Math 2300: Midterm 3 Practice Solutions

1. Evaluate $\lim_{n\to\infty} \frac{\sqrt{n^2+\cos n}}{n}$.

Solution:

$$\lim_{n\to\infty} \frac{\sqrt{n^2+\cos n}}{n} = \lim_{n\to\infty} \sqrt{\frac{n^2+\cos n}{n^2}} = \lim_{n\to\infty} \sqrt{1+\frac{\cos n}{n^2}} = \sqrt{1+0} = 1.$$

2. Suppose $S_n = \sum_{k=1}^{n} a_k = \frac{n^2}{\ln(n+1)+3n}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

Solution: We examine the sequence of partial sums:

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{n^2}{\ln(n+1)+3n} \\ &= \lim_{n\to\infty} \frac{n}{3+\frac{\ln(n+1)}{n}} = \infty.\end{aligned}$$

Since the partial sums diverge, the series diverges.

3. Suppose $S_n = \sum_{k=1}^{n} a_k = \frac{4n^2}{\ln(n+1)+4n^2}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

Solution: We examine the sequence of partial sums:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4n^2}{\ln(n+1)+4n^2} = \lim_{n\to\infty} \frac{4}{\frac{\ln(n+1)}{n^2}+4} = 1.$$

Thus, the sequence of partial sums converges to 1. Therefore, the series $\sum_{n=1}^{\infty} a_n$ converges, and the sum is 1.

4. Suppose $S_n = \sum_{k=1}^{n} a_k = \dfrac{4n^2+1}{2n^2+5n+1}$. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and, if so, to what value.

Solution: We compute the limit of the partial sums:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4n^2+1}{2n^2+5n+1} = \frac{4}{2} = 2.$$

Therefore, the series converges, and

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 2.$$

5. If $\sum a_n = \pi$ and $\sum b_n = 4$, find $\sum (2a_n - 3b_n)$.

Solution: Using linearity of infinite series:

$$\begin{aligned} \sum (2a_n - 3b_n) &= 2\sum a_n - 3\sum b_n \\ &= 2(\pi) - 3(4) = 2\pi - 12. \end{aligned}$$

6. Find the sum of the geometric series: $\displaystyle\sum_{n=2}^{\infty} \frac{6}{4^n}$.

Solution: This is a geometric series with first term

$$a = \frac{6}{4^2} = \frac{6}{16} = \frac{3}{8}, \quad r = \frac{1}{4}.$$

The sum is:

$$\sum_{n=2}^{\infty} \frac{6}{4^n} = \frac{a}{1-r} = \frac{3/8}{1-1/4} = \frac{3/8}{3/4} = \frac{1}{2}.$$

7. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^4+3}{n(n+1)^2}$ converges. If it converges, find its sum.

Solution: As $n \to \infty$,

$$\frac{n^4+3}{n(n+1)^2} \sim \frac{n^4}{n \cdot n^2} = n.$$

Since $\frac{n^4+3}{n(n+1)^2} \sim n$, the terms do not tend to zero. Therefore, the series diverges by the Test for Divergence.

8. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{5^n-1}$ converges.

Solution: Compare with $\sum b_n = \sum \left(\frac{4}{5}\right)^n$:

$$\lim_{n\to\infty} \frac{4^n}{5^n} \cdot \frac{5^n-1}{4^n} = \lim_{n\to\infty} 1 - \frac{1}{5^n} = 1.$$

Since $\sum \left(\frac{4}{5}\right)^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

9. Let $a_n = \dfrac{n^2}{2^n}$. Use the Ratio Test to determine whether $\sum a_n$ converges or diverges.

Solution: We have

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\ &= \frac{1}{2} \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{2}.\end{aligned}$$

Since the limit is less than 1, the series converges absolutely.

10. Let $a_n = \dfrac{n!}{2^n}$. Use the Ratio Test to determine whether $\sum a_n$ converges or diverges.

Solution: We apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} &= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{1}{2} \\
&= \lim_{n\to\infty} (n+1) \cdot \frac{1}{2} \\
&= \infty.
\end{aligned}$$

Since the limit is greater than 1, the series diverges by the Ratio Test.

11. Use the Ratio Test to determine for which values of $c > 0$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{c^n}{n}$ converges.

Solution: Let $a_n = \frac{c^n}{n}$. Then:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{c^{n+1}}{n+1} \cdot \frac{n}{c^n} = c \cdot \lim_{n\to\infty} \frac{n}{n+1} = c.$$

By the Ratio Test:

- If $c < 1$, the limit is less than 1, so the series converges.
- If $c > 1$, the limit is greater than 1, so the series diverges.
- If $c = 1$, the limit is 1, so the Ratio Test is inconclusive.

12. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^3 + \cos n}$ converges or diverges.

Solution: We compare the given series to a simpler one using the Limit Comparison Test, which applies since all terms are positive for $n \geq 1$.

Note that for all n, we have $-1 \leq \cos n \leq 1$, so

$$n^3 - 1 \leq n^3 + \cos n \leq n^3 + 1.$$

This implies that for large n, the term $n^3 + \cos n \sim n^3$. Thus, we consider the comparison series

$$b_n = \frac{2n}{n^3} = \frac{2}{n^2},$$

which is a convergent p-series with $p = 2 > 1$.

We now compute the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{2n}{n^3+\cos n}}{\dfrac{2}{n^2}} = \lim_{n\to\infty} \frac{2n\cdot n^2}{2(n^3+\cos n)}$$
$$= \lim_{n\to\infty} \frac{n^3}{n^3+\cos n}$$
$$= \lim_{n\to\infty} \frac{n^3}{n^3+\cos n}\cdot\frac{1/n^3}{1/n^3}$$
$$= \lim_{n\to\infty} \frac{1}{1+\frac{\cos n}{n^3}}$$
$$= \frac{1}{1+0} = 1$$

Because the limit is a positive finite number and the comparison series $\sum b_n = \sum \frac{2}{n^2}$ converges, the original series

$$\sum_{n=1}^{\infty} \frac{2n}{n^3+\cos n}$$

also **converges** by the Limit Comparison Test.

13. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^3)}{n^3+1}$ converges absolutely, conditionally, or diverges.

Solution: We first test for absolute convergence by considering the series

$$\sum_{n=1}^{\infty} \left|\frac{\sin(n^3)}{n^3+1}\right|.$$

Since $|\sin(n^3)| \leq 1$ for all n, we have:

$$\left|\frac{\sin(n^3)}{n^3+1}\right| \leq \frac{1}{n^3+1} < \frac{1}{n^3}.$$

Because $\sum \frac{1}{n^3}$ is a convergent p-series with $p = 3 > 1$, the Comparison Test implies that

$$\sum_{n=1}^{\infty} \left|\frac{\sin(n^3)}{n^3+1}\right|$$

converges absolutely.

14. Determine whether $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n^2-4}$ converges absolutely, conditionally, or diverges.

Solution: Let

$$a_n = \frac{(-1)^n}{n^2-4}, \quad b_n = \frac{1}{n^2}.$$

Then

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{1}{n^2-4} \cdot n^2 = \lim_{n\to\infty} \frac{n^2}{n^2-4} = 1.$$

Since $\sum b_n = \sum \frac{1}{n^2}$ converges, the series $\sum |a_n|$ also converges by the Limit Comparison Test.

Therefore, the series converges absolutely.

15. Use the Alternating Series Remainder Theorem to estimate the error when approximating $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3}$ using the first 4 terms.

Solution:

$$|R_4| \leq \left|\frac{1}{5^3}\right| = \frac{1}{125} = 0.008.$$

16. Use the Integral Test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{(n^3+1)^3}$ converges. Remember to verify that the corresponding function $\displaystyle\sum_{n=1}^{\infty} \frac{3x^2}{(x^3+1)^3}$ is positive, continuous, and decreasing for $x \geq 1$. Estimate the error when approximating the series with the first 4 terms.

Solution: Let $f(x) = \dfrac{3x^2}{(x^3+1)^3}$. To apply the Integral Test, we must verify the following on the interval $[2, \infty)$:

- **Positive:** Since both the numerator and denominator are positive for $x \geq 2$, it follows that $f(x) > 0$ on this interval.
- **Continuous:** The function $f(x)$ is a rational function with no discontinuities for $x \geq 2$, so it is continuous.

- **Decreasing:** We compute the derivative to confirm that $f(x)$ is decreasing:

$$\begin{aligned} f(x) &= \frac{3x^2}{(x^3+1)^3}, \\ f'(x) &= \frac{6x(x^3+1)^3 - 3x^2 \cdot 3(x^3+1)^2 \cdot 3x^2}{(x^3+1)^6} \\ &= \frac{6x(x^3+1)^3 - 27x^4(x^3+1)^2}{(x^3+1)^6} \\ &= \frac{6x - 21x^4}{(x^3+1)^4} \end{aligned}$$

The numerator is negative for $x \geq 1$, so $f'(x) < 0$, and $f(x)$ is decreasing.

Since $f(x)$ is positive, continuous, and decreasing for $x \geq 1$, we may apply the Integral Test. Let Let $u = x^3 + 1 \Rightarrow du = 3x^2\, dx$.

$$\begin{aligned} \int_1^\infty \frac{3x^2}{(x^3+1)^3}\, dx = \int_{u=1^3+1}^\infty \frac{1}{u^3}\, du &= \int_2^\infty u^{-3}\, du \\ &= \left[\frac{u^{-2}}{-2}\right]_2^\infty = \frac{1}{2 \cdot 2^2} = \frac{1}{8}. \end{aligned}$$

The integral converges, so by the Integral Test, the series

$$\sum_{n=1}^{\infty} \frac{3n^2}{(n^3+1)^3}$$

converges.

To estimate the error using the first 4 terms, let

$$S_4 = \sum_{n=1}^{4} \frac{3n^2}{(n^3+1)^3}.$$

Note: since $n = 2$ to start, the first 4 terms stop at $n = 5$. The remainder satisfies

$$R_4 \leq \int_4^\infty \frac{3x^2}{(x^3+1)^3}\, dx.$$

Again, using the substitution $u = x^3 + 1$, $du = 3x^2\, dx$, and when $x = 4$, $u = 65$, we get

$$R_4 \leq \int_{65}^\infty \frac{1}{u^3}\, du = \left[\frac{u^{-2}}{-2}\right]_{65}^\infty = \frac{1}{2 \cdot 65^2} = \frac{1}{8450}.$$

The series converges by the Integral Test, and the error from using the first 4 terms is less than $\frac{1}{8450}$.

17. Use the Integral Test to show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$ converges. Remember to verify that the corresponding function $\displaystyle\sum_{n=2}^{\infty} \frac{1}{x\ln(x)^2}$ is positive, continuous, and decreasing for $x \geq 2$. Estimate the error when approximating the series with the first 5 terms.

Solution: Let $f(x) = \frac{1}{x(\ln x)^2}$. For $x \geq 2$, the function is continuous, positive, and decreasing (check!), so the Integral Test applies:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx.$$

Substitute $u = \ln x$, so $du = \frac{1}{x}dx$. Then:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\, du = \left[-\frac{1}{u}\right]_{\ln 2}^{\infty} = \frac{1}{\ln 2}.$$

Since the integral converges, the series converges by the Integral Test.

To estimate the error after the first 5 terms ($n = 2$ through 6), we use the remainder estimate:

$$R_6 \leq \int_6^{\infty} \frac{1}{x(\ln x)^2}\, dx = \left[-\frac{1}{\ln x}\right]_6^{\infty} = \frac{1}{\ln 6}.$$

Thus,

$$R_6 \leq \frac{1}{\ln 6} \approx \frac{1}{1.7918} \approx 0.5583.$$

The series converges, and the error in approximating the sum using the first 5 terms is at most $\frac{1}{\ln 6} \approx 0.5583$.

18. If the third degree Taylor polynomial of $f(x)$ about $x = 4$ is

$$T_3(x) = 5 + 4(x-4) - 3(x-4)^2 - 2(x-4)^3$$

is $f(x)$ increasing or decreasing at $x = 4$? Is $f(x)$ concave up or concave down at $x = 4$?

Solution:

- Since $f'(4) > 0$, the function is increasing at $x = 4$.
- Since $f''(4) < 0$, the function is concave down at $x = 4$.