Midterm 2 Study Guide

MATH2300 - Calculus II

Spring 2026

Contents

Overview

Exam information:

- **Date:** Monday, March 9
- **Time:** 5:45pm – 7:15pm
- **Location:** CHEM 140

Topics covered:

- §6.2: Solids of Revolution
- §6.3: Volumes by Shells
- §6.4: Work
- §8.3: Center of Mass
- §10.1: Parametric Equations
- §10.2: Tangents to Parametric Curves
- §8.1 & 10.2: Arc Length
- §10.3: Introduction to Polar Coordinates
- §10.4: Calculus in Polar Coordinates
- §11.1: Sequences
- §11.2: Series, partial sums, test for divergence

Not on the exam: Telescoping series, geometric series.

Suggested things to study:

- Fall 2022 practice exam on Canvas; Spring 2023 practice exam on Canvas
- Quizzes (Quiz 5, Quiz 6, Quiz 7, Quiz 8)
- WebAssign
- Written homeworks
- The problems in this study guide

A good way to prepare is to practice actively: try problems without notes first, check your work, and then redo similar problems until each method feels routine. Focus on choosing the right method, writing clean setups, and finishing with a correct final answer.

VOLUMES OF REVOLUTION

Disk/Washer (slices *perpendicular* to axis):

$$V = \pi \int \left(R^2 - r^2\right) (\text{thickness})$$

- **Horizontal axis $y = c$.** $R(x)$ is the distance of the outer curve to $y = c$ and $r(x)$ is the distance of the inner curve to $y = c$.
- **Vertical axis $x = c$.** $R(y)$ is the distance of the outer curve to $x = c$ and $r(y)$ is the distance of the inner curve to $x = c$.
- *Disk method:* Special case where inner radius $r = 0$, and there is no gap to axis.

Shells (slices *parallel* to axis):

$$V = 2\pi \int (\text{radius})(\text{height}) (\text{thickness})$$

- **Horizontal axis $y = c$:** The shell radius is the distance from y to $y = c$, the height is (right curve) − (left curve), and the thickness is dy.
- **Vertical axis $x = c$:** The shell radius is the distance from x to $x = c$, the height is (top curve)−(bottom curve), and the thickness is dx.

WORK

Variable force along a line:

$$W = \int_a^b F(x)\, dx$$

Springs (Hooke's Law): $F(x) = kx, \quad k = \frac{F_0}{x_0}$

$$W = \int_a^b kx\, dx = \frac{k}{2}\left(b^2 - a^2\right)$$

Cables/Ropes/Chains (uniform linear density λ):
A slice of thickness dy has mass $\lambda\, dy$ and weight $\lambda g\, dy$.

$$W = \int (\text{weight of slice})(\text{distance lifted})$$

Pumping liquids (Tanks):
Note: You must integrate over the **water** (from the bottom of the liquid to the top of the liquid), not the entire height of the tank!

$$W = \int_{y_{\text{bottom}}}^{y_{\text{top}}} \rho g\, A(y)\, D(y)\, dy$$

Here, ρ = density, g = gravity, $A(y)$ is the cross-sectional area of a thin slice, and $D(y)$ is the distance the slice is lifted to the top.

CENTER OF MASS

Point masses:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \qquad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

Region under $y = f(x)$ above the x-axis on $[a, b]$:

$$A = \int_a^b f(x)\, dx,$$
$$M_y = \int_a^b x\, f(x)\, dx, \qquad M_x = \frac{1}{2}\int_a^b (f(x))^2\, dx,$$
$$\bar{x} = \frac{M_y}{A}, \qquad \bar{y} = \frac{M_x}{A}.$$

Symmetry: If the region is symmetric about the y-axis, then $\bar{x} = 0$.

PARAMETRIC CURVES

Given $x = f(t),\ y = g(t),\ t \in [\alpha, \beta]$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \left(\frac{dx}{dt} \neq 0\right)$$

At $t = t_0$, the point is $(x_0, y_0) = \big(f(t_0), g(t_0)\big)$ and the slope is $m = \left.\dfrac{dy}{dx}\right|_{t_0}$, so the tangent line is $y - y_0 = m(x - x_0)$.

- **Horizontal tangent:** $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
- **Vertical tangent:** $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

ARC LENGTH

If $y = f(x)$ on $[a, b]$:

$$L = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx$$

If $x = g(y)$ on $[c, d]$:

$$L = \int_c^d \sqrt{1 + \big(g'(y)\big)^2}\, dy$$

If $x = f(t),\ y = g(t)$ on $[\alpha, \beta]$:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

POLAR COORDINATES

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = \sqrt{x^2+y^2}$$

Equivalent coordinate representations:

$$(r,\theta) = (r, \theta + 2\pi k) = (-r, \theta + \pi + 2\pi k)$$

Symmetry tests for $r = f(\theta)$:

$$f(-\theta) = f(\theta) \implies \text{symmetry about polar axis}$$
$$f(\pi-\theta) = f(\theta) \implies \text{symmetry about } \theta = \pi/2$$

Area:

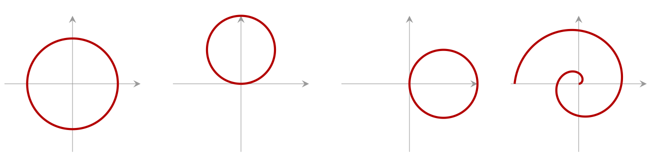
$$A = \frac{1}{2}\int_a^b \big(r(\theta)\big)^2\, d\theta$$
$$A = \frac{1}{2}\int_a^b \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta$$

Slope for $r = f(\theta)$:

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

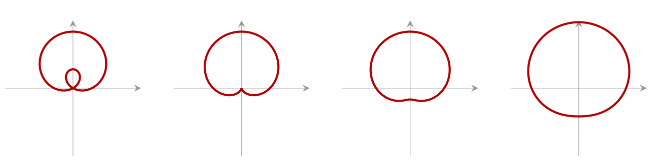
Common Polar Curves:

- **Circles and Spirals:** $r = a$, $r = a\sin\theta$, $r = a\cos\theta$; $r = a\theta$.



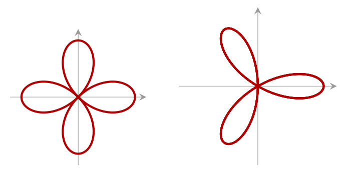
$r = a$ $\quad r = a\sin\theta$ $\quad r = a\cos\theta$ $\quad r = a\theta$

- **Limaçons:** $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$ $(a > 0, b > 0)$.
  - $a < b$: inner loop
  - $a = b$: cardioid
  - $a > b$: dimpled
  - $a \ge 2b$: convex



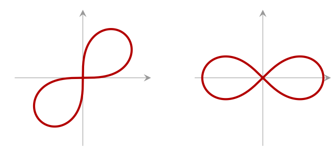
$a < b$ $\quad a = b$ $\quad a > b$ $\quad a \ge 2b$

- **Roses:** $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$.
  - n odd $\implies n$-leaved
  - n even $\implies 2n$-leaved



$r = a\cos(2\theta)$ $\quad r = a\cos(3\theta)$

- **Lemniscates:** $r^2 = a^2\sin(2\theta)$ or $r^2 = a^2\cos(2\theta)$.



$r^2 = a^2\sin(2\theta)$ $\quad r^2 = a^2\cos(2\theta)$

SEQUENCES

A sequence $\{a_n\}$ **converges** to L if $\lim\limits_{n\to\infty} a_n = L$ exists.

Limit laws (if limits exist):

$$\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n$$
$$\lim(ca_n) = c\lim a_n$$
$$\lim(a_n b_n) = (\lim a_n)(\lim b_n)$$
$$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} \quad (\lim b_n \neq 0)$$

Using a related function: If $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$. If $f(x)$ yields $\frac{\infty}{\infty}$ or $\frac{0}{0}$, L'Hôpital's Rule may be used.

Squeeze Theorem: $a_n \le b_n \le c_n$ and $\lim a_n = \lim c_n = L \implies \lim b_n = L$.

Monotone Bounded Theorem: Bounded above and increasing or bounded below and decreasing $\implies$ convergent.

SERIES AND PARTIAL SUMS

A **series** is $\sum_{n=1}^{\infty} a_n$, and its **partial sums** are $S_n = \sum_{k=1}^{n} a_k$.

Convergence. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} S_n = S$ exists; in that case, $\sum_{n=1}^{\infty} a_n = S$.

If you are given S_n **explicitly:**

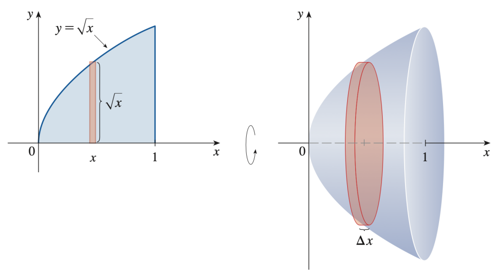
$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

Divergence Test. If $\lim\limits_{n\to\infty} a_n \neq 0$ (or does not exist), then $\sum_{n=1}^{\infty} a_n$ diverges.

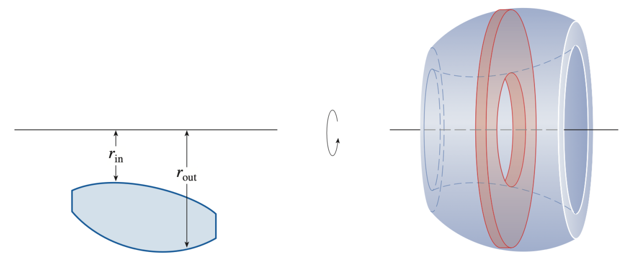
6.2-6.3 Solids of Revolution

There are three primary slicing methods used to compute the volume of a solid of revolution:

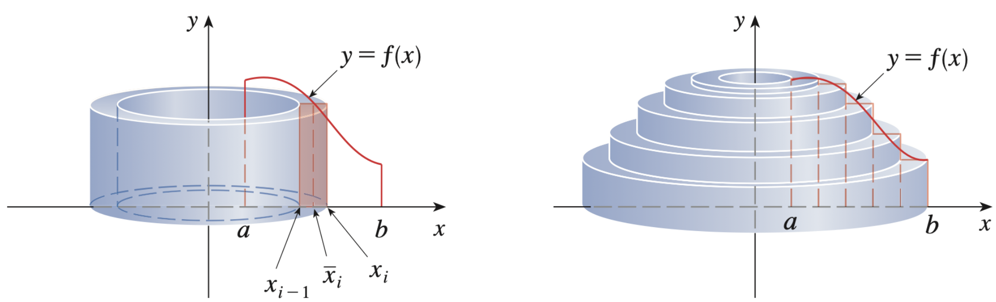
Disk Method: Slices are drawn perpendicular to the axis of rotation. If there is no gap between the region and the axis, each slice is disk.



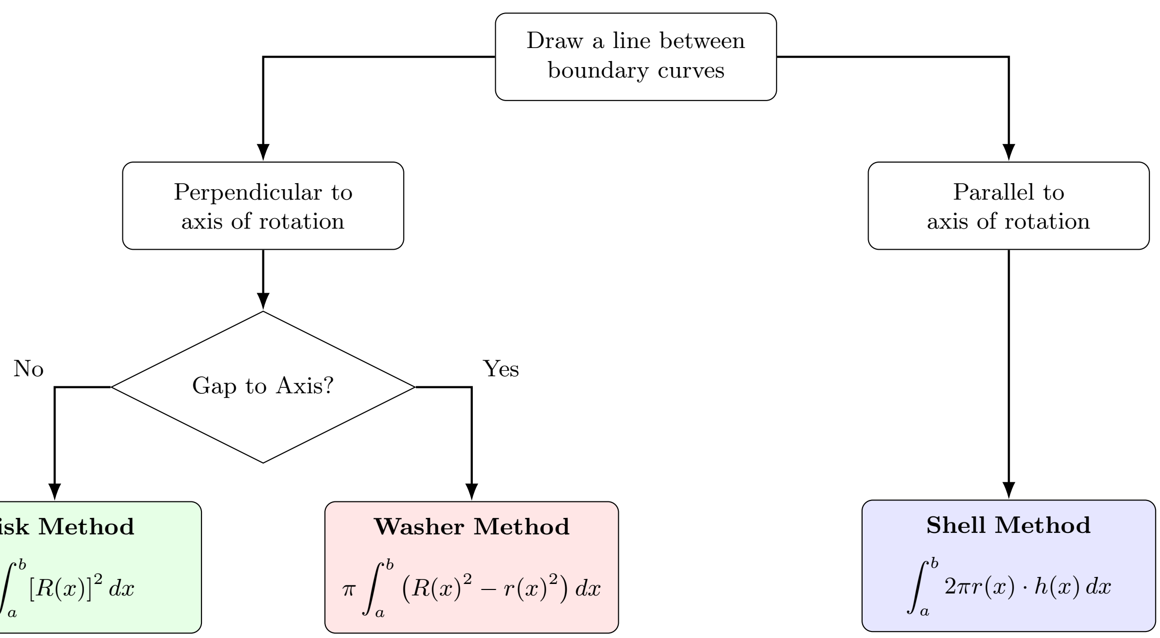
Washer Method: Slices are drawn perpendicular to the axis of rotation. If there is a gap between the region and the axis, each slice is washer.



Shell Method: Slices are drawn parallel to the axis of rotation. Each slice forms a cylindrical shell, and its volume is determined by its radius, height, and thickness.



This flowchart helps you choose the correct setup for a volume of revolution problem.



Volume Problems

Disk Method

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x-axis.

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x^2, \quad y = 0$$

about the x-axis.

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 0, \quad y = 3$$

about the y-axis.

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2x, \quad x = 0, \quad y = 4$$

about the y-axis.

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x-3}, \quad y = 0, \quad x = 7$$

about the vertical line $x = 7$.

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x+1}, \quad y = 0, \quad x = 3$$

about the vertical line $x = -1$.

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - \sqrt{x}, \quad y = 2, \quad x = 1$$

about the horizontal line $y = 2$.

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = -1 + \sqrt{x}, \quad y = -1, \quad x = 1$$

about the horizontal line $y = -1$.

Washer Method

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = x^2$$

about the x-axis.

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = x, \quad x = 0$$

about the x-axis.

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y, \quad x = y^2$$

about the y-axis.

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = 4 - x, \quad \text{and} \quad y = 0$$

about the y-axis.

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 5 - y^2, \quad y = -1, \quad \text{and} \quad y = 1$$

about the vertical line $x = 7$.

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x - 1, \quad y = 4 - x, \quad y = 0, \quad \text{and} \quad y = 1$$

about the vertical line $x = -1$.

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the horizontal line $y = 2$.

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -1$.

Cylindrical Shell Method

1. Find the volume of the solid obtained by rotating the region bounded by
$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$
about the x-axis.

2. Find the volume of the solid obtained by rotating the region bounded by
$$y = 2 - x, \quad y = 0, \quad x = 0$$
about the x-axis.

3. Find the volume of the solid obtained by rotating the region bounded by
$$y = x^2, \quad y = 4, \quad x = 0$$
about the y-axis.

4. Find the volume of the solid obtained by rotating the region bounded by
$$y = \sqrt{x}, \quad y = 0, \quad x = 9$$
about the y-axis.

5. Find the volume of the solid obtained by rotating the region bounded by
$$y = \sqrt{x}, \quad y = 0, \quad x = 1, \quad x = 4$$
about the vertical line $x = -1$.

6. Find the volume of the solid obtained by rotating the region bounded by
$$y = x + 2, \quad y = 0, \quad x = -1, \quad x = 3$$
about the vertical line $x = 5$.

7. Find the volume of the solid obtained by rotating the region bounded by
$$x = y^2, \quad x = 4, \quad y = 0$$
about the horizontal line $y = 3$.

8. Find the volume of the solid obtained by rotating the region bounded by
$$y = \sqrt{x}, \quad y = \frac{x}{2}$$
about the horizontal line $y = -2$.

Additional Problems

1. Find the volume of the solid obtained by rotating the region bounded by
$$x = y^2, \quad x = 4$$
about the y-axis.

2. Find the volume of the solid obtained by rotating the region bounded by
$$y = x^2, \quad y = 2 - x$$
about the vertical line $x = -2$.

3. Find the volume of the solid obtained by rotating the region bounded by
$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$
about the line $y = 1$.

4. Find the volume of the solid obtained by rotating the region bounded by
$$y = x^2, \quad y = \sqrt{x}$$
about the y-axis.

5. Find the volume of the solid obtained by rotating the region bounded by
$$x = 1 - y^2, \quad x = 0$$
about the y-axis.

6. Find the volume of the solid obtained by rotating the region bounded by
$$y = x^2, \quad y = 2 - x^2$$
about the line $y = -1$.

7. Find the volume of the solid obtained by rotating the region bounded by
$$y = \ln x, \quad y = 0, \quad x = 1, \quad x = e^2$$
about the y-axis.

8. Find the volume of the solid obtained by rotating the region bounded by
$$y = \frac{4}{x}, \quad y = 0, \quad x = 1, \quad x = 4$$
about the line $x = -2$.

6.4 Work

Overview

Definition (Force and Newton's Second Law)**.** If an object of mass m moves along a straight line with position function $s(t)$, then its *acceleration* is

$$a(t) = \frac{d^2 s}{dt^2}.$$

By Newton's Second Law of Motion, the *force* F on the object is

$$F = m\,a = m\,\frac{d^2 s}{dt^2}.$$

Definition (Work for Constant Force)**.** If a constant force F acts on an object and moves it through a distance d in the direction of the force, then the *work* W done by the force is

$$W = F \cdot d \quad (\text{force} \times \text{distance}).$$

Definition (Units of Work)**.**

- In the **SI (metric) system**, force is measured in *newtons* (N) and distance in *meters* (m). Hence work is measured in *newton-meters*, also called *joules* (J).

$$1\,\text{joule} = 1\,\text{newton} \cdot 1\,\text{meter}.$$

- In the **US Customary system**, force is measured in *pounds* (lb) and distance in *feet* (ft). Hence work is measured in *foot-pounds* (ft-lb).

$$1\,\text{ft-lb} \approx 1.36\,\text{joules}.$$

Definition (Weight vs. Mass)**.**

- An object's *mass* (in kilograms, kg) times the acceleration due to gravity ($\approx 9.8\,\text{m/s}^2$) gives its *weight* (in newtons).

$$\text{weight} = m \times 9.8 \text{ (in newtons)}.$$

- In the US system, the word "pound" itself denotes a force (weight). Hence an object that weighs W pounds has mass $m = W/g$ (in "slugs") if $g \approx 32\,\text{ft/s}^2$.

Theorem (Work for a Variable Force)**.** Let an object move along the x-axis from $x = a$ to $x = b$, acted upon by a *continuous* force $f(x)$ that depends on the position x. The *work* W done by this force is given by the definite integral

$$W = \int_a^b f(x)\,dx.$$

Interpretation: We partition the interval $[a, b]$ into small segments, approximate the (nearly constant) force on each segment, multiply by the small distance, and then let the partition become finer. The limit of these Riemann sums is the above integral.

Definition (Hooke's Law for Springs)**.** If a spring is stretched (or compressed) x units from its natural length (where x is not too large), the *force* F required to hold it there obeys

$$F(x) = k\,x,$$

where k is a positive constant called the *spring constant.* Consequently, the work required to stretch a spring from $x = a$ to $x = b$ (beyond its natural length) is

$$W = \int_a^b kx\,dx = \frac{k}{2}\left[b^2 - a^2\right].$$

Definition (Work for Pumping Fluids (Tanks))**.** Let y measure height (choose $y = 0$ at the bottom or at the top, but be consistent). A horizontal slice of fluid at height y with cross-sectional area $A(y)$ and thickness dy has volume $A(y)\,dy$ and weight $\rho g\,A(y)\,dy$. If that slice must be lifted a distance $d(y)$ to reach the outlet, then

$$dW = \rho g\,A(y)\,d(y)\,dy, \qquad W = \int_{y=a}^{y=b} \rho g\,A(y)\,d(y)\,dy,$$

where $[a, b]$ is the interval of heights containing the fluid.

Work Problems

Spring Problems

1. A spring has a natural length of 20 m. A force of 12 N is required to stretch the spring to 25 m. Determine the work required to stretch the spring from 20 m to 30 m.

2. A spring has a natural length of 15 m. A force of 10 N is required to stretch the spring to 18 m. Determine the work required to stretch the spring from 16 m to 22 m.

3. A spring has a natural length of 30 m. A force of 8 N is required to stretch the spring to 35 m. Determine the work required to compress the spring from 30 m to 20 m.

Cable Problems

1. A 100-meter-long cable with a linear density of 5 kg/m is hanging from a winch at the top of a well. The cable is initially fully extended into the well and is lifted to the top. Compute the work required to lift the entire cable.

2. A 50-meter-long chain with a linear density of 8 kg/m is hanging from a pulley at the top of a mine shaft. The chain is initially fully extended into the shaft and is lifted to the top. Compute the work required to lift the entire chain.

3. A 60-meter-long rope with a linear density of 3 kg/m is hanging over the edge of a cliff, with one end secured at the top and the other end dangling freely. The rope is slowly lifted until it is fully coiled at the top of the cliff. Compute the work required to lift the rope.

4. A 30-meter-long anchor chain with a linear density of 12 kg/m is hanging from the side of a ship, with one end attached to the ship and the other submerged in the water. The chain is hoisted onto the deck of the ship. Compute the work required to lift the entire chain onto the ship.

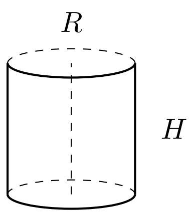
Tank Problems

For each problem, set up (but do not solve) the work integral for pumping water out of the tank. In all setups, use the density of water as 1000 (kg/m^3) and gravity as 9.8 (m/s^2).

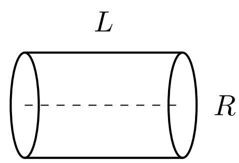
1. **Rectangular Tank with Triangular Ends:** A tank is 6 m long, and its end view is an isosceles triangle with a base of 2 m and a height of 3 m. Water is pumped out through a spout located 0.5 m above the top of the tank.
2. **Cylindrical Tank with a Spout:** A vertical cylindrical tank is 4 m high with a circular cross section of radius 1.5 m. Water is pumped out through a spout that is 0.3 m above the top of the tank.
3. **Inverted Conical Tank:** An inverted conical tank has a height of 3 m and an open top with a radius of 1 m. Water is pumped out to a spout 0.2 m above the top.
4. ***Spherical Tank:** A spherical tank of radius 2 m is completely filled with water. Water is pumped out through a spout located 0.1 m above the top of the sphere.
5. ***Composite Tank – Cylinder with Hemispherical Top:** The tank consists of a cylindrical section 3 m high with a circular cross section of radius 1 m, topped by a hemispherical dome of radius 1 m. Water is pumped out through a spout located 0.15 m above the dome.

More Tanks to Know

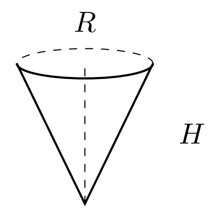
Vertical Cylinder



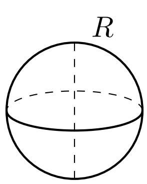
Horizontal Cylinder



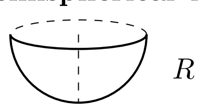
Conical Tank



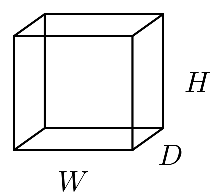
***Spherical Tank**



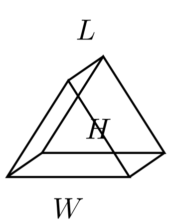
***Hemispherical Tank**



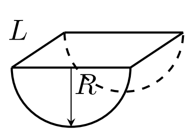
Rectangular Box Tank



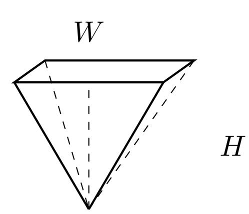
Triangular Prism



***Semi-Cylindrical Trough**



Square Pyramidal Tank



* denotes challenge.

8.3 Center of Mass

Discrete Systems

Theorem (Law of the Lever)**.** If two masses m_1 and m_2 are placed on opposite sides of a fulcrum at distances d_1 and d_2, respectively, then the rod will balance provided that

$$m_1 d_1 = m_2 d_2.$$

In the case where m_1 is located at x_1, m_2 at x_2, and the center of mass is at $\bar{x}$, the balancing condition can be written as

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}),$$

which implies

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Definition (Center of Mass for a System of Particles)**.** The **center of mass** of a system of particles is the point

$$(\bar{x}, \bar{y}).$$

For particles on a line, the center of mass is

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.$$

For particles in the plane with masses $m_1, m_2, \ldots, m_n$ located at

$$(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n),$$

the center of mass is

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}, \qquad \bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n}.$$

Moments

Definition (Moment)**.** The *moment* of a mass m about a point (or an axis) is the product of the mass and its distance from that point (or axis). For a system of particles, the moment about the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i.$$

In the plane, we define:

$$M_y = \sum_{i=1}^{n} m_i x_i \quad \text{(moment about the } y\text{–axis)},$$

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \text{(moment about the } x\text{–axis)}.$$

Regions Bounded by Curves

Region Above the x-Axis: Suppose a lamina has **uniform density** and occupies the region bounded above by $y = f(x)$, below by the x-axis, and from $x = a$ to $x = b$. Then

$$A = \int_a^b f(x)\,dx,$$

$$M_y = \int_a^b x\,f(x)\,dx, \qquad M_x = \int_a^b \frac{[f(x)]^2}{2}\,dx.$$

Hence the center of mass (centroid) is

$$\bar{x} = \frac{1}{A}\int_a^b x\,f(x)\,dx, \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{[f(x)]^2}{2}\,dx.$$

Region Between Two Curves: Suppose a lamina has **uniform density** and occupies the region bounded above by $y = f(x)$, below by $y = g(x)$, and from $x = a$ to $x = b$, where $f(x) \ge g(x)$ on $a \le x \le b$. Then

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx,$$

$$M_y = \int_a^b x\big(f(x) - g(x)\big)\,dx, \qquad M_x = \int_a^b \frac{[f(x)]^2 - [g(x)]^2}{2}\,dx.$$

Hence the center of mass (centroid) is

$$\bar{x} = \frac{1}{A}\int_a^b x\big(f(x) - g(x)\big)\,dx, \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{[f(x)]^2 - [g(x)]^2}{2}\,dx.$$

Symmetry Principle

Theorem (Symmetry Principle)**.** If a region D is symmetric with respect to a line ℓ, then the centroid of D lies on ℓ.

Center of Mass Problems

1. A system consists of three point masses:
   - $m_1 = 2$ kg at $(1, 3)$
   - $m_2 = 3$ kg at $(4, 5)$
   - $m_3 = 4$ kg at $(6, 2)$

   Compute the center of mass of the system.

2. A system consists of four point masses:
   - $m_1 = 1$ kg at $(0, 0)$
   - $m_2 = 2$ kg at $(2, 4)$
   - $m_3 = 3$ kg at $(5, 1)$
   - $m_4 = 4$ kg at $(3, 3)$

   Compute the center of mass of the system.

3. Compute the center of mass of the region bounded by

$$y = x^2, \quad y = 0, \quad x = 1, \quad \text{and} \quad x = 2.$$

4. Compute the center of mass of the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = \pi,$$

5. Use symmetry to find the center of mass of a uniform semicircular lamina of radius R.

6. Use symmetry to determine the center of mass of a uniform triangular lamina with vertices at $(0, 0)$, $(a, 0)$, and (a, a).

10.1 Parametric Curves

Definition (Parametric Equations)**.** A curve in the plane can be described by a pair of parametric equations

$$x = f(t), \quad y = g(t),$$

where t is a parameter (often representing time). The set of all points $\big(f(t), g(t)\big)$ traced out as t varies over an interval $[a, b]$ is called a *parametric curve.*

Eliminating the Parameter: Sometimes we can solve one of the equations (e.g. $x = f(t)$) for t and substitute into the other ($y = g(t)$) to obtain a Cartesian equation $F(x, y) = 0$. This process is called *eliminating the parameter.* However, doing so can lose information about direction and speed.

Remark. Different parametric equations can represent the same geometric curve. However, the direction of traversal and the speed at which the curve is traced depend on how x and y change with respect to the parameter t.

Parametric Curves Problems

1. Sketch and identify the curve.

$$x = t^2 - 4t, \qquad y = t + 2, \qquad 0 \le t \le 5.$$

2. Sketch and identify the curve.

$$x = 2\cos t - 1, \qquad y = 2\sin t + 3, \qquad 0 \le t \le 2\pi.$$

3. Sketch and identify the curve.

$$x = \sin(2t), \qquad y = \cos(2t), \qquad 0 \le t \le 2\pi.$$

4. Sketch and identify the curve.

$$x = t^3, \qquad y = t, \qquad -2 \le t \le 2.$$

5. Sketch and identify the curve.

$$x = \sin t, \qquad y = \sin^2 t, \qquad 0 \le t \le 2\pi.$$

6. Find a parametrization $\mathbf{r}(t) = (x(t), y(t))$ of the line segment from

$$P_0 = (-2, 1) \quad \text{to} \quad P_1 = (4, -3),$$

using $0 \le t \le 1$.

7. Give parametric equations for the circle of radius 3 centered at $(2, -1)$ that
   - starts at the *rightmost* point of the circle, and
   - travels *clockwise* exactly once.

8. Give a parametrization of the upper semicircle

$$x^2 + y^2 = 9, \qquad y \ge 0,$$

traced from $(-3, 0)$ to $(3, 0)$ (left to right). State an interval for t.

10.2 Tangents to Parametric Curves

Suppose $x = f(t)$ and $y = g(t)$ are differentiable functions. If $\frac{dx}{dt} \neq 0$, then the slope of the curve is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

To find the equation of the tangent line at a specific parameter value $t = t_0$:

- Find the point on the curve: $x_0 = f(t_0)$ and $y_0 = g(t_0)$.
- Calculate the slope at that point: $m = \left.\frac{dy}{dx}\right|_{t=t_0}$.
- Use the point-slope form:

$$y - y_0 = m(x - x_0).$$

From the slope formula, we can determine the orientation of the tangents:

- **Horizontal Tangent:** Occurs when $\frac{dy}{dx} = 0$. This happens if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
- **Vertical Tangent:** Occurs when $\frac{dy}{dx}$ is undefined. This happens if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

Tangents to Parametric Curves Problems

1. Given the parametric equations:

$$x = t^2 + 1, \quad y = 2t - 3,$$

find the slope of the curve at $t = 1$.

2. Consider the curve defined by:

$$x = \sin t, \quad y = \cos t, \quad 0 \leq t \leq 2\pi.$$

Find the points where the curve has a horizontal tangent.

3. Find the equation of the tangent line to the curve:

$$x = e^t, \quad y = e^{-t}$$

at $t = 0$.

4. Determine the slope of the tangent line for the parametric curve:

$$x = t - \sin t, \quad y = 1 - \cos t$$

at $t = \frac{\pi}{4}$.

5. Compute the equation of the tangent line to the curve:

$$x = \ln(t), \quad y = t^2$$

at $t = 1$.

6. Given the parametric equations:

$$x = 3t^2 + 2, \quad y = 4t^3 - 5,$$

find the slope of the tangent line at $t = -1$.

7. Find the equation of the tangent line to the parametric curve:

$$x = t^2 + 2t, \quad y = 3t - 1$$

at the point corresponding to $t = 2$.

8.1, 10.2 Arc Length

Theorem (Arc Length Formula for $y = f(x)$)**.** If f is differentiable on $[a, b]$ and f' is continuous, then the arc length L of the curve $y = f(x)$ is given by

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Theorem (Arc Length for $x = g(y)$)**.** If a curve is given by $x = g(y)$ for $y \in [c, d]$ with g differentiable and g' continuous, then the arc length L is

$$L = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Theorem (Arc Length for Parametric Equations)**.** Let a curve C be defined by the parametric equations

$$x = f(t), \quad y = g(t), \quad t \in [\alpha, \beta],$$

where f and g are differentiable and their derivatives are continuous. Then the arc length L of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Arc Length Problems

1. Find the arc length of the curve $f(x) = \ln(\cos x)$ over the interval $0 \le x \le \frac{\pi}{4}$.[1]
2. Find the arc length of the curve $f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$ over the interval $0 \le x \le 2$.
3. Find the arc length of the curve $x = \frac{1}{3}y^{3/2} - y^{1/2}$ over the interval $1 \le y \le 4$.
4. Find the arc length of the curve $x = \frac{2}{3}y^{3/2}$ over the interval $0 \le y \le 4$.
5. Find the arc length of the curve defined by

$$x = t^2, \quad y = t^3, \quad 0 \le t \le 1.$$

6. Find the arc length of the cycloid given by

$$x = t - \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi.$$

[1]These problems are hard to come up with. That is, how can we ensure that $\sqrt{1 + [f'(x)]^2}$ can be integrated? For a discussion of this, see `https://math.colorado.edu/~chda1090/arclengthprobs.pdf`

10.3 Intro to Polar Coordinates

Polar Coordinates and Conversion

Definition (Polar Coordinate System)**.** A point in the plane is represented by *polar coordinates* (r, θ) where:

- r is the distance from the point to a fixed point O (the *pole*).
- θ is the angle (measured in radians) between the polar axis (usually the positive x-axis) and the ray from O to the point.

When $r < 0$, the point (r, θ) is equivalent to $(-r, \theta + \pi)$.

Theorem (Conversion Between Polar and Cartesian Coordinates)**.** If a point has polar coordinates (r, θ) and Cartesian coordinates (x, y), then:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Conversely, if (x, y) are given, then:

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Properties of Polar Coordinates

Theorem (Multiple Representations)**.** A point can have infinitely many polar representations. In particular,

$$(r, \theta) = \Big(r,\, \theta + 2\pi k\Big) \quad \text{and} \quad (r, \theta) = \Big(-r,\, \theta + (2k+1)\pi\Big)$$

for any integer k.

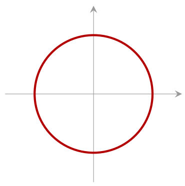
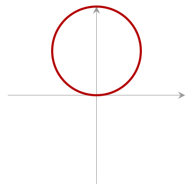
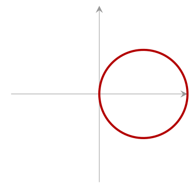
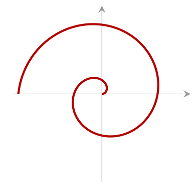
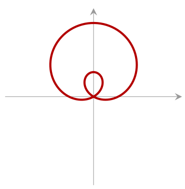
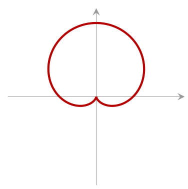
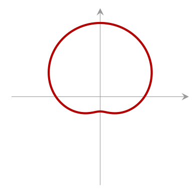
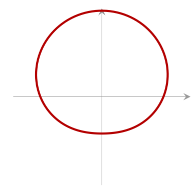
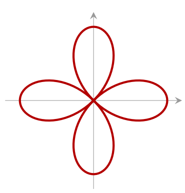
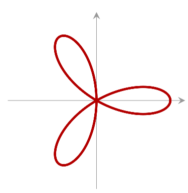
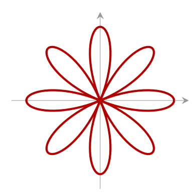
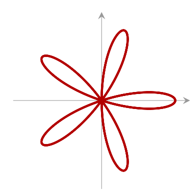
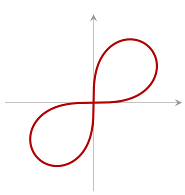
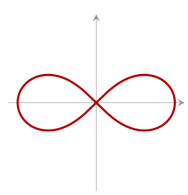
Theorem (Symmetry of Polar Graphs)**.** Let $r = f(\theta)$ be a polar equation. Then:

1. If $f(-\theta) = f(\theta)$ for all θ, the graph is symmetric about the polar axis.
2. If the equation is unchanged when r is replaced by $-r$ (or equivalently when θ is replaced by $\theta + \pi$), then the graph is symmetric about the pole.
3. If $f(\pi - \theta) = f(\theta)$ for all θ, the graph is symmetric about the line $\theta = \frac{\pi}{2}$.

Additional Notes

- A polar curve is defined by an equation of the form $r = f(\theta)$; its graph consists of all points (r, θ) satisfying the equation.
- Many common curves (such as circles, cardioids, limaçons, roses, and lemniscates) have elegant representations in polar coordinates.

Common Polar Graphs

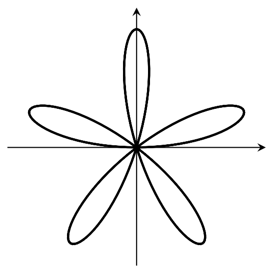
Circles and Spiral	 $r = a$ circle	 $r = a\sin\theta$ circle	 $r = a\cos\theta$ circle	 $r = a\theta$ spiral
Limaçons $r = a \pm b\sin\theta$ $r = a \pm b\cos\theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b	 $a < b$ limaçon with inner loop	 $a = b$ cardioid	 $a > b$ dimpled limaçon	 $a \geq 2b$ convex limaçon
Roses $r = a\sin n\theta$ $r = a\cos n\theta$ n-leaved if n is odd $2n$-leaved if n is even	 $r = a\cos 2\theta$ four-leaved rose	 $r = a\cos 3\theta$ three-leaved rose	 $r = a\cos 4\theta$ eight-leaved rose	 $r = a\cos 5\theta$ five-leaved rose
Lemniscates Figure-eight-shaped curves	 $r^2 = a^2\sin 2\theta$ lemniscate	 $r^2 = a^2\cos 2\theta$ lemniscate		

Polar Coordinates Problems

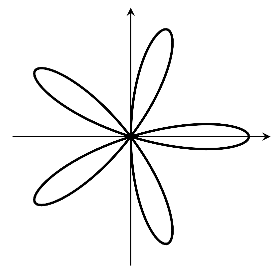
Multiple Choice Matching Problems

For each question, choose the letter (A, B, C, or D) that correctly matches the given description.

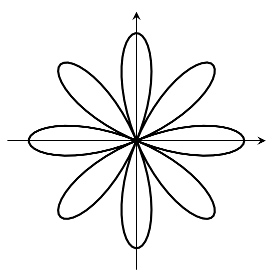
1. Which of the following graphs represents the polar function $r = \sin(5\theta)$?



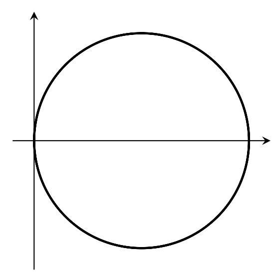
A.



B.

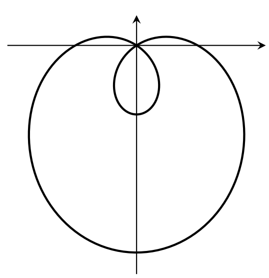


C.

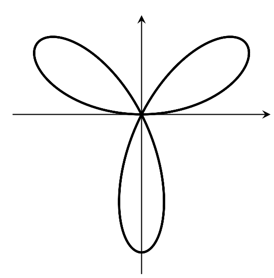


D.

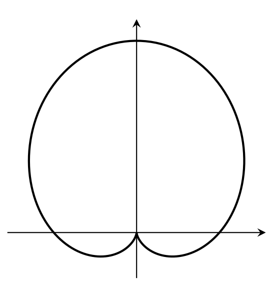
2. Which of the following graphs represents the polar function $r = 1 + \sin\theta$?



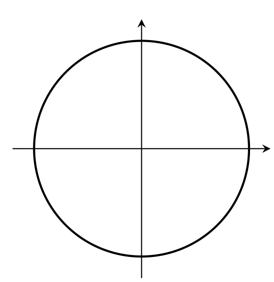
A.



B.

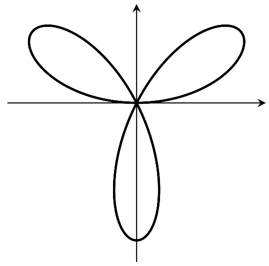


C.

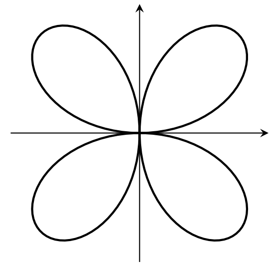


D.

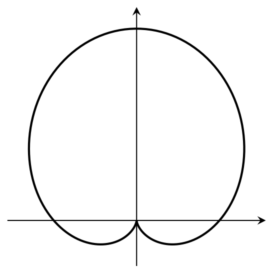
3. Which of the following graphs represents the polar function $r^2 = 2\cos(2\theta)$?



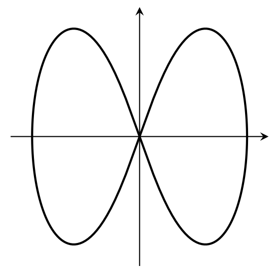
A.



B.

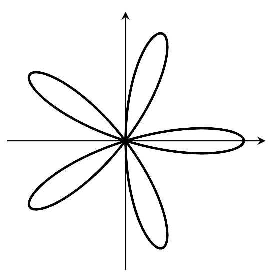


C.

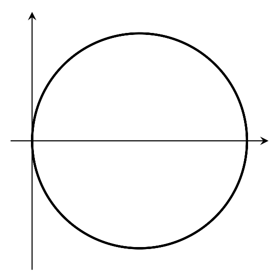


D.

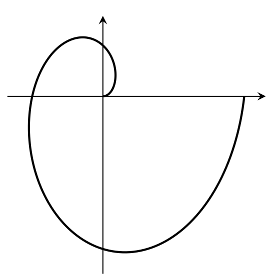
4. Which of the following graphs represents the polar function $r = 2\cos\theta$?



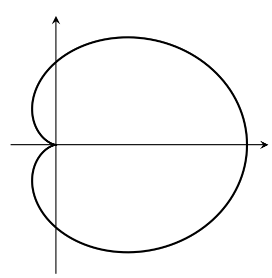
A.



B.

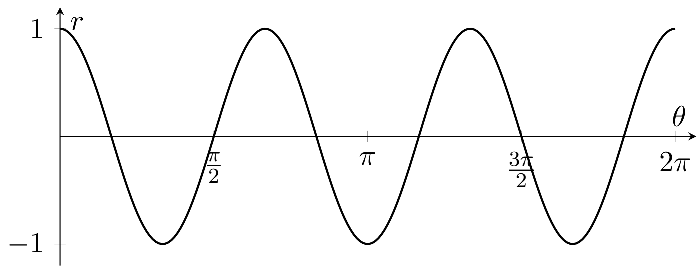


C.

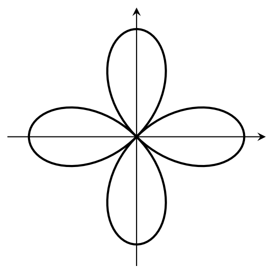


D.

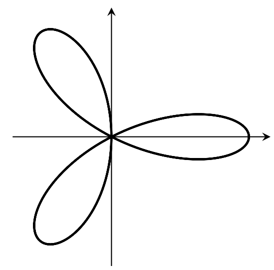
5. The following figure shows a graph of r as a function of θ.



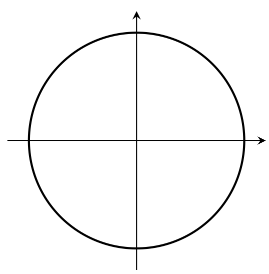
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



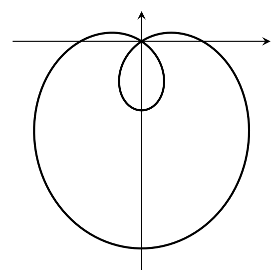
A.



B.

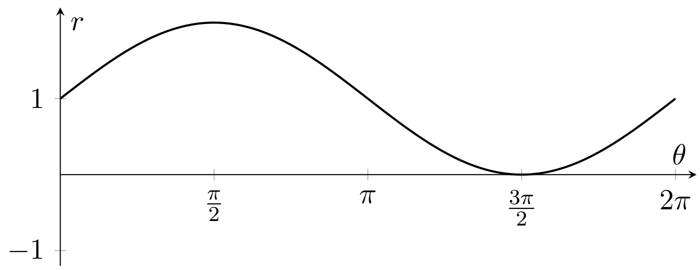


C.

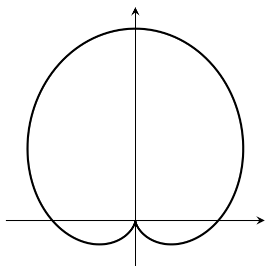


D.

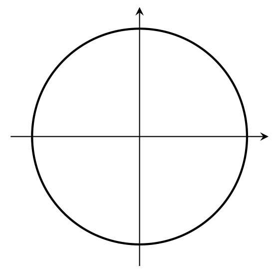
6. The following figure shows a graph of r as a function of θ.



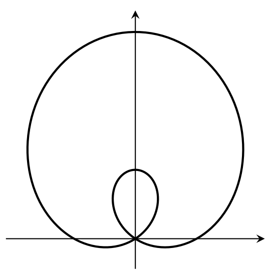
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



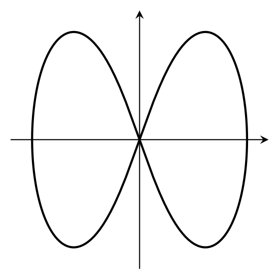
A.



B.

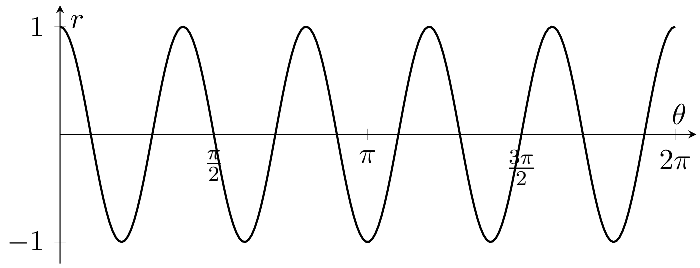


C.

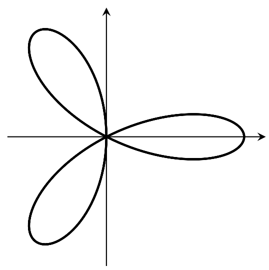


D.

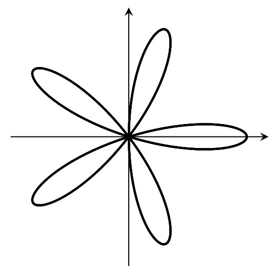
7. The following figure shows a graph of r as a function of θ.



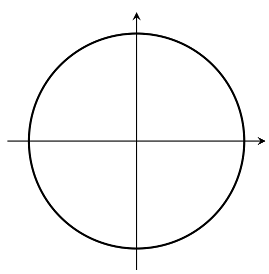
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



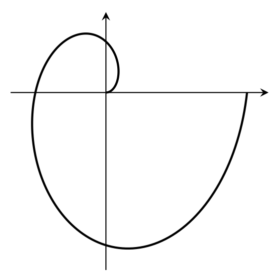
A.



B.

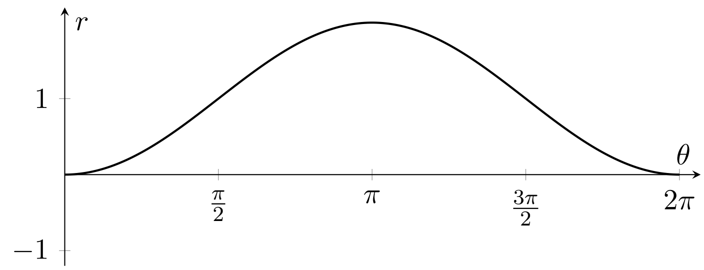


C.

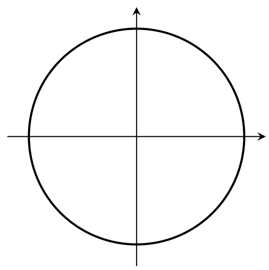


D.

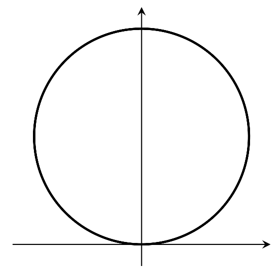
8. The following figure shows a graph of r as a function of θ.



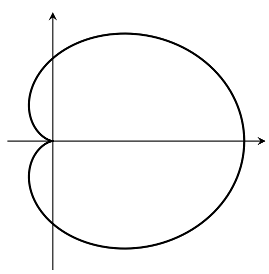
Which of the following corresponds to the curve on the Cartesian xy-plane, using polar coordinates?



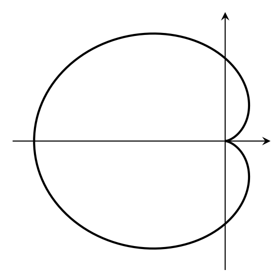
A.



B.



C.



D.

10.4 Calculus in Polar Coordinates

Area in Polar Coordinates

Theorem (Area of a Polar Region)**.** Let $r = f(\theta)$ be a non-negative, continuous function on the interval $\theta \in [a, b]$, with $b - a \leq 2\pi$. Then the area A of the region bounded by the curve $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ is given by

$$A = \frac{1}{2} \int_a^b [f(\theta)]^2 \, d\theta.$$

Theorem. The area of a sector of a circle with radius r and central angle θ (in radians) is given by

$$\text{Area} = \frac{1}{2} r^2 \theta.$$

This result underlies the derivation of the polar area formula.

Tangents and Slopes in Polar Coordinates

Theorem (Slope of the Tangent Line for a Polar Curve)**.** For a polar curve defined by $r = f(\theta)$, the Cartesian coordinates are

$$x = f(\theta) \cos\theta, \quad y = f(\theta) \sin\theta.$$

Differentiating with respect to θ gives

$$\frac{dx}{d\theta} = f'(\theta) \cos\theta - f(\theta) \sin\theta, \quad \frac{dy}{d\theta} = f'(\theta) \sin\theta + f(\theta) \cos\theta.$$

Thus, the slope of the tangent line is

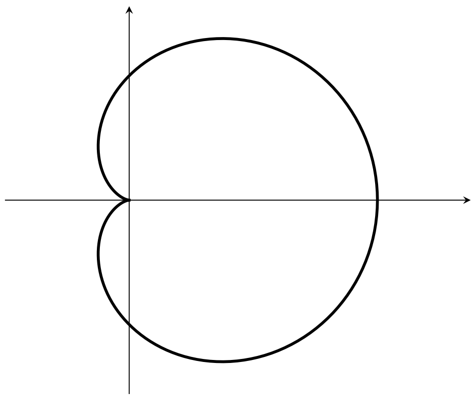
$$\frac{dy}{dx} = \frac{f'(\theta) \sin\theta + f(\theta) \cos\theta}{f'(\theta) \cos\theta - f(\theta) \sin\theta}.$$

Polar Coordinates Problems

Areas Between Polar Curves

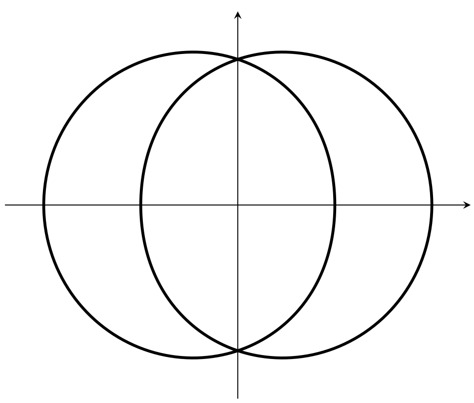
1. Set up an integral to find the area enclosed by the cardioid:

$$r = 2(1 + \cos\theta).$$



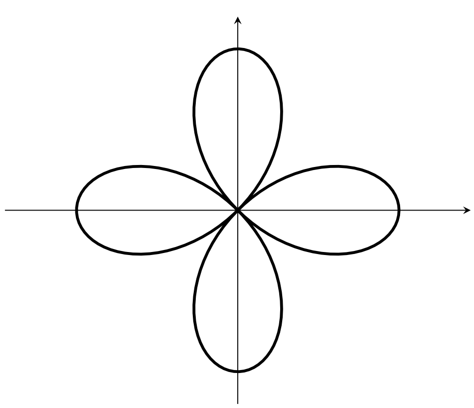
2. Find the area common to both polar curves:

$$r = 3 + \cos\theta, \quad r = 3 - \cos\theta.$$



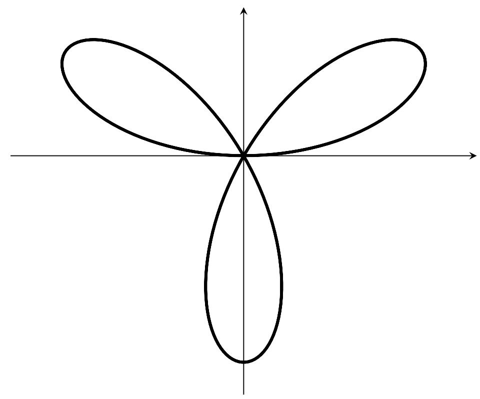
3. Find the area enclosed by the four-leaved rose:

$$r = 3\cos(2\theta).$$

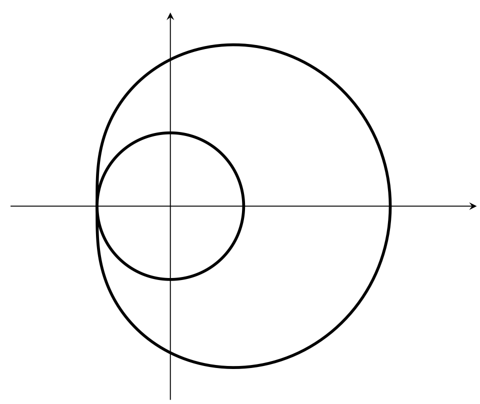


4. Compute the area inside one petal of the rose curve:

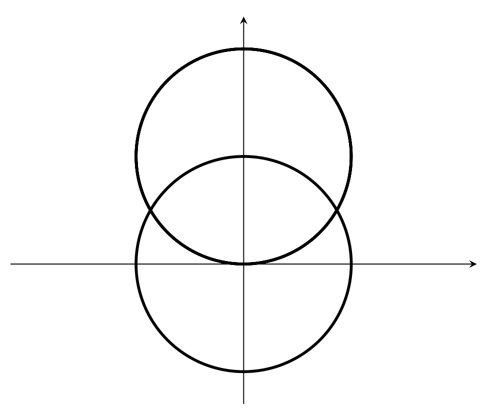
$$r = 2\sin(3\theta).$$



5. Find the area inside $r = 2 + \cos\theta$ and outside $r = 1$.



6. Find the area inside $r = 6\sin\theta$ and outside $r = 3$.



Tangent Lines and Intersection Points

1. Find all points of intersection of the curves:

$$r = 1 + \sin\theta, \quad r = 1 - \cos\theta.$$

2. Find all points of intersection of the curves:

$$r = 2\cos 2\theta, \quad r = 1.$$

3. Find the slope of the tangent line to the curve:

$$r = 1 + 2\sin\theta, \quad \theta = \frac{\pi}{6}.$$

4. Find points where the tangent line is horizontal or vertical for:

$$r = 2(1 - \cos\theta).$$

11.1 Sequences

Overview

Basic Definitions

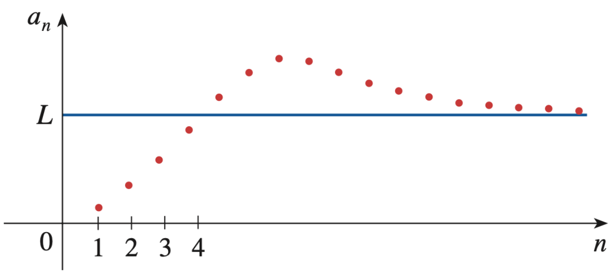
Definition. An *infinite sequence* is a function $a : \mathbb{N} \to \mathbb{R}$, usually written as $a_1, a_2, a_3, \ldots$ or equivalently as $\{a_n\}_{n=1}^{\infty}$.

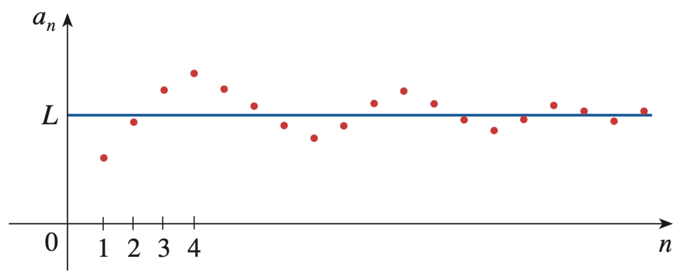
Definition. A sequence $\{a_n\}$ has the **limit** L, and we write:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty,$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, the sequence is said to converge (or be convergent). Otherwise, the sequence diverges (or is divergent).





Limit Laws for Sequences

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences and let c be a constant. Then:

1. **Sum Law:**
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$
2. **Difference Law:**
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$
3. **Constant Multiple Law:**
$$\lim_{n\to\infty} (c\, a_n) = c \lim_{n\to\infty} a_n.$$
4. **Product Law:**
$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$
5. **Quotient Law:** If $\lim_{n\to\infty} b_n \neq 0$, then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$
6. **Power Law:** For any real number $p > 0$ (and with $a_n > 0$),
$$\lim_{n\to\infty} (a_n^p) = \left(\lim_{n\to\infty} a_n\right)^p.$$

Convergence Theorems

Theorem. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$

Theorem (Squeeze Theorem for Sequences)**.** If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences satisfying $a_n \leq b_n \leq c_n$ for all $n \geq N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem (Absolute Value Theorem)**.** If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem (Limits and Continuity)**.** If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Theorem (Geometric Sequences)**.** Consider the sequence $\{r^n\}$.

- If $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$.
- If $r = 1$, then $\lim_{n\to\infty} r^n = 1$.
- If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ diverges (in the case $r \leq -1$, the terms oscillate without approaching a single value).

Theorem (Monotonic Sequence Theorem)**.** Every bounded, monotonic sequence converges.
A sequence is said to be:

- **Increasing** if $a_{n+1} \geq a_n$ for all n.
- **Decreasing** if $a_{n+1} \leq a_n$ for all n.
- **Monotonic** if it is either increasing or decreasing.

A sequence $\{a_n\}$ is *bounded above* if there exists a number M such that $a_n \leq M$ for all n, and *bounded below* if there exists m such that $a_n \geq m$ for all n. If both conditions hold, the sequence is called *bounded*.

Sequences Problems

1. Determine whether the sequence $a_n = \dfrac{n}{n+1}$ converges or diverges. If it converges, find its limit.

2. Determine whether the sequence $b_n = (-1)^n$ converges or diverges. If it converges, find its limit.

3. Determine whether the sequence $c_n = \dfrac{1}{n^2}$ converges or diverges. If it converges, find its limit.

4. Determine whether the sequence $a_n = \dfrac{n^2}{n^2+1}$. converges or diverges. If it converges, find its limit.

5. Determine whether the sequence $b_n = \dfrac{\ln n}{n}$ converges or diverges. If it converges, find its limit.

6. Determine whether the sequence $c_n = \dfrac{n}{\sqrt{n^2+1}}$ converges or diverges. If it converges, find its limit.

7. Determine whether the sequence $d_n = \dfrac{(-1)^n}{n}$ converges or diverges. If it converges, find its limit.

8. Determine whether the sequence $a_n = \dfrac{\cos(3n+1)}{n^2}$ converges or diverges. If it converges, find its limit.

9. Determine whether the sequence $c_n = \dfrac{n^2-3n}{n^3+5}$ converges or diverges. If it converges, find its limit.

10. Determine whether the sequence $a_n = \dfrac{\sqrt{9n^5+4n^2}}{n^3}$ converges or diverges. If it converges, find its limit.

11. Determine whether the sequence $b_n = \dfrac{5n+\sin(n)}{n+10}$. converges or diverges. If it converges, find its limit.

12. Determine whether the sequence $c_n = \dfrac{n^3+2}{\sqrt{n^6+5n^2}}$ converge or diverge? If it converges, find its limit.

13. Determine whether the sequence $a_n = \dfrac{\sin(5n)}{n^3}$ converges or diverges. If it converges, find its limit.

14. Determine whether the sequence $b_n = \dfrac{3\sqrt{n}+n^3}{n^3+\sqrt{n}}$ converges or diverges. If it converges, find its limit.

15. Determine whether the sequence $c_n = \dfrac{n^3-2n}{\sqrt{4n^6+7n}}$ converges or diverges. If it converges, find its limit.

11.2 Series & Partial Sums

- A **sequence** is a list of real numbers $\{a_n\}_{n=1}^{\infty}$. The associated **series** is the infinite sum

$$\sum_{n=1}^{\infty} a_n \;=\; a_1 + a_2 + a_3 + \cdots .$$

- For each $n \geq 1$, the n**th partial sum** is

$$S_n \;=\; a_1 + a_2 + \cdots + a_n \;=\; \sum_{k=1}^{n} a_k.$$

  The sequence $\{S_n\}$ is called the **sequence of partial sums**.
- The series $\sum_{n=1}^{\infty} a_n$ **converges** if the sequence of partial sums converges to a finite limit:

$$\lim_{n\to\infty} S_n \;=\; S \quad \text{(a real number)}.$$

  In that case, S is the **sum** of the series, and we write

$$\sum_{n=1}^{\infty} a_n \;=\; S.$$

  If $\lim_{n\to\infty} S_n$ does not exist or is not finite, then the series **diverges**.
- **Finding a sum from partial sums:** If you are given a formula for S_n, then

$$\sum_{n=1}^{\infty} a_n \;=\; \lim_{n\to\infty} S_n,$$

  provided the limit exists.

Each problem gives the partial sum $S_n = \sum_{k=1}^{n} a_k$. Find the value of the infinite series $\sum_{n=1}^{\infty} a_n$ by computing $\lim_{n\to\infty} S_n$.

1. $S_n = 3 - \dfrac{1}{n}$
2. $S_n = \dfrac{5n}{n+1}$
3. $S_n = \dfrac{2n^2+1}{n^2+2n+1}$
4. $S_n = \dfrac{4n^2}{n^2+1}$
5. $S_n = 1 - \dfrac{2}{n^2+1}$
6. $S_n = \dfrac{6n^2+3n+2}{2n^2+5n+1}$
7. $S_n = \dfrac{5n}{\ln(n+1)+2n}$
8. $S_n = \dfrac{n+4}{\sqrt{n^2+9}}$
9. $S_n = \dfrac{4n+\ln(n)}{3n+1}$
10. $S_n = \dfrac{n^2+1}{n\ln n+1}$