)$$
  and $f(x)$ is the corresponding continuous, positive, and decreasing function.

- **Step 2: Check the conditions for the Integral Test.**
  The function $f(x)$ must satisfy:
  - $f(x)$ is **continuous** for $x \geq 1$.
  - $f(x)$ is **positive** for $x \geq 1$.
  - $f(x)$ is **decreasing** for $x \geq 1$.

- **Step 3: Set up the improper integral.**
  Compute the corresponding improper integral:
  $$\int_1^{\infty} f(x)\, dx$$

- **Step 4: Evaluate the integral.**
  Use appropriate integration techniques (substitution, integration by parts, etc.) to compute the improper integral.

- **Step 5: Apply the results of the integral:**
  - If $\int_1^{\infty} f(x)\, dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\int_1^{\infty} f(x)\, dx$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Use the Integral Test to determine whether each series converges or diverges.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ (general case)
6. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{\ln n}}{n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^4+25}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^6+36}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3}{n^8+64}$

Direct Comparison Test

- **Step 1: Identify the general term a_n of the series.**
  You are given a series $\sum a_n$ with $a_n > 0$.
- **Step 2: Choose a known comparison series $\sum b_n$.**
  Select a benchmark series with known convergence behavior—usually a geometric series or a p-series.
- **Step 3: Establish an inequality between a_n and b_n.**
  - To show convergence, prove $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum b_n$ converges.
  - To show divergence, prove $0 \le b_n \le a_n$ for all sufficiently large n, and $\sum b_n$ diverges.
- **Step 4: Conclude the behavior of $\sum a_n$.**
  - If $a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.
  - If $a_n \ge b_n$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.
- **Important Notes:**
  - The comparison must involve **non-negative terms**.
  - The inequality must go in the correct direction depending on whether you're trying to prove convergence or divergence.
  - If you cannot establish a valid inequality, try the Limit Comparison Test instead.

Use the Direct Comparison Test to determine whether each series converges or diverges by comparing it to a known p-series or geometric series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2}{n^4+n^2+1}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4^n+n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+2^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin^2\left(\frac{1}{n}\right)}{n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Limit Comparison Test

- **Step 1: Identify the given series.**
  You are given a series of the form $\sum a_n$, where $a_n > 0$ for all n sufficiently large.
- **Step 2: Choose a comparison series $\sum b_n$.**
  Pick a series with known convergence behavior (typically a p-series or geometric series) that resembles a_n for large n.
- **Step 3: Compute the limit of the ratio.**
  Evaluate:
  $$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$
- **Step 4: Apply the conclusion of the test.**
  If $0 < L < \infty$, then either both series converge or both diverge.
- **Important Notes:**
  - Use this test when a_n and b_n are **positive** for large n.
  - The Limit Comparison Test often works when the Direct Comparison Test fails (e.g., when inequalities are hard to establish).

Use the Limit Comparison Test to determine whether each series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3-4}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4-1}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+5}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+3n}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+2^n}{4^n+1}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2+(-1)^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+\tan^{-1}(n)}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2+(-1)^n}$

Alternating Series Test

- **Step 1: Identify the alternating form.**
  The series must have alternating signs:
  $$\sum(-1)^n b_n \quad \text{or} \quad \sum(-1)^{n+1} b_n, \quad \text{with } b_n > 0.$$
- **Step 2: Check the two AST conditions.**
  - b_n is decreasing: $b_{n+1} \leq b_n$ for all n (or eventually).
  - $\lim_{n\to\infty} b_n = 0$.
- **Step 3: Conclude convergence.**
  If both conditions are met, then the series **converges**.
- **Step 4: If AST fails, try another test.**
  - If b_n is not decreasing or $\lim b_n \neq 0$, AST does not apply.
  - Use the **Test for Divergence** if needed.

Determine whether each series converges or diverges. If the Alternating Series Test does not apply, state why.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(-1)^n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+\ln n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sin(1/n)}{n}$
10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{0.9}}$

Absolute Convergence

- **Step 1: Start with the given series $\sum a_n$**, which may include negative or alternating terms.
- **Step 2: Test for absolute convergence by analyzing $\sum |a_n|$.**
  - Use any appropriate convergence test:
    - p-series, geometric series, comparison, limit comparison, ratio, root, etc.
  - If $\sum |a_n|$ converges, then the original series **converges absolutely**.
- **Step 3: If $\sum |a_n|$ diverges, try the Alternating Series Test (if applicable).**
  - AST requires:
    - $b_n = |a_n| > 0$
    - b_n is decreasing (eventually)
    - $\lim_{n\to\infty} b_n = 0$
  - If AST applies, the series **converges conditionally**.
- **Step 4: If neither test shows convergence, the series diverges.**
  - For example, if $\lim_{n\to\infty} a_n \neq 0$, or neither $\sum |a_n|$ nor AST applies.
- **Summary:**
  - $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.
  - $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.
  - $\sum a_n$ **diverges** if it fails both.

For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2+1)}{n^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}}$
9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Ratio Test

- **Step 1: Identify the general term.** Let a_n be the general term of the series $\sum a_n$.
- **Step 2: Compute the ratio.** Find the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- **Step 3: Interpret the result.**
  - If $L < 1$, the series **converges absolutely**.
  - If $L > 1$ or $L = \infty$, the series **diverges**.
  - If $L = 1$, the test is **inconclusive**.
- **Step 4: Use absolute values.** Always apply the test to $\left|\frac{a_{n+1}}{a_n}\right|$ to determine **absolute convergence**. If the test shows convergence, the original series converges (even if alternating).
- **Step 5: If inconclusive, try another test.** Consider:
  - Alternating Series Test (AST)
  - Comparison / Limit Comparison Test

Determine whether each series converges absolutely, converges conditionally, or diverges using the Ratio Test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^2}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n^n}$
9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{2^n}$
10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2}{n!}$

Remainder Estimates

Alternating Series Remainder

- **Applies to:** An alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, where:
  - $b_n > 0$
  - b_n is decreasing
  - $\lim_{n\to\infty} b_n = 0$
- **Remainder estimate:** The error when approximating the sum by the first n terms satisfies:
$$|R_n| = |S - S_n| \leq b_{n+1}$$
- **Steps:**
  - Confirm the series meets the conditions of the Alternating Series Test.
  - Compute or estimate b_{n+1} to bound the error.
  - Use this bound to describe how accurate the partial sum S_n is.

Integral Test Remainder

- **Applies to:** A positive, decreasing, continuous function $f(n)$ where $a_n = f(n)$, and $\sum a_n$ is approximated using the Integral Test.
- **Remainder estimate:** If $S_n = \sum_{k=1}^{n} a_k$, then the remainder $R_n = S - S_n$ satisfies:
$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$$
- **Steps:**
  - Confirm that $f(x)$ is positive, continuous, and decreasing for $x \geq n$.
  - Evaluate (or estimate) both bounds:
$$\text{Lower bound: } \int_{n+1}^{\infty} f(x)\,dx,$$
$$\text{Upper bound: } \int_{n}^{\infty} f(x)\,dx$$
  - Conclude that the true error lies between the two.

For Problems 1–5, use the Alternating Series Remainder Theorem. For Problems 6–10, use the Integral Test Remainder estimate. Show all reasoning.

Alternating Series Remainder

1. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ using the first 4 terms. Estimate the error.
2. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^3+1}$ using the first 3 terms. How accurate is your approximation?
3. Find how many terms are needed to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ to within 0.0001.
4. Use the first 5 terms of $\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n+1)}{n^2}$ to approximate the sum. Estimate the error.
5. Determine the minimum number of terms needed to estimate $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ with error less than 0.01.

Integral Test Remainder

6. Approximate $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ using the first 5 terms. Use the Integral Test to bound the error.
7. Use the Integral Test to estimate the remainder when approximating $\sum_{n=1}^{\infty} \frac{1}{n\ln n}$ from $n = 2$ to $n = 10$.
8. Estimate $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ using the first 8 terms, and bound the error using the Integral Test.
9. Find the smallest n such that the partial sum $\sum_{k=2}^{n} \frac{1}{k(\ln k)^2}$ approximates the infinite series to within 0.05.
10. Approximate $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ using the first 6 terms. Estimate how close your approximation is to the true value.

Taylor Polynomials

1. **Recall the Taylor polynomial formula:**

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

2. **Compute the derivatives of $f(x)$:**
   - Find $f^{(k)}(x)$ for $k = 0, 1, 2, \ldots, n$.
3. **Evaluate each derivative at $x = a$:**

$$f(a), \quad f'(a), \quad f''(a), \quad \ldots, \quad f^{(n)}(a)$$

4. **Substitute into the Taylor formula:**
   - Plug the values from step 3 into the formula from step 1.
5. **Simplify:**
   - Expand and combine like terms to write the polynomial in standard form.

Optional Checks

- **Check the number of terms:** Make sure you include the first n nonzero terms.
- **Use known Taylor series:**
  - For functions like e^x, $\sin x$, $\cos x$, $\ln(1+x)$, use known Maclaurin series if centered at $a = 0$.

Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

1. $f(x) = x^2 e^x, \quad a = 0$
2. $f(x) = \dfrac{1}{1+x^2}, \quad a = 1$
3. $f(x) = \sqrt[4]{x}, \quad a = 16$
4. $f(x) = \cos(x^2), \quad a = 0$
5. $f(x) = \sin(2x), \quad a = \dfrac{\pi}{4}$
6. $f(x) = \ln(1+2x), \quad a = 0$
7. $f(x) = \tan^{-1}(x), \quad a = 0$
8. $f(x) = \dfrac{x}{1-x}, \quad a = 0$
9. $f(x) = e^{x^2}, \quad a = 0$
10. $f(x) = \dfrac{1}{\sqrt{1-x}}, \quad a = 0$

Mixed Series Practice

For each problem below, determine convergence/divergence, evaluate limits or sums, or justify the requested conclusion. Show clear reasoning.

1. Evaluate $\lim_{n\to\infty} \dfrac{\sqrt{n^2+\cos n}}{n}$.

2. Evaluate $\lim_{n\to\infty} \left(1-\dfrac{1}{2n}\right)^{3n}$.

3. Suppose $S_n = \sum_{k=1}^{n} a_k = \dfrac{n^2}{\ln(n+1)+3n}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

4. Suppose $S_n = \sum_{k=1}^{n} a_k = \dfrac{4n^2+1}{2n^2+5n+1}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

5. If $\sum a_n = 5$ and $\sum b_n = 4$, find $\sum(2a_n - 3b_n)$.

6. Find the sum of the geometric series: $\sum_{n=2}^{\infty} \dfrac{6}{4^n}$.

7. Determine whether the series $\sum_{n=2}^{\infty} \dfrac{n^4+3}{n(n+1)^2}$ converges. If it converges, find its sum.

8. Determine whether the series $\sum_{n=1}^{\infty} \dfrac{4^n}{5^n-1}$ converges.

9. Let $a_n = \dfrac{n^2}{2^n}$. Use the Ratio Test to determine whether $\sum a_n$ converges.

10. Use the Ratio Test to determine for which values of $c > 0$ the series $\sum_{n=1}^{\infty} \dfrac{c^n}{n}$ converges.

11. Determine whether the series $\sum_{n=1}^{\infty} \dfrac{2n}{n^3+\cos n}$ converges.

12. Determine whether $\sum_{n=1}^{\infty} \dfrac{\sin(n^3)}{n^3+1}$ converges absolutely, conditionally, or diverges.

13. Determine whether $\sum_{n=3}^{\infty} \dfrac{(-1)^n}{n^2-4}$ converges absolutely, conditionally, or diverges.

14. Use the Alternating Series Remainder Theorem to estimate the error when approximating $\sum_{n=1}^{\infty} (-1)^n \dfrac{1}{n^3}$ using the first 4 terms.

15. Use the Integral Test to show that $\sum_{n=2}^{\infty} \dfrac{1}{n\ln(n)^2}$ converges. Estimate the error when approximating the series with the first 5 terms.

Solutions

Sequences

1. $\lim_{n\to\infty} \frac{2n^2+1}{3n^2-5}$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{3-\frac{5}{n^2}} = \frac{2}{3}$$

2. $\lim_{n\to\infty} \frac{n^2+3n}{4n^2+1}$

$$= \lim_{n\to\infty} \frac{1+\frac{3}{n}}{4+\frac{1}{n^2}} = \frac{1}{4}$$

3. $\lim_{n\to\infty} \frac{1}{n^{1/2}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

4. $\lim_{n\to\infty} \frac{\ln(n+1)}{n}$

   Let $f(x) = \frac{\ln(x+1)}{x}$. Then:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x+1} = 0$$

   Therefore,

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n} = 0$$

5. $\lim_{n\to\infty} \frac{\sin(n)}{n}$

   Since $-1 \le \sin(n) \le 1$ for all n, we have:

$$-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$$

   Taking the limit of all three expressions:

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

6. $\lim_{n\to\infty} n^{1/n}$

   Let $a_n = n^{1/n}$. Take logarithms:

$$\ln a_n = \frac{\ln n}{n} \to 0 \Rightarrow a_n = e^0 = 1$$

   Therefore,

$$\lim_{n\to\infty} n^{1/n} = 1$$

7. $\lim_{n\to\infty} \frac{n^2}{n!}$

   For $n \ge 5$, we have:

$$n! \ge n(n-1)(n-2) \ge \frac{n^3}{6}$$

   Then:

$$0 < \frac{n^2}{n!} \le \frac{n^2}{\frac{n^3}{6}} = \frac{6}{n}$$

   Since $\lim_{n\to\infty} \frac{6}{n} = 0$, the squeeze theorem gives:

$$\lim_{n\to\infty} \frac{n^2}{n!} = 0$$

8. $\lim_{n\to\infty} \frac{\cos(n)+n}{\sqrt{n^2+1}}$

   Write:

$$\frac{\cos(n)+n}{\sqrt{n^2+1}} = \frac{n+\cos(n)}{n\sqrt{1+\frac{1}{n^2}}}$$

   Then:

$$= \frac{1+\frac{\cos(n)}{n}}{\sqrt{1+\frac{1}{n^2}}} \to \frac{1+0}{1} = 1$$

9. $\lim_{n\to\infty} \frac{\sin(n)}{\sqrt{n^2+4}-n}$

$$\frac{\sin(n)}{\sqrt{n^2+4}-n} = \frac{\sin(n)\cdot(\sqrt{n^2+4}+n)}{4}$$

   Since $\sin(n)$ is bounded and $\sqrt{n^2+4}+n \to \infty$, the numerator oscillates without settling. The limit does not exist.

10. $\lim_{n\to\infty} \frac{\sqrt{n^2+\sin(n)}}{n}$

   Use bounds:

$$\sqrt{n^2-1} \le \sqrt{n^2+\sin(n)} \le \sqrt{n^2+1}$$

   Then:

$$\frac{\sqrt{n^2-1}}{n} \le \frac{\sqrt{n^2+\sin(n)}}{n} \le \frac{\sqrt{n^2+1}}{n}$$

$$\Rightarrow \sqrt{1-\frac{1}{n^2}} \le \frac{\sqrt{n^2+\sin(n)}}{n} \le \sqrt{1+\frac{1}{n^2}}$$

   Taking limits of both bounds gives:

$$\lim_{n\to\infty} \frac{\sqrt{n^2+\sin(n)}}{n} = 1$$

Series & Partial Sums

1. $S_n = 3 - \dfrac{1}{n}$

$$\lim_{n\to\infty} S_n = 3$$

2. $S_n = \dfrac{5n}{n+1}$

$$= \frac{5}{1+\frac{1}{n}} \to 5$$

3. $S_n = \dfrac{2n^2+1}{n^2+2n+1}$

$$= \frac{2+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} \to \frac{2}{1} = 2$$

4. $S_n = \dfrac{4n^2}{n^2+1}$

$$= \frac{4}{1+\frac{1}{n^2}} \to 4$$

5. $S_n = 1 - \dfrac{2}{n^2+1}$

$$\frac{2}{n^2+1} \to 0 \quad \Rightarrow \quad \lim S_n = 1$$

6. $S_n = \dfrac{6n^2+3n+2}{2n^2+5n+1}$

$$= \frac{6+\frac{3}{n}+\frac{2}{n^2}}{2+\frac{5}{n}+\frac{1}{n^2}} \to \frac{6}{2} = 3$$

7. $S_n = \dfrac{5n}{\ln(n+1)+2n}$

$$= \frac{5}{\frac{\ln(n+1)}{n}+2} \to \frac{5}{0+2} = \frac{5}{2}$$

8. $S_n = \dfrac{n+4}{\sqrt{n^2+9}}$

$$= \frac{1+\frac{4}{n}}{\sqrt{1+\frac{9}{n^2}}} \to \frac{1}{1} = 1$$

9. $S_n = \dfrac{4n+\ln(n)}{3n+1}$

$$= \frac{4+\frac{\ln n}{n}}{3+\frac{1}{n}} \to \frac{4}{3}$$

10. $S_n = \dfrac{n^2+1}{n\ln n+1}$

$$= \frac{n+\frac{1}{n}}{\ln n+\frac{1}{n}} \to \frac{n}{\ln n} \to \infty$$

The series diverges

Telescoping Series

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$$\begin{aligned}
S_N &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{N}-\frac{1}{N+1}\right)\\
&= 1-\frac{1}{N+1}\\
\lim_{N\to\infty} S_N &= 1
\end{aligned}$$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n(n+1)}$

$$\begin{aligned}
\frac{4}{n(n+1)} &= 4\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
S_N &= 4\left(1-\frac{1}{N+1}\right)\\
\lim_{N\to\infty} S_N &= 4
\end{aligned}$$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-1}$

$$\begin{aligned}
\frac{1}{n^2-1} &= \frac{1}{(n-1)(n+1)} = \frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\\
S_N &= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{N}-\frac{1}{N+1}\right)\\
\lim_{N\to\infty} S_N &= \frac{1}{2}\left(1+\frac{1}{2}\right)=\frac{3}{4}
\end{aligned}$$

4. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$

$$\begin{aligned}
S_N &= \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\\
&= \frac{1}{2}+\frac{1}{3}-\frac{1}{N+2}-\frac{1}{N+3}\\
\lim_{N\to\infty} S_N &= \frac{5}{6}
\end{aligned}$$

5. $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

$$\begin{aligned}
\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{\infty}\left(\ln(n+1)-\ln(n)\right)\\
&= \ln(2)-\ln(1)+\ln(3)-\ln(2)+\cdots+\ln(N+1)-\ln(N)\\
&= \ln(N+1)\\
\lim_{N\to\infty}\ln(N+1) &= \infty
\end{aligned}$$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

$$\begin{aligned}
\frac{2}{n(n+2)} &= \frac{1}{n} - \frac{1}{n+2} \\
S_N &= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \\
\lim_{N\to\infty} S_N &= \frac{3}{2}
\end{aligned}$$

7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\
&= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{N^2} - \frac{1}{(N+1)^2}\right) \\
&= 1 - \frac{1}{(N+1)^2}
\end{aligned}$$

$$\lim_{N\to\infty} S_N = 1$$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

$$\begin{aligned}
\frac{1}{n(n-1)} &= \frac{1}{n-1} - \frac{1}{n} \\
S_N &= 1 - \frac{1}{N} \\
\lim_{N\to\infty} S_N &= 1
\end{aligned}$$

9. $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

$$\begin{aligned}
S_N &= \left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \cdots + \left(\sqrt{N+1} - \sqrt{N}\right) \\
&= \sqrt{N+1} - 1 \\
\lim_{N\to\infty} S_N &= \infty
\end{aligned}$$

10. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

$$\begin{aligned}
S_N &= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \\
\lim_{N\to\infty} S_N &= \frac{3}{2}
\end{aligned}$$

Geometric Series

1. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

$$a=1,\quad r=\frac{1}{2},\quad |r|<1$$
$$\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2$$

2. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$

$$a=\left(\frac{3}{4}\right)^1=\frac{3}{4},\quad r=\frac{3}{4}$$
$$\frac{a}{1-r}=\frac{\frac{3}{4}}{1-\frac{3}{4}}=\frac{\frac{3}{4}}{\frac{1}{4}}=3$$

3. $\displaystyle\sum_{n=0}^{\infty}\frac{5}{10^n}$

$$a=5,\quad r=\frac{1}{10}$$
$$\frac{5}{1-\frac{1}{10}}=\frac{5}{\frac{9}{10}}=\frac{50}{9}$$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n}$

$$a=\frac{1}{3},\quad r=\frac{1}{3}$$
$$\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{\frac{1}{3}}{\frac{2}{3}}=\frac{1}{2}$$

5. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

$$a=\frac{2}{3},\quad r=\frac{2}{3}$$
$$\frac{\frac{2}{3}}{1-\frac{2}{3}}=\frac{\frac{2}{3}}{\frac{1}{3}}=2$$

6. $\displaystyle\sum_{n=0}^{\infty}7\cdot\left(\frac{2}{3}\right)^n$

$$a=7,\quad r=\frac{2}{3}$$
$$\frac{7}{1-\frac{2}{3}}=\frac{7}{\frac{1}{3}}=21$$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}$

Let $m=n+1\Rightarrow n=m-1,\quad m\geq 2$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}&=\sum_{m=2}^{\infty}\frac{1}{2^m}\\&=\sum_{k=0}^{\infty}\frac{1}{2^k}-\left(\frac{1}{2^0}+\frac{1}{2^1}\right)\\&=2-1-\frac{1}{2}=\frac{1}{2}\end{aligned}$$

8. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{5^n}$

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{1}{5^n}-\left(\frac{1}{5^0}+\frac{1}{5^1}\right)\\&=\frac{1}{1-\frac{1}{5}}-\left(1+\frac{1}{5}\right)\\&=\frac{5}{4}-\frac{6}{5}=\frac{25-24}{20}=\frac{1}{20}\end{aligned}$$

9. $\displaystyle\sum_{n=0}^{\infty}\frac{4}{3^{n+2}}$

$$\frac{4}{3^{n+2}}=\frac{4}{9}\cdot\left(\frac{1}{3}\right)^n$$
$$a=\frac{4}{9},\quad r=\frac{1}{3}$$
$$\frac{\frac{4}{9}}{1-\frac{1}{3}}=\frac{\frac{4}{9}}{\frac{2}{3}}=\frac{4}{9}\cdot\frac{3}{2}=\frac{2}{3}$$

10. $\displaystyle\sum_{n=2}^{\infty}\frac{4^n}{7^{n-2}}$

Rewriting, $\frac{4^n}{7^{n-2}}=\frac{4^n}{7^n}\cdot 7^2=49\cdot\left(\frac{4}{7}\right)^n$. Now,

$$a=49\cdot\left(\frac{4}{7}\right)^2=49\cdot\frac{16}{49}=16,\quad r=\frac{4}{7}$$
$$\frac{a}{1-r}=\frac{16}{1-\frac{4}{7}}=\frac{16}{\frac{3}{7}}=\frac{112}{3}$$

p-Series

1. $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\begin{aligned}\text{This is the harmonic series: } p = 1\\ \text{Since } p \leq 1,\ \text{the series diverges}\end{aligned}$$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$p = 2 > 1 \Rightarrow \text{The series converges}$$

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$= \sum \frac{1}{n^{1/2}}, \quad p = \frac{1}{2} < 1$$
$$\text{The series diverges}$$

4. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$$p = 3 > 1 \Rightarrow \text{The series converges}$$

5. $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$

$$p = 0.9 < 1 \Rightarrow \text{The series diverges}$$

6. $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

$$p = \frac{5}{4} > 1 \Rightarrow \text{The series converges}$$

7. $\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$

$$p = 1.0001 > 1 \Rightarrow \text{The series converges}$$

8. $\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$

$$p = 0.99 < 1 \Rightarrow \text{The series diverges}$$

9. $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$

$$p = \frac{4}{3} > 1 \Rightarrow \text{The series converges}$$

10. $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^9}}$

$$= \sum \frac{n^2}{n^{9/2}} = \sum \frac{1}{n^{\frac{5}{2}}}$$
$$p = \frac{5}{2} > 1 \Rightarrow \text{The series converges}$$

Test for Divergence

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

Series diverges

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{2n^2+1}$

$$\lim_{n\to\infty} \frac{n^2+3}{2n^2+1} = \frac{1}{2}$$

Series diverges

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$

Since $\sin n$ oscillates, limit does not exist

Series diverges

4. $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

Series diverges

5. $\displaystyle\sum_{n=1}^{\infty} \frac{5n-4}{n}$

$$\frac{5n-4}{n} = 5 - \frac{4}{n}$$

$$\lim_{n\to\infty} \frac{5n-4}{n} = 5$$

Series diverges

6. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^2+1}$

$$\lim_{n\to\infty} \frac{3n^2}{n^2+1} = \frac{3}{1} = 3$$

Series diverges

7. $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \tan\left(\frac{1}{n}\right) = \tan(0) = 0$$

Test is inconclusive

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Test is inconclusive

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

Test is inconclusive

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Test is inconclusive

Integral Test

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Let $f(x) = \frac{1}{x \ln x}$, which is continuous, positive, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \ln(\ln x)\Big|_2^{\infty} = \infty$$

The improper integral diverges, so the series diverges by the Integral Test.

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Let $f(x) = \frac{1}{x(\ln x)^2}$, continuous, positive, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \left(-\frac{1}{\ln x}\right)\Big|_2^{\infty} = \frac{1}{\ln 2}$$

The improper integral converges, so the series converges by the Integral Test.

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$

Let $f(x) = \frac{1}{x(\ln x)(\ln \ln x)}$, which is continuous, positive, and decreasing for $x \geq 3$. We apply the substitution

$$u = \ln \ln x \quad \Rightarrow \quad \ln x = e^u, \quad x = e^{e^u}, \quad dx = x \ln x\, du$$

Then,

$$\int \frac{1}{x(\ln x)(\ln \ln x)}\, dx = \int \frac{1}{u} \cdot \frac{1}{x \ln x} \cdot x \ln x\, du = \int \frac{1}{u}\, du$$

$$\int_3^{\infty} \frac{1}{x(\ln x)(\ln \ln x)}\, dx = \int_{\ln \ln 3}^{\infty} \frac{1}{u}\, du = \infty$$

Therefore, the integral diverges, so the series diverges by the Integral Test.

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let $f(x) = \frac{1}{x\sqrt{\ln x}}$, which is continuous, positive, and decreasing for $x \geq 2$.

Use the substitution $u = \sqrt{\ln x}$, which leads to a divergent integral.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx \text{ diverges}$$

Therefore, the series diverges by the Integral Test.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ (general case)

Let $f(x) = \frac{1}{x^p}$, which is continuous, positive, and decreasing for all $x \geq 1$ when $p > 0$. We apply the Integral Test by evaluating the improper integral:

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{t \to \infty} \int_1^t x^{-p}\, dx$$

Case 1: $p \neq 1$

$$\int_1^t x^{-p}\, dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{t^{-p+1} - 1}{-p+1}$$

$$\lim_{t \to \infty} \frac{t^{-p+1} - 1}{-p+1} = \begin{cases} \text{Converges to } \frac{1}{p-1} & \text{if } p > 1 \\ \text{Diverges} & \text{if } p < 1 \end{cases}$$

Case 2: $p = 1$

$$\int_1^t \frac{1}{x}\, dx = \ln t \quad \Rightarrow \quad \lim_{t \to \infty} \ln t = \infty$$

Conclusion: The integral converges if and only if $p > 1$. Therefore, by the Integral Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1$$

6. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n}$

Let $f(x) = \frac{\sqrt{\ln x}}{x}$, continuous, positive, and decreasing for $x \geq 2$.

Use the substitution $u = \ln x \Rightarrow du = \frac{1}{x}dx$, so:

$$\int_2^{\infty} \frac{\sqrt{\ln x}}{x}\, dx = \int_{\ln 2}^{\infty} \sqrt{u}\, du = \infty$$

The integral diverges, so the series diverges by the Integral Test.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \frac{1}{x^2+1}$, which is continuous, positive, and decreasing for $x \geq 1$. Apply the Integral Test:

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Since the improper integral converges, the series converges by the Integral Test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^4+25}$

Let $f(x) = \frac{2x}{x^4+25}$, which is continuous, positive, and decreasing for $x \geq 1$. Make the substitution $u = x^2$, so $du = 2x\,dx$:

$$\begin{aligned}\int_1^{\infty} \frac{2x}{x^4+25}\,dx &= \int_{1^2}^{\infty} \frac{1}{u^2+25}\,du \\ &= \frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right)\Big|_1^{\infty} \\ &= \frac{1}{5}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{5}\right)\right)\end{aligned}$$

Since the improper integral converges, the series converges by the Integral Test.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^6+36}$

Let $f(x) = \frac{3x^2}{x^6+36}$, which is continuous, positive, and decreasing for $x \geq 1$. Let $u = x^3$, so $du = 3x^2\,dx$:

$$\begin{aligned}\int_1^{\infty} \frac{3x^2}{x^6+36}\,dx &= \int_{1^3}^{\infty} \frac{1}{u^2+36}\,du \\ &= \frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right)\Big|_1^{\infty} \\ &= \frac{1}{6}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{6}\right)\right)\end{aligned}$$

Since the improper integral converges, the series converges by the Integral Test.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3}{n^8+64}$

Let $f(x) = \frac{4x^3}{x^8+64}$, which is continuous, positive, and decreasing for $x \geq 1$. Let $u = x^4$, so $du = 4x^3\,dx$:

$$\begin{aligned}\int_1^{\infty} \frac{4x^3}{x^8+64}\,dx &= \int_{1^4}^{\infty} \frac{1}{u^2+64}\,du \\ &= \frac{1}{8}\tan^{-1}\left(\frac{u}{8}\right)\Big|_1^{\infty} \\ &= \frac{1}{8}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{8}\right)\right)\end{aligned}$$

Since the improper integral converges, the series converges by the Integral Test.

Direct Comparison Test

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

   We compare to $\frac{1}{n^2}$:

   $$\frac{1}{n^2+1} < \frac{1}{n^2}$$

   The series $\sum \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$). Therefore, the original series converges by the Comparison Test.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$

   We compare to $\left(\frac{2}{3}\right)^n$:

   $$\frac{2^n}{3^n+5} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

   The geometric series converges (common ratio $r = \frac{2}{3} < 1$). Therefore, the original series converges by the Comparison Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n}$

   We compare to $\frac{1}{2n}$:

   $$\frac{1}{\sqrt{n}+n} > \frac{1}{2n}$$

   The series $\sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ diverges (harmonic series). Therefore, the original series diverges by the Comparison Test.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

   We compare to $\frac{1}{n^2}$:

   $$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$$

   The series $\sum \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$). Therefore, the original series converges by the Comparison Test.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2}{n^4+n^2+1}$

   We compare to $\frac{4}{n^2}$:

   $$\frac{3n^2+2}{n^4+n^2+1} < \frac{4n^2}{n^4} = \frac{4}{n^2}$$

   The series $\sum \frac{1}{n^2}$ converges. Therefore, the original series converges by the Comparison Test.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4^n+n}$

   We compare to $\left(\frac{1}{2}\right)^n$:

   $$\frac{2^n}{4^n+n} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n$$

   The geometric series converges. Therefore, the original series converges by the Comparison Test.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$

   We compare to $\frac{1}{2n}$:

   $$\frac{n}{n^2+2} > \frac{n}{2n^2} = \frac{1}{2n}$$

   The series $\sum \frac{1}{n}$ diverges. Therefore, the original series diverges by the Comparison Test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+2^n}$

   We compare to $\left(\frac{5}{3}\right)^n$:

   $$\frac{5^n}{3^n+2^n} > \frac{5^n}{2\cdot 3^n} = \frac{1}{2}\left(\frac{5}{3}\right)^n$$

   The geometric series diverges ($r = \frac{5}{3} > 1$). Therefore, the original series diverges by the Comparison Test.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin^2\left(\frac{1}{n}\right)}{n}$

   We compare to $\frac{1}{n}$:

   $$\frac{1}{n} < \frac{1+\sin^2\left(\frac{1}{n}\right)}{n}$$

   The series $\sum \frac{1}{n}$ diverges. Therefore, the original series diverges by the Comparison Test.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

    For $n \geq 4$, we have:

    $$\begin{aligned}\frac{n!}{n^n} &= \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n}\\ &\leq 1\cdot 1\cdot 1\cdots 1\cdot\frac{2}{n^2}\\ &= \frac{2}{n^2}\end{aligned}$$

    The series $\sum \frac{2}{n^2}$ converges. Therefore, the original series converges by the Comparison Test.

Limit Comparison Test

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3-4}$

Let $b_n = \frac{1}{n}$. Then,

$$\lim_{n\to\infty} \frac{\frac{n^2+3n}{n^3-4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+3n^2}{n^3-4} = 1$$

Since $\sum b_n = \sum \frac{1}{n}$ diverges and the limit is positive and finite, the original series diverges by the Limit Comparison Test.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4-1}$

Let $b_n = \frac{1}{n^2}$. Then,

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4-1} = 1$$

Since $\sum b_n = \sum \frac{1}{n^2}$ converges, the original series converges by the Limit Comparison Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+5}$

Let $b_n = \frac{1}{n^{3/2}}$. Then,

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}+1}{n^2+5}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}(\sqrt{n}+1)}{n^2+5} = \lim_{n\to\infty} \frac{n^2+n^{3/2}}{n^2+5} = 1$$

Since $\sum b_n = \sum \frac{1}{n^{3/2}}$ converges, the original series converges by the Limit Comparison Test.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

Let $b_n = \frac{1}{n}$. Then,

$$\lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = 1$$

Since $\sum b_n = \sum \frac{1}{n}$ diverges, the original series diverges by the Limit Comparison Test.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Let $b_n = \frac{1}{\sqrt{n}}$. Then,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}}$$
$$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2}$$

Since $\sum b_n = \sum \frac{1}{\sqrt{n}}$ diverges (p-series with $p = \frac{1}{2} < 1$), the original series diverges by the Limit Comparison Test.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+3n}}$

Let $b_n = \frac{1}{n^2}$. Then,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^4+3n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+3n}} = 1$$

Since $\sum b_n = \sum \frac{1}{n^2}$ converges, the original series converges by the Limit Comparison Test.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+2^n}{4^n+1}$

Let $b_n = \left(\frac{3}{4}\right)^n$. Then,

$$\lim_{n\to\infty} \frac{\frac{3^n+2^n}{4^n+1}}{\left(\frac{3}{4}\right)^n} = \lim_{n\to\infty} \frac{3^n+2^n}{4^n+1}\cdot\frac{4^n}{3^n}$$
$$= \lim_{n\to\infty} \frac{3^n+2^n}{3^n}\cdot\frac{4^n}{4^n+1}$$
$$= \lim_{n\to\infty} \left(1+\left(\frac{2}{3}\right)^n\right)\cdot\left(\frac{1}{1+\frac{1}{4^n}}\right) = 1$$

Since $\sum b_n = \sum \left(\frac{3}{4}\right)^n$ is a convergent geometric series, the original series converges by the Limit Comparison Test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2+(-1)^n}$

Let $b_n = \frac{1}{n^2}$. Then,

$$\lim_{n\to\infty} \frac{\frac{5}{n^2+(-1)^n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5n^2}{n^2+(-1)^n} = 5$$

Since $\sum b_n = \sum \frac{1}{n^2}$ converges, the original series converges by the Limit Comparison Test.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2 + \tan^{-1}(n)}$

Let $b_n = \frac{1}{n^2}$. Then,

$$\lim_{n\to\infty} \frac{\frac{4}{n^2+\tan^{-1}(n)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{4n^2}{n^2 + \tan^{-1}(n)} = 4$$

Since $\sum b_n = \sum \frac{1}{n^2}$ converges, the original series converges by the Limit Comparison Test.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2 + (-1)^n}$

Let $b_n = \frac{1}{n^2}$. Then,

$$\lim_{n\to\infty} \frac{\frac{5}{n^2+(-1)^n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5n^2}{n^2 + (-1)^n} = 5$$

Since $\sum b_n = \sum \frac{1}{n^2}$ converges, the original series converges by the Limit Comparison Test.

Alternating Series Test

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

   Let $b_n = \frac{1}{n}$. Then:

   - $b_n > 0$, decreasing
   - $\lim_{n\to\infty} b_n = 0$

   The series converges by the Alternating Series Test.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

   Let $b_n = \frac{1}{n^2}$. Then:

   - $b_n > 0$, decreasing
   - $\lim_{n\to\infty} b_n = 0$

   The series converges by the Alternating Series Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   Let $b_n = \frac{1}{\sqrt{n}}$. Then:

   - $b_n > 0$, decreasing
   - $\lim_{n\to\infty} b_n = 0$

   The series converges by the Alternating Series Test.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

   Let $b_n = \frac{1}{\ln(n+1)}$. Then:

   - $b_n > 0$, decreasing
   - $\lim_{n\to\infty} b_n = 0$

   The series converges by the Alternating Series Test.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

   Let $b_n = \frac{n}{n+1}$. Then:

   - $\lim_{n\to\infty} b_n = 1 \neq 0$

   The Alternating Series Test does not apply. The series diverges by the Test for Divergence.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(-1)^n}$

   Let $b_n = \frac{1}{n+(-1)^n}$. Then:

   - b_n is not monotonic (alternates between even and odd terms)

   The Alternating Series Test does not apply.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$

   Let $b_n = \frac{1}{n^{1/3}}$. Then:

   - $b_n > 0$, decreasing
   - $\lim_{n\to\infty} b_n = 0$

   The series converges by the Alternating Series Test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+\ln n}$

   Let $b_n = \frac{1}{n^2+\ln n}$. Then:

   - $b_n > 0$, decreasing for $n \geq 2$
   - $\lim_{n\to\infty} b_n = 0$

   The series converges by the Alternating Series Test.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sin(1/n)}{n}$

   Let $b_n = \frac{\sin(1/n)}{n}$. Then:

   - $b_n > 0$
   - $\lim_{n\to\infty} b_n = 0$
   - To show b_n is decreasing for $n \geq 1$, define $f(x) = \frac{\sin(1/x)}{x}$. Then

     $$f'(x) = -\frac{\cos(1/x)}{x^3} - \frac{\sin(1/x)}{x^2} < 0 \text{ for } x \geq 1,$$

     so $b_n = f(n)$ is decreasing.

   The series converges by the Alternating Series Test.

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{0.9}}$

    Let $b_n = \frac{1}{n^{0.9}}$. Then:

    - $b_n > 0$, decreasing
    - $\lim_{n\to\infty} b_n = 0$

    The series converges by the Alternating Series Test.

Absolute Convergence

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

   This is the alternating harmonic series. Since:

   $$\lim_{n\to\infty} \frac{1}{n} = 0, \quad \text{and} \quad \frac{1}{n} \text{ is decreasing,}$$

   it converges by the Alternating Series Test. But $\sum \frac{1}{n}$ diverges, so it does not converge absolutely.

   Conclusion: Converges conditionally.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

   The series passes the Alternating Series Test. Also,

   $$\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}, \quad \text{and} \quad \sum \frac{1}{n^2} \text{ converges.}$$

   Conclusion: Converges absolutely.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   This is an alternating series with decreasing terms and limit zero, so it converges conditionally. However,

   $$\sum \frac{1}{\sqrt{n}} \text{ diverges.}$$

   Conclusion: Converges conditionally.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

   For $n \geq 2$, the terms $\frac{\ln n}{n}$ are positive, decreasing, and approach 0. So the series converges by the Alternating Series Test. But

   $$\sum \frac{\ln n}{n} \text{ diverges.}$$

   Conclusion: Converges conditionally.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2+1)}{n^3}$

   Since $|\sin(n^2+1)| \leq 1$, we have

   $$\left|\frac{\sin(n^2+1)}{n^3}\right| \leq \frac{1}{n^3},$$

   and $\sum \frac{1}{n^3}$ converges.

   Conclusion: Converges absolutely.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$

   Let $b_n = \frac{1}{(n+1)\ln(n+1)}$. Then b_n is positive, decreasing, and $\lim b_n = 0$, so the alternating series converges. To test for absolute convergence, consider

   $$\sum \frac{1}{(n+1)\ln(n+1)} \sim \sum \frac{1}{n \ln n}, \text{ which diverges.}$$

   Conclusion: Converges conditionally.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

   Let $a_n = \frac{(-1)^n}{n!}$. Since $\sum \frac{1}{n!}$ converges (rapid decay), the series converges absolutely.

   Conclusion: Converges absolutely.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}}$

   Since $|\cos(n)| \leq 1$, we have

   $$\left|\frac{\cos(n)}{n^{3/2}}\right| \leq \frac{1}{n^{3/2}},$$

   and $\sum \frac{1}{n^{3/2}}$ converges.

   Conclusion: Converges absolutely.

9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

   Let $b_n = \frac{1}{n(\ln n)^2}$. Then b_n is positive, decreasing, and tends to 0. The series converges by the Alternating Series Test. For absolute convergence, apply the Integral Test:

   $$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx < \infty.$$

   Conclusion: Converges absolutely.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

    Since $|\sin(n)| \leq 1$, we have

    $$\left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2},$$

    and $\sum \frac{1}{n^2}$ converges.

    Conclusion: Converges absolutely.

Ratio Test

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^2}$

   Let $a_n = \frac{5^n}{n^2}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)^2}\cdot\frac{n^2}{5^n} = 5\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 5.$$

   Since the limit is greater than 1, the series diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

   Let $a_n = \frac{n^2}{2^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \frac{1}{2}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}.$$

   The series converges absolutely.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

   Let $a_n = \frac{2^n}{n^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n} = 2\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1} = 0.$$

   The series converges absolutely.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

   Let $a_n = \frac{2^n}{n!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \frac{2}{n+1}\to 0.$$

   The series converges absolutely.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$

   Let $a_n = \frac{n!}{3^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!} = \lim_{n\to\infty}\frac{n+1}{3} = \infty.$$

   The series diverges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\cdot n!}{n^n}$

   Let $a_n = \frac{3^n n!}{n^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{3^n n!} = 3\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = 3\cdot\frac{1}{e}.$$

   Since $3/e > 1$, the series diverges.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

   Let $a_n = \frac{n!}{(2n)!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{(2n+2)!}\cdot\frac{(2n)!}{n!} = \lim_{n\to\infty}\frac{(n+1)}{(2n+2)(2n+1)} = 0.$$

   The series converges absolutely.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\cdot n^n}$

   Let $a_n = \frac{(2n)!}{n!\,n^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(2n+2)!}{(n+1)!\cdot(n+1)^{n+1}}\cdot\frac{n!\cdot n^n}{(2n)!} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)\cdot(n+1)^{n+1}}\cdot n^n = 0.$$

   The series converges absolutely.

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\cdot\frac{n}{2^n}$

   Let $a_n = \left|(-1)^n\cdot\frac{n}{2^n}\right| = \frac{n}{2^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n} = \frac{1}{2}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{2}.$$

   The series converges absolutely.

10. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n^2}{n!}$

Let $a_n = \left|(-1)^n \cdot \frac{n^2}{n!}\right| = \frac{n^2}{n!}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{n+1}\cdot\frac{1}{n^2} \\
&= \lim_{n\to\infty}\frac{n+1}{n^2} = 0.
\end{aligned}$$

The series converges absolutely.

Remainder Estimates

1. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ using the first 4 terms.

$$S_4 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} = \frac{205}{144} \approx 1.4236.$$

The remainder satisfies

$$|R_4| \le \left|\frac{1}{25}\right| = 0.04.$$

2. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$ using the first 3 terms.

$$S_3 = -\frac{1}{2} + \frac{1}{9} - \frac{1}{28} = -\frac{103}{252} \approx -0.4087.$$

The error satisfies

$$|R_3| \le \left|\frac{1}{4^3+1}\right| = \frac{1}{65} \approx 0.0154.$$

3. How many terms are needed to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ to within 0.0001?

We want to solve for n so that $b_{n+1} < 0.0001$

$$\frac{1}{(n+1)^5} < 0.0001 \quad \Rightarrow \quad n+1 > \sqrt[5]{10000} \approx 6.3.$$

Thus, $n > 5.3$, so 6 terms are needed.

4. Use the first 5 terms of $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n+1)}{n^2}$ to approximate the sum.

$$\begin{aligned} S_5 &= -\frac{\ln 2}{1^2} + \frac{\ln 3}{2^2} - \frac{\ln 4}{3^2} + \frac{\ln 5}{4^2} - \frac{\ln 6}{5^2} \\ &\approx -0.6931 + 0.2747 - 0.1540 + 0.0866 - 0.0553 \\ &= -0.5411. \end{aligned}$$

The error satisfies

$$|R_5| \le b_6 = \frac{\ln(7)}{6^2} \approx 0.054053.$$

5. Determine the minimum number of terms needed to estimate $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ with error less than 0.01.

We need to solve for n so that $b_{n+1} < 0.01$:

$$\frac{1}{(n+1)\ln(n+1)} < 0.01.$$

We can solve this by guessing and checking. Indeed, $\frac{1}{29\ln(29)} > 0.01$, but $\frac{1}{30\ln(30)} < 0.01$. This means that n should be 29 in order for $b_{n+1} < 0.01$. Since n starts at 2, we use the first 28 terms.

6. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ using the first 5 terms.

$$\begin{aligned} S_5 &= \sum_{n=1}^{5} \frac{1}{n^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} \\ &\approx 0.5 + 0.2 + 0.1 + 0.0588 + 0.0385 = 0.8973. \end{aligned}$$

Use the remainder bound:

$$\begin{aligned} \int_5^{\infty} \frac{1}{x^2+1}\,dx &= \tan^{-1}(x)\Big|_5^{\infty} \\ &= \frac{\pi}{2} - \tan^{-1}(5) \\ &\approx 1.5708 - 1.3734 \\ &= 0.1974. \end{aligned}$$

So, error < 0.1974.

7. Estimate the remainder when approximating $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ from $n=2$ to $n=10$.

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x \ln x}\,dx.$$

Let $u = \ln x$, $du = \frac{1}{x}dx$:

$$\int_{10}^{\infty} \frac{1}{x \ln x}dx = \int_{\ln 10}^{\infty} \frac{1}{u}du = \infty.$$

So, the series diverges and the error is unbounded.

8. Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ using the first 8 terms.

$$\begin{aligned} S_8 &= \sum_{n=1}^{8} \frac{1}{n^{3/2}} \\ &\approx 1 + 0.3536 + 0.1925 + 0.125 \\ &\qquad + 0.0894 + 0.0680 + 0.0535 + 0.0442 \\ &= 1.9262. \end{aligned}$$

Estimate error:

$$\int_8^{\infty} \frac{1}{x^{3/2}}dx = -2x^{-1/2}\Big|_8^{\infty} = 2 \cdot \frac{1}{\sqrt{8}} \approx 0.7071.$$

9. Find the smallest n such that $\sum_{k=2}^{n} \frac{1}{k(\ln k)^2}$ approximates the full sum within 0.05.

We want:

$$\int_n^\infty \frac{1}{x(\ln x)^2} dx < 0.05.$$

Use substitution $u = \ln x$, $du = \frac{1}{x} dx$:

$$\int_n^\infty \frac{1}{x(\ln x)^2} dx = \int_{\ln n}^\infty \frac{1}{u^2} du = \frac{1}{\ln n}.$$

Set $\frac{1}{\ln n} < 0.05 \Rightarrow \ln n > 20 \Rightarrow n > e^{20}$.

So, $n = 485{,}165{,}195$ terms are needed.

10. Approximate $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ using the first 6 terms.

$$\begin{aligned} S_6 &= \sum_{n=2}^{7} \frac{\ln n}{n^3} \\ &\approx 0.0866 + 0.0459 + 0.0277 + 0.0181 + 0.0122 + 0.0086 \\ &= 0.1991. \end{aligned}$$

Estimate error:

$$\int_7^\infty \frac{\ln x}{x^3} dx.$$

Use integration by parts:

$$u = \ln x, \quad dv = x^{-3} dx, \quad du = \frac{1}{x} dx, \quad v = -\frac{1}{2x^2}$$

$$\int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2} + \int \frac{1}{2x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}.$$

Evaluating from 7 to ∞:

$$\begin{aligned} -\frac{\ln x}{2x^2} - \frac{1}{4x^2} \Big|_7^\infty &= 0 + \frac{\ln 7}{2 \cdot 49} + \frac{1}{4 \cdot 49} \\ &\approx \frac{1.9459}{98} + \frac{1}{196} \\ &\approx 0.0199 + 0.0051 \\ &= 0.025. \end{aligned}$$

So, the error is less than 0.025.

Taylor Polynomials

1. $f(x) = x^2e^x, \quad a = 0$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$x^2e^x = x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \cdots$$

2. $f(x) = \dfrac{1}{1+x^2}, \quad a = 1$

   Let $u = x - 1$, then $x = 1 + u$. Expand $f(1+u)$:

$$f(1+u) = \frac{1}{1+(1+u)^2} = \frac{1}{2+2u+u^2}$$

   Use polynomial division or binomial expansion to find:

$$f(x) = \frac{1}{2} - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{4} + \cdots$$

$$= \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(x-1)^3 + \cdots$$

3. $f(x) = \sqrt[4]{x}, \quad a = 16$

   Let $u = x - 16$, then $f(x) = (16+u)^{1/4}$. Use binomial expansion:

$$f(x) = 16^{1/4}\left(1 + \frac{u}{16}\right)^{1/4}$$

$$= 2 + \frac{1}{32}(x-16) - \frac{3}{2048}(x-16)^2 + \frac{21}{131072}(x-16)^3 + \cdots$$

4. $f(x) = \cos(x^2), \quad a = 0$

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

5. $f(x) = \sin(2x), \quad a = \dfrac{\pi}{4}$

   Let $u = x - \frac{\pi}{4}$, then

$$f(x) = \sin\left(2 \cdot \left(\frac{\pi}{4} + u\right)\right) = \sin\left(\frac{\pi}{2} + 2u\right) = \cos(2u)$$

$$= 1 - \frac{(2u)^2}{2!} + \frac{(2u)^4}{4!} - \frac{(2u)^6}{6!} + \cdots$$

$$= 1 - 2(x - \frac{\pi}{4})^2 + \frac{2}{3}(x - \frac{\pi}{4})^4 - \frac{4}{45}(x - \frac{\pi}{4})^6 + \cdots$$

6. $f(x) = \ln(1+2x), \quad a = 0$

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots$$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \cdots$$

7. $f(x) = \tan^{-1}(x), \quad a = 0$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

8. $f(x) = \dfrac{x}{1-x}, \quad a = 0$

$$\frac{x}{1-x} = x + x^2 + x^3 + x^4 + \cdots$$

9. $f(x) = e^{x^2}, \quad a = 0$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

10. $f(x) = \dfrac{1}{\sqrt{1-x}}, \quad a = 0$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$$

Mixed Series Practice

1. We have:

$$\lim_{n\to\infty} \frac{\sqrt{n^2+\cos n}}{n} = \lim_{n\to\infty} \sqrt{1+\frac{\cos n}{n^2}} = \sqrt{1+0} = 1.$$

2. Let $x_n = \left(1-\frac{1}{2n}\right)^{3n}$. Then:

$$x_n = \left[\left(1-\frac{1}{2n}\right)^{2n}\right]^{3/2} \to \left(e^{-1}\right)^{3/2} = e^{-3/2}.$$

3. We examine the sequence of partial sums:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2}{\ln(n+1)+3n} = \lim_{n\to\infty} \frac{n}{3+\frac{\ln(n+1)}{n}} = \infty.$$

Since the partial sums diverge, the series diverges.

4. We compute the limit of the partial sums:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4n^2+1}{2n^2+5n+1} = \frac{4}{2} = 2.$$

Therefore, the series converges, and

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 2.$$

5. Using linearity of infinite series:

$$\sum(2a_n - 3b_n) = 2\sum a_n - 3\sum b_n = 2(5) - 3(4) = 10 - 12 = -2.$$

6. This is a geometric series with first term

$$a = \frac{6}{4^2} = \frac{6}{16} = \frac{3}{8}, \quad r = \frac{1}{4}.$$

The sum is:

$$\sum_{n=2}^{\infty} \frac{6}{4^n} = \frac{a}{1-r} = \frac{3/8}{1-1/4} = \frac{3/8}{3/4} = \frac{1}{2}.$$

7. As $n \to \infty$,

$$\frac{n^4+3}{n(n+1)^2} \sim \frac{n^4}{n\cdot n^2} = n.$$

Since $\frac{n^4+3}{n(n+1)^2} \sim n$, the terms do not tend to zero. Therefore, the series diverges by the Test for Divergence.

8. Compare with $\sum b_n = \sum \left(\frac{4}{5}\right)^n$:

$$\lim_{n\to\infty} \frac{4^n}{5^n} \cdot \frac{5^n-1}{4^n} = \lim_{n\to\infty} 1 - \frac{1}{5^n} = 1.$$

Since $\sum \left(\frac{4}{5}\right)^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

9. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{2}.$$

Since the limit is less than 1, the series converges absolutely.

10. Let $a_n = \frac{c^n}{n}$. Then:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{c^{n+1}}{n+1} \cdot \frac{n}{c^n} = c \cdot \lim_{n\to\infty} \frac{n}{n+1} = c.$$

By the Ratio Test:

- If $c < 1$, the limit is less than 1, so the series converges.
- If $c > 1$, the limit is greater than 1, so the series diverges.
- If $c = \pm 1$, the limit is 1, so the Ratio Test is inconclusive.

11. We compare with $b_n = \frac{2n}{n^3-1} \sim \frac{2n}{n^3} = \frac{2}{n^2}$.

For large n, $\cos n$ is bounded and negligible. So:

$$\frac{2n}{n^3+\cos n} < \frac{2n}{n^3-1} \sim \frac{2}{n^2}.$$

Since $\sum \frac{2}{n^2}$ converges, the original series converges by the Comparison Test.

12. Since $|\sin(n^3)| \le 1$, we have

$$\left|\frac{\sin(n^3)}{n^3+1}\right| \le \frac{1}{n^3+1} < \frac{1}{n^3}.$$

Since $\sum \frac{1}{n^3}$ converges, the series converges absolutely.

13. Let
$$a_n = \frac{(-1)^n}{n^2-4}, \quad b_n = \frac{1}{n^2}.$$
Then
$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\frac{1}{n^2-4}\cdot n^2 = \lim_{n\to\infty}\frac{n^2}{n^2-4} = 1.$$
Since $\sum b_n = \sum \frac{1}{n^2}$ converges, the series $\sum |a_n|$ also converges by the Limit Comparison Test.

Therefore, the series converges absolutely.

14. The remainder satisfies:
$$|R_4| \le \left|\frac{1}{5^3}\right| = \frac{1}{125} = 0.008.$$

15. Let $f(x) = \frac{1}{x(\ln x)^2}$. For $x \ge 2$, the function is continuous, positive, and decreasing, so the Integral Test applies:
$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx.$$
Substitute $u = \ln x$, so $du = \frac{1}{x}dx$. Then:
$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \int_{\ln 2}^\infty \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]_{\ln 2}^\infty = \frac{1}{\ln 2}.$$
Since the integral converges, the series converges by the Integral Test.

To estimate the error after the first 5 terms ($n = 2$ through 6), we use the remainder estimate:
$$R_6 \le \int_6^\infty \frac{1}{x(\ln x)^2}\,dx = \left[-\frac{1}{\ln x}\right]_6^\infty = \frac{1}{\ln 6}.$$
Thus,
$$R_6 \le \frac{1}{\ln 6} \approx \frac{1}{1.7918} \approx 0.5583.$$
The series converges, and the error in approximating the sum using the first 5 terms is at most $\frac{1}{\ln 6} \approx 0.5583$.