Midterm 2 Study Guide (Solutions)

MATH2300 - Calculus II

Spring 2026

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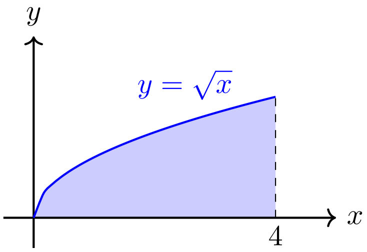
6.2-6.3 Solids of Revolution (Solutions)

6.2-6.3 Disk Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x-axis.

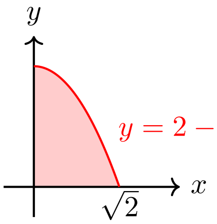


$$\begin{aligned} V &= \pi \int_0^4 \left(\sqrt{x}\right)^2 dx \\ &= \pi \int_0^4 x\, dx \\ &= \pi \left[\frac{x^2}{2}\right]_0^4 \\ &= \pi \left(\frac{16}{2}\right) \\ &= 8\pi. \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x^2, \quad y = 0, \quad x \in \left[0, \sqrt{2}\right]$$

about the x-axis.

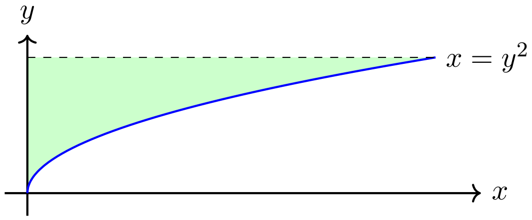


$$\begin{aligned} V &= \pi \int_0^{\sqrt{2}} \left(2 - x^2\right)^2 dx \\ &= \pi \int_0^{\sqrt{2}} \left(4 - 4x^2 + x^4\right) dx \\ &= \pi \left[4x - \frac{4x^3}{3} + \frac{x^5}{5}\right]_0^{\sqrt{2}} \\ &= \pi \left(4\sqrt{2} - \frac{4(\sqrt{2})^3}{3} + \frac{(\sqrt{2})^5}{5}\right) \\ &= \pi\sqrt{2} \left(4 - \frac{8}{3} + \frac{4}{5}\right) \\ &= \pi\sqrt{2} \left(\frac{60 - 40 + 12}{15}\right) \\ &= \pi\sqrt{2} \cdot \frac{32}{15} \\ &= \frac{32\sqrt{2}\pi}{15}. \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 0, \quad y = 3$$

about the y-axis.

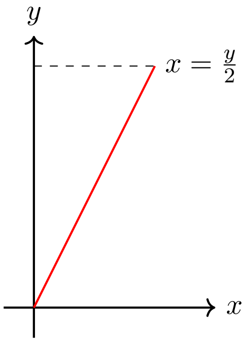


$$\begin{aligned} V &= \pi \int_0^3 \left(y^2\right)^2 dy \\ &= \pi \int_0^3 y^4 dy \\ &= \pi \left[\frac{y^5}{5}\right]_0^3 \\ &= \pi \cdot \frac{3^5}{5} \\ &= \frac{243\pi}{5}. \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$x = \frac{y}{2}, \quad x = 0, \quad y = 4$$

about the y-axis.

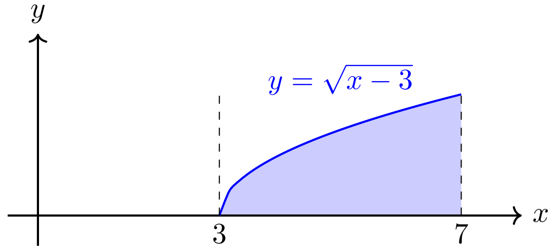


$$\begin{aligned} V &= \pi \int_0^4 \left(\frac{y}{2}\right)^2 dy \\ &= \pi \int_0^4 \frac{y^2}{4} dy \\ &= \frac{\pi}{4} \int_0^4 y^2 dy \\ &= \frac{\pi}{4} \left[\frac{y^3}{3}\right]_0^4 \\ &= \frac{\pi}{4} \cdot \frac{64}{3} \\ &= \frac{16\pi}{3}. \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x-3}, \quad y = 0, \quad x = 7$$

about the vertical line $x = 7$.



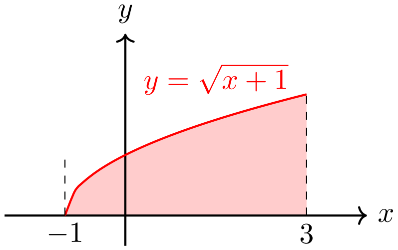
First, rewrite $y = \sqrt{x-3}$ as $x = y^2 + 3$. When $x = 7$, $y = \sqrt{7-3} = 2$, so using horizontal slices:

$$\begin{aligned} V &= \pi \int_0^2 \left[\left(7 - (y^2+3)\right)^2 \right] dy \\ &= \pi \int_0^2 \left(4 - y^2\right)^2 dy \\ &= \pi \int_0^2 16 - 8y^2 + y^4 dy \\ &= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\ &= \pi \cdot \left[32 - \frac{64}{3} + \frac{32}{5} \right] \\ &= \frac{256\pi}{15}. \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x+1}, \quad y = 0, \quad x = 3$$

about the vertical line $x = 3$.



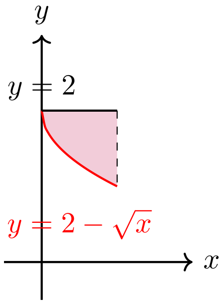
Rewrite $y = \sqrt{x+1}$ as $x = y^2 - 1$. When $x = 3$, $y = \sqrt{3+1} = 2$. Then,

$$\begin{aligned} V &= \pi \int_0^2 \left[\left(3 - (y^2-1)\right)^2 \right] dy \\ &= \pi \int_0^2 \left(4 - y^2\right)^2 dy \\ &= \pi \int_0^2 16 - 8y^2 + y^4 dy \\ &= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\ &= \pi \cdot \left[32 - \frac{64}{3} + \frac{32}{5} \right] \\ &= \frac{256\pi}{15}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - \sqrt{x}, \quad y = 2, \quad x = 1$$

about the horizontal line $y = 2$.

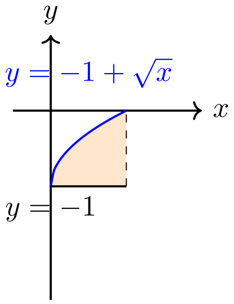


$$\begin{aligned}
V &= \pi \int_0^1 \left[2 - \left(2 - \sqrt{x}\right)\right]^2 dx \\
&= \pi \int_0^1 \left(\sqrt{x}\right)^2 dx \\
&= \pi \int_0^1 x\, dx \\
&= \pi \left[\frac{x^2}{2}\right]_0^1 \\
&= \frac{\pi}{2}
\end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = -1 + \sqrt{x}, \quad y = -1, \quad x = 1$$

about the horizontal line $y = -1$.



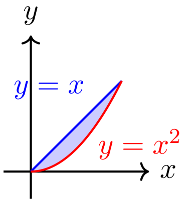
$$\begin{aligned}
V &= \pi \int_0^1 \left[(-1 + \sqrt{x}) - (-1)\right]^2 dx \\
&= \pi \int_0^1 \left(\sqrt{x}\right)^2 dx \\
&= \pi \int_0^1 x\, dx \\
&= \pi \left[\frac{x^2}{2}\right]_0^1 \\
&= \frac{\pi}{2}
\end{aligned}$$

6.2-6.3 Washer Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = x^2$$

about the x-axis.

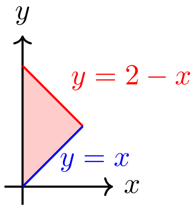


$$\begin{aligned}
V &= \pi \int_0^1 \left[\big(R(x)\big)^2 - \big(r(x)\big)^2 \right] dx \\
&= \pi \int_0^1 \left[\big(x\big)^2 - \big(x^2\big)^2 \right] dx \\
&= \pi \int_0^1 \left[x^2 - x^4 \right] dx \\
&= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\
&= \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \pi \left(\frac{5-3}{15} \right) \\
&= \frac{2\pi}{15}
\end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = x, \quad x = 0$$

about the x-axis.

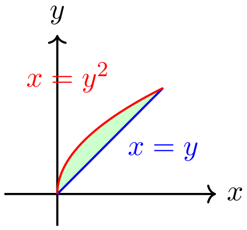


$$\begin{aligned}
V &= \pi \int_0^1 \left[\big(R(x)\big)^2 - \big(r(x)\big)^2 \right] dx \\
&= \pi \int_0^1 \left[\big(2 - x\big)^2 - \big(x\big)^2 \right] dx \\
&= \pi \int_0^1 \left[(4 - 4x + x^2) - x^2 \right] dx \\
&= \pi \int_0^1 (4 - 4x) dx \\
&= \pi \left[4x - 2x^2 \right]_0^1 \\
&= \pi(4 - 2) \\
&= 2\pi
\end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y, \quad x = y^2$$

about the y-axis.

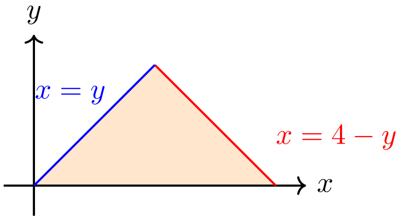


$$\begin{aligned}
V &= \pi \int_0^1 \left[\big(R(y)\big)^2 - \big(r(y)\big)^2\right] dy \\
&= \pi \int_0^1 \left[\big(y\big)^2 - \big(y^2\big)^2\right] dy \\
&= \pi \int_0^1 \left[y^2 - y^4\right] dy \\
&= \pi \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 \\
&= \pi \left(\frac{1}{3} - \frac{1}{5}\right) \\
&= \frac{2\pi}{15}
\end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = 4 - x, \quad \text{and} \quad y = 0$$

about the y-axis.

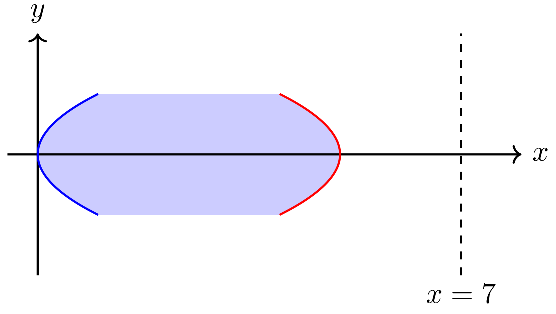


$$\begin{aligned}
V &= \pi \int_0^2 \left[\big(R(y)\big)^2 - \big(r(y)\big)^2\right] dy \\
&= \pi \int_0^2 \left[\big((4-y)\big)^2 - \big(y\big)^2\right] dy \\
&= \pi \int_0^2 \left[(16 - 8y + y^2) - y^2\right] dy \\
&= \pi \int_0^2 (16 - 8y) dy \\
&= \pi \left[16y - 4y^2\right]_0^2 \\
&= \pi\big(32 - 16\big) \\
&= 16\pi
\end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 5 - y^2, \quad y = -1, \quad \text{and} \quad y = 1$$

about the vertical line $x = 7$.



For a given y, the distances from the vertical line $x = 7$ to the curves are:

$$\begin{aligned} r(y) &= 7 - (5 - y^2) = 2 + y^2 \\ R(y) &= 7 - y^2 \end{aligned}$$

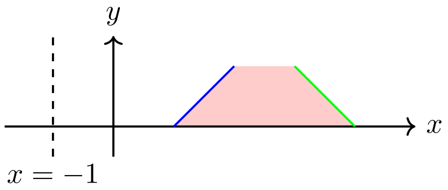
Thus, the volume is

$$\begin{aligned} V &= \pi \int_{-1}^{1} \Big[(7 - y^2)^2 - (2 + y^2)^2\Big]\, dy \\ &= \pi \int_{-1}^{1} \Big[(49 - 14y^2 + y^4) - (4 + 4y^2 + y^4)\Big]\, dy \\ &= \pi \int_{-1}^{1} \Big(45 - 18y^2\Big)\, dy \\ &= \pi \left[45y - 18\left(\frac{y^3}{3}\right)\right]_{-1}^{1} \\ &= \pi \left[45y - 6y^3\right]_{-1}^{1} \\ &= \pi \left(\left[45(1) - 6(1)^3\right] - \left[45(-1) - 6(-1)^3\right]\right) \\ &= \pi \left((45 - 6) - (-45 + 6)\right) \\ &= \pi \left(39 + 39\right) \\ &= 78\pi \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x - 1, \quad y = 4 - x, \quad y = 0, \quad \text{and} \quad y = 1$$

about the vertical line $x = -1$.



For a fixed y with $0 \leq y \leq 1$, the region extends horizontally from

$$x = y + 1 \quad \text{to} \quad x = 4 - y.$$

Since the axis of rotation is $x = -1$, the distances from $x = -1$ to these curves are:

$$\begin{aligned} r(y) &= (y + 1) - (-1) = y + 2 \\ R(y) &= (4 - y) - (-1) = 5 - y. \end{aligned}$$

Thus, the cross-sectional area (washer) is:

$$A(y) = \pi\Big[(5 - y)^2 - (y + 2)^2\Big].$$

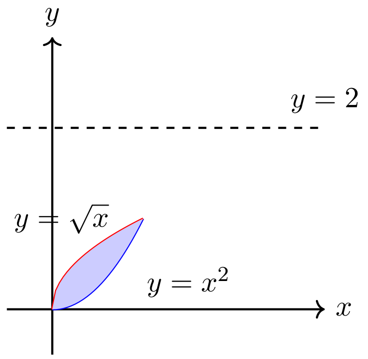
Hence, the volume is

$$\begin{aligned} V &= \pi \int_{0}^{1} \Big[(5 - y)^2 - (y + 2)^2\Big] dy \\ &= \pi \int_{0}^{1} \Big[(25 - 10y + y^2) - (y^2 + 4y + 4)\Big] dy \\ &= \pi \int_{0}^{1} \Big[25 - 10y + y^2 - y^2 - 4y - 4\Big] dy \\ &= \pi \int_{0}^{1} \Big(21 - 14y\Big) dy \\ &= \pi \left[21y - \frac{14y^2}{2}\right]_0^1 \\ &= \pi \left[21y - 7y^2\right]_0^1 \\ &= \pi \left((21 - 7) - 0\right) \\ &= 14\pi \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the horizontal line $y = 2$.



Using the washer method with vertical slices, note that the region in x runs from $x = 0$ to $x = 1$. For a fixed x in this interval, the y-values range from

$$y = x^2 \quad \text{(lower curve)} \quad \text{to} \quad y = \sqrt{x} \quad \text{(upper curve)}.$$

When this slice is rotated about the horizontal line $y = 2$, the distances from $y = 2$ to the curves are:

$$\text{Outer radius: } R(x) = 2 - x^2$$
$$\text{Inner radius: } r(x) = 2 - \sqrt{x}.$$

Thus, the cross-sectional area is:

$$A(x) = \pi\Big[(2 - x^2)^2 - (2 - \sqrt{x})^2\Big].$$

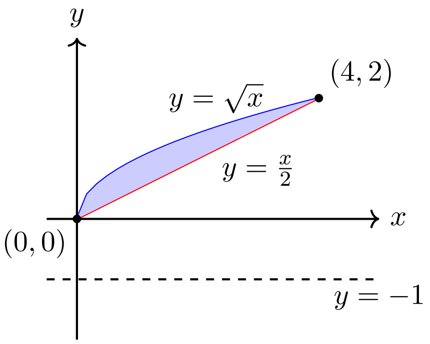
Hence, the volume is

$$\begin{aligned} V &= \pi \int_0^1 \Big[(2 - x^2)^2 - (2 - \sqrt{x})^2\Big]\,dx \\ &= \pi \int_0^1 \Big[(4 - 4x^2 + x^4) - (4 - 4\sqrt{x} + x)\Big]\,dx \\ &= \pi \int_0^1 \Big(-4x^2 + x^4 + 4\sqrt{x} - x\Big)\,dx \\ &= \pi \left[-\frac{4x^3}{3} + \frac{x^5}{5} + \frac{8x^{3/2}}{3} - \frac{x^2}{2}\right]_0^1 \\ &= \pi \left(-\frac{4}{3} + \frac{1}{5} + \frac{8}{3} - \frac{1}{2}\right) \\ &= \frac{31\pi}{30}. \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -1$.



For a fixed x in $[0, 4]$, the region extends vertically from

$$y = \frac{x}{2} \quad \text{to} \quad y = \sqrt{x}.$$

When this slice is rotated about $y = -1$, it forms a washer with

$$\text{Outer radius: } R(x) = \sqrt{x} - (-1) = \sqrt{x} + 1$$
$$\text{Inner radius: } r(x) = \frac{x}{2} - (-1) = \frac{x}{2} + 1.$$

Thus, the cross-sectional area is

$$A(x) = \pi\left[\left(\sqrt{x} + 1\right)^2 - \left(\frac{x}{2} + 1\right)^2\right].$$

Hence, the volume is

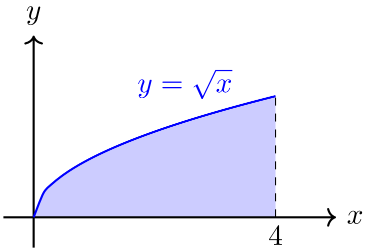
$$\begin{aligned} V &= \pi \int_0^4 \left[\left(\sqrt{x} + 1\right)^2 - \left(\frac{x}{2} + 1\right)^2\right] dx \\ &= \pi \int_0^4 \left[\left(x + 2\sqrt{x} + 1\right) - \left(\frac{x^2}{4} + x + 1\right)\right] dx \\ &= \pi \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx \\ &= \pi \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 \\ &= \pi \left[\frac{4}{3}(4^{3/2}) - \frac{4^3}{12}\right] \\ &= \pi \left[\frac{32}{3} - \frac{16}{3}\right] \\ &= \frac{16\pi}{3}. \end{aligned}$$

6.2-6.3 Cylindrical Shells Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x-axis.



Using horizontal shells (with y as the variable), note that:

- Express $y = \sqrt{x}$ as $x = y^2$.
- For a given horizontal slice at height y, the shell extends from $x = y^2$ to $x = 4$. Thus, the *height* is $h(y) = 4 - y^2$.
- The *radius* of the shell is the distance from y to the x-axis, i.e. $r(y) = y$.

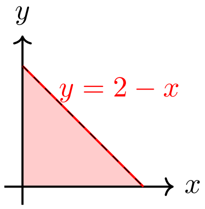
Then,

$$\begin{aligned} V &= 2\pi \int_0^2 (\text{radius}) \cdot (\text{height})\, dy \\ &= 2\pi \int_0^2 y\,(4 - y^2)\, dy \\ &= 2\pi \left[\int_0^2 (4y - y^3)\, dy \right] \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi\, [8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = 0, \quad x \in [0, 2]$$

about the x-axis.



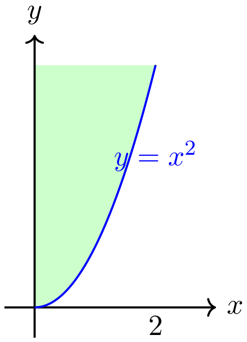
Rewriting $y = 2 - x$ as $x = 2 - y$, the region in y is from $y = 0$ to $y = 2$ with horizontal length $h(y) = 2 - y$ and radius $r(y) = y$. Then,

$$\begin{aligned} V &= 2\pi \int_0^2 y\,(2 - y)\, dy \\ &= 2\pi \int_0^2 (2y - y^2)\, dy \\ &= 2\pi \left[y^2 - \frac{y^3}{3} \right]_0^2 \\ &= 2\pi \left[4 - \frac{8}{3} \right] \\ &= 2\pi \left(\frac{12 - 8}{3} \right) \\ &= 2\pi \left(\frac{4}{3} \right) \\ &= \frac{8\pi}{3} \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 4, \quad x \in [0, 2]$$

about the y-axis.



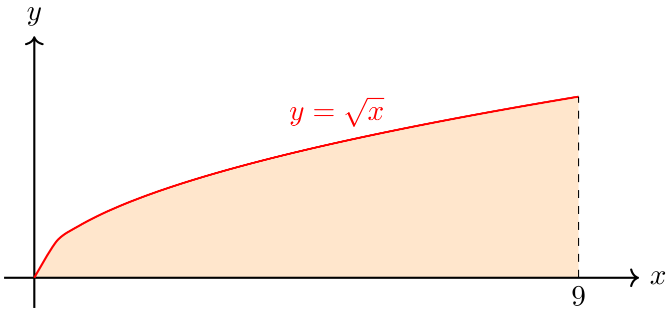
Using vertical shells, the radius is $r(x) = x$ and the height is $h(x) = 4 - x^2$. Thus,

$$\begin{aligned} V &= 2\pi \int_0^2 x\,(4 - x^2)\,dx \\ &= 2\pi \left[\int_0^2 (4x - x^3)\,dx \right] \\ &= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi\,[8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 9$$

about the y-axis.



For vertical shells, the radius is $r(x) = x$ and the height is $h(x) = \sqrt{x}$. Then,

$$\begin{aligned} V &= 2\pi \int_0^9 x \cdot \sqrt{x}\,dx \\ &= 2\pi \int_0^9 x^{3/2}\,dx \\ &= 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^9 \\ &= 2\pi \cdot \frac{2}{5} \cdot 9^{5/2} \\ &= \frac{4\pi}{5} \cdot (9^{5/2}). \end{aligned}$$

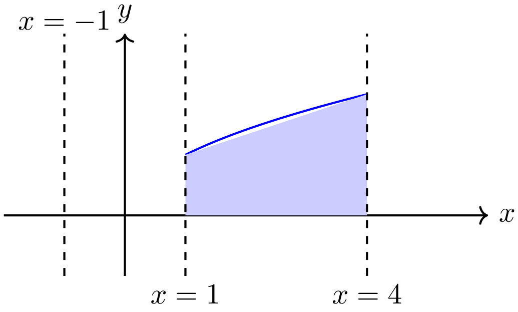
Since $9^{5/2} = (\sqrt{9})^5 = 3^5 = 243$, we have

$$V = \frac{4\pi}{5} \cdot 243 = \frac{972\pi}{5}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the vertical line $x = -1$.



Since the region is bounded vertically by $y = 0$ and $y = \sqrt{x}$ and horizontally by $x = 1$ and $x = 4$, we use the *shell method* with vertical slices. For a typical slice at position x, the height is

$$h(x) = \sqrt{x} - 0 = \sqrt{x},$$

and the radius (distance from x to the axis $x = -1$) is

$$r(x) = x - (-1) = x + 1.$$

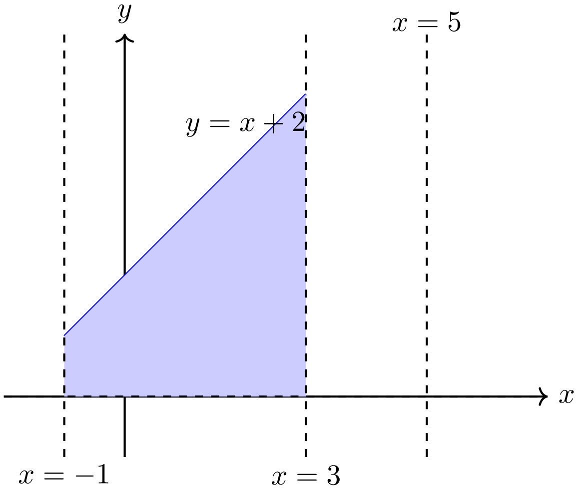
Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 r(x)\, h(x)\, dx \\ &= 2\pi \int_1^4 (x+1)\sqrt{x}\, dx \\ &= 2\pi \int_1^4 \left(x^{3/2} + x^{1/2}\right) dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2}\right]_{x=1}^{4} \\ &= 2\pi \left\{ \left[\frac{2}{5}(4^{5/2}) + \frac{2}{3}(4^{3/2})\right] - \left[\frac{2}{5}(1^{5/2}) + \frac{2}{3}(1^{3/2})\right] \right\} \\ &= \frac{512\pi}{15}. \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x + 2, \quad y = 0, \quad x = -1, \quad x = 3$$

about the vertical line $x = 5$.



Since the region is defined by $-1 \le x \le 3$ and $0 \le y \le x+2$, we use the *cylindrical shells* method. A typical vertical slice at x has:

$$\text{Height: } h(x) = x + 2,$$

and its distance from the axis $x = 5$ is

$$\text{Radius: } r(x) = 5 - x.$$

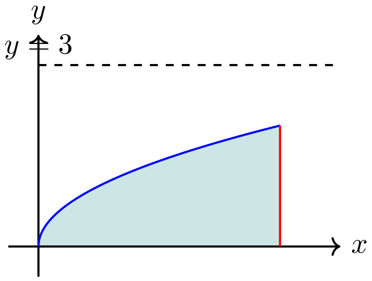
Thus, the volume is given by

$$\begin{aligned} V &= 2\pi \int_{-1}^{3} r(x)\, h(x)\, dx \\ &= 2\pi \int_{-1}^{3} (5 - x)(x + 2)\, dx \\ &= 2\pi \int_{-1}^{3} \Big[5(x+2) - x(x+2)\Big] dx \\ &= 2\pi \int_{-1}^{3} \Big[5x + 10 - x^2 - 2x\Big] dx \\ &= 2\pi \int_{-1}^{3} \Big[-x^2 + 3x + 10\Big] dx \\ &= 2\pi \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 10x\right]_{x=-1}^{3} \\ &= \frac{256\pi}{3}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4, \quad y = 0$$

about the horizontal line $y = 3$.



For a horizontal slice at y (with $0 \le y \le 2$):

- The *radius* is the distance from y to the line $y = 3$: $r(y) = 3 - y$.
- The *height* is the horizontal length: $h(y) = 4 - x$ where x runs from $x = y^2$ to $x = 4$; thus, $h(y) = 4 - y^2$.

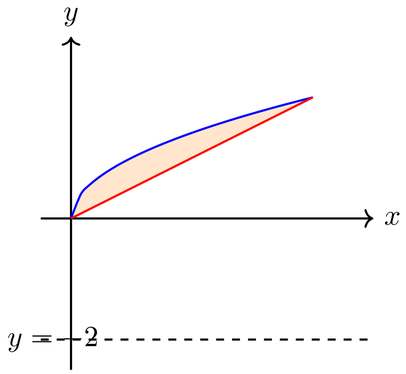
Then,

$$\begin{aligned} V &= 2\pi \int_{y=0}^{2} (3-y)(4-y^2)\,dy \\ &= 2\pi \int_0^2 \left[12 - 3y^2 - 4y + y^3\right] dy \\ &= 2\pi \left[12y - \frac{3y^3}{3} - 2y^2 + \frac{y^4}{4}\right]_0^2 \\ &= 2\pi \left[12(2) - y^3|_2 - 2(2)^2 + \frac{(2)^4}{4}\right] \\ &= 2\pi\,[24 - 8 - 8 + 4] \\ &= 2\pi(12) \\ &= 24\pi \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -2$.



Using horizontal shells, for a slice at height y (with y from the intersection of the curves, $0 \le y \le 2$):

- The *radius* is the distance from y to $y = -2$: $r(y) = y - (-2) = y + 2$.
- The *height* is the horizontal length between the curves: $h(y) = 2y - y^2$.

Then,

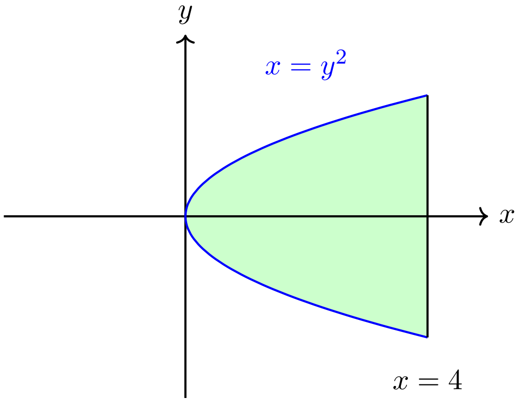
$$\begin{aligned} V &= 2\pi \int_{y=0}^{2} (y+2)\left[2y - y^2\right] dy \\ &= 2\pi \int_0^2 \left[(y+2)(2y-y^2)\right] dy \\ &= 2\pi \int_0^2 \left[2y^2 - y^3 + 4y - 2y^2\right] dy \\ &= 2\pi \int_0^2 \left(4y - y^3\right) dy \\ &= 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4}\right] \\ &= 2\pi\,[8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

6.2-6.3 Additional Problems (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4$$

about the y-axis.



Solution via Washer Method:

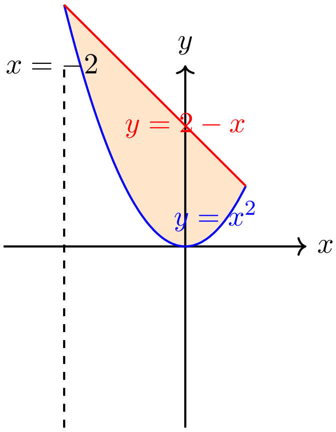
Using the Washer Method with y-limits $y = -2$ to 2, outer radius $R(y) = 4$ and inner radius $r(y) = y^2$:

$$\begin{aligned} V &= \pi \int_{-2}^{2} \left[4^2 - \left(y^2\right)^2\right] dy = \pi \int_{-2}^{2} \left(16 - y^4\right) dy \\ &= 2\pi \int_{0}^{2} \left(16 - y^4\right) dy = 2\pi \left[16y - \frac{y^5}{5}\right]_0^2 \\ &= 2\pi \left(32 - \frac{32}{5}\right) = 2\pi \left(\frac{160 - 32}{5}\right) = \frac{256\pi}{5}. \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x$$

about the vertical line $x = -2$.



Solution via the Shell Method:

Using vertical slices, for x from -2 to 1 the height of a typical slice is

$$h(x) = (2 - x) - x^2,$$

and its distance from the axis $x = -2$ (the shell radius) is

$$r(x) = x - (-2) = x + 2.$$

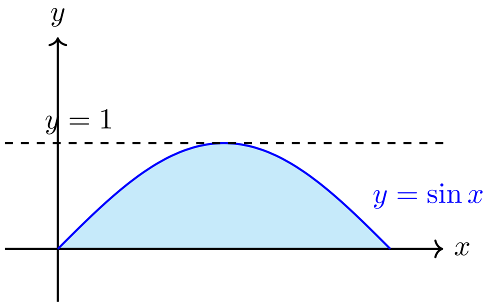
Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_{-2}^{1} \left[(x + 2)\Big((2 - x) - x^2\Big)\right] dx \\ &= 2\pi \int_{-2}^{1} (x + 2)(2 - x - x^2)\, dx \\ &= 2\pi \int_{-2}^{1} \left[4 - 3x^2 - x^3\right] dx \\ &= 2\pi \left[4x - x^3 - \frac{x^4}{4}\right]_{x=-2}^{1} \\ &= 2\pi \left[\frac{11}{4} - (-4)\right] = 2\pi \left(\frac{11}{4} + 4\right) = 2\pi \left(\frac{11 + 16}{4}\right) \\ &= 2\pi \left(\frac{27}{4}\right) = \frac{27\pi}{2} \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

about the line $y = 1$.



Solution via the Washer Method:

Since the region lies entirely below $y = 1$, a vertical slice at a fixed x (with $0 \le x \le \pi$) extends from $y = 0$ to $y = \sin x$. When rotated about $y = 1$, this slice generates a washer with

$$\text{Outer radius: } R = 1 - 0 = 1$$
$$\text{Inner radius: } r = 1 - \sin x$$

Thus, the area of the washer is

$$A(x) = \pi\Big[R^2 - r^2\Big] = \pi\Big[1^2 - (1 - \sin x)^2\Big].$$

Expanding the inner square,

$$(1 - \sin x)^2 = 1 - 2\sin x + \sin^2 x,$$

so that

$$A(x) = \pi\Big[1 - (1 - 2\sin x + \sin^2 x)\Big] = \pi\Big(2\sin x - \sin^2 x\Big).$$

Hence, the volume is given by

$$\begin{aligned} V &= \int_{x=0}^{\pi} A(x)\,dx \\ &= \pi \int_0^{\pi} \Big(2\sin x - \sin^2 x\Big)dx. \end{aligned}$$

We evaluate the integrals separately:

$$\int_0^{\pi} 2\sin x\,dx = 2\Big[-\cos x\Big]_0^{\pi} = 2\Big[-(-1) + 1\Big] = 4,$$

and

$$\int_0^{\pi} \sin^2 x\,dx = \frac{\pi}{2}.$$

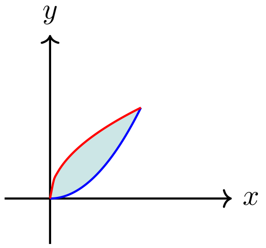
Thus,

$$V = \pi\Big(4 - \frac{\pi}{2}\Big) = \frac{\pi}{2}(8 - \pi).$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the y-axis.



Solution via the Washer Method:

$$\begin{aligned} V_{\text{washers}} &= \int_{y=0}^{1} A(y)\,dy \\ &= \pi \int_0^1 (y - y^4)\,dy \\ &= \pi \left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{5}\right) \\ &= \pi \left(\frac{5-2}{10}\right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

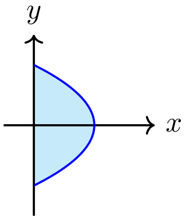
Solution via the Shell Method:

$$\begin{aligned} V_{\text{shells}} &= 2\pi \int_0^1 \left(x^{3/2} - x^3\right)dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1 \\ &= 2\pi \left(\frac{2}{5} - \frac{1}{4}\right) \\ &= 2\pi \left(\frac{8-5}{20}\right) \\ &= 2\pi \left(\frac{3}{20}\right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = 1 - y^2, \quad x = 0, \quad y \in [-1, 1]$$

about the y-axis.



Solution Using the Disk Method:

Since rotation is about the y-axis, the radius of a typical disk at height y is

$$R(y) = 1 - y^2.$$

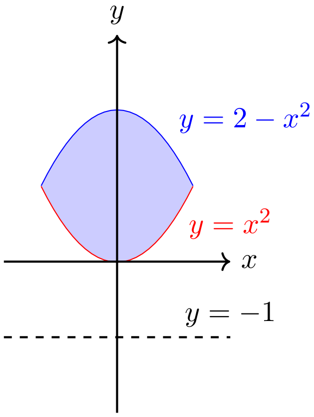
Thus, the volume is

$$\begin{aligned}
V &= \pi \int_{y=-1}^{1} (1 - y^2)^2 dy \\
&= \pi \int_{-1}^{1} \left(1 - 2y^2 + y^4\right) dy \\
&= \pi \left[\int_{-1}^{1} 1\, dy - 2\int_{-1}^{1} y^2\, dy + \int_{-1}^{1} y^4\, dy\right] \\
&= \pi \left[2 - 2\left(\frac{2}{3}\right) + \frac{2}{5}\right] \\
&= \pi \left[2 - \frac{4}{3} + \frac{2}{5}\right] \\
&= \pi \left[\frac{30}{15} - \frac{20}{15} + \frac{6}{15}\right] \\
&= \pi \left(\frac{16}{15}\right).
\end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x^2$$

about the line $y = -1$.



Solution Using the Washer Method:

For a fixed x with $x \in [-1, 1]$, the region extends vertically from the lower curve $y = x^2$ to the upper curve $y = 2 - x^2$. When rotated about $y = -1$, the distances from $y = -1$ are:

$$\begin{aligned}
\text{Inner radius: } r(x) &= x^2 - (-1) = x^2 + 1 \\
\text{Outer radius: } R(x) &= (2 - x^2) - (-1) = 3 - x^2.
\end{aligned}$$

Thus, the washer cross-sectional area is

$$A(x) = \pi \Big[(3 - x^2)^2 - (x^2 + 1)^2\Big].$$

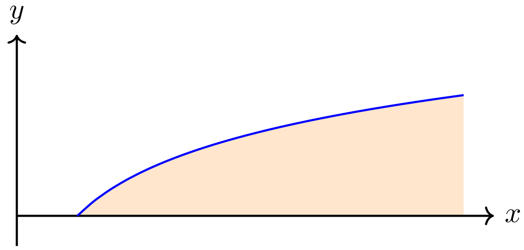
The volume is

$$\begin{aligned}
V &= \pi \int_{x=-1}^{1} \Big[(3 - x^2)^2 - (x^2 + 1)^2\Big] dx \\
&= \pi \int_{-1}^{1} (8 - 8x^2) dx \\
&= 8\pi \int_{-1}^{1} (1 - x^2) dx \\
&= 8\pi \left[\int_{-1}^{1} 1\, dx - \int_{-1}^{1} x^2\, dx\right] \\
&= 8\pi \left[(1 - (-1)) - \left(\frac{1^3 - (-1)^3}{3}\right)\right] \\
&= \frac{32\pi}{3}.
\end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln x, \quad y = 0, \quad x = 1, \quad x = e^2$$

about the y-axis.



Solution Using the Washer Method:

Since we are rotating about the y-axis, it is convenient to express x in terms of y. From

$$y = \ln x \quad \Longrightarrow \quad x = e^y,$$

and note that when $x = e^2$ we have $y = \ln(e^2) = 2$. Thus, for a fixed y between 0 and 2, the region extends in x from the curve $x = e^y$ to the vertical line $x = e^2$. When rotated about the y-axis, the horizontal slice produces an annular cross section with

Inner radius: $R_{\text{in}}(y) = e^y$, Outer radius: $R_{\text{out}}(y) = e^2$.

The area of the washer is then

$$A(y) = \pi\Big[(e^2)^2 - (e^y)^2\Big] = \pi\Big(e^4 - e^{2y}\Big).$$

Thus, the volume by washers is

$$\begin{aligned}
V_{\text{washers}} &= \pi \int_{y=0}^{2} \Big(e^4 - e^{2y}\Big)\,dy \\
&= \pi \left[e^4 y - \frac{e^{2y}}{2}\right]_{y=0}^{2} \\
&= \pi \left[e^4(2) - \frac{e^4}{2} - \left(0 - \frac{1}{2}\right)\right] \\
&= \pi \left(2e^4 - \frac{e^4}{2} + \frac{1}{2}\right) \\
&= \pi \left(\frac{4e^4 - e^4 + 1}{2}\right) \\
&= \frac{\pi(3e^4 + 1)}{2}.
\end{aligned}$$

Solution Using the Shell Method:

Using vertical slices, the region is described by x from 1 to e^2 and, for each x, y runs from $y = 0$ to $y = \ln x$. A typical vertical slice at x has height

$$h(x) = \ln x,$$

and its distance (radius) from the y-axis is

$$r(x) = x.$$

Thus, the volume by shells is given by

$$V_{\text{shells}} = 2\pi \int_{x=1}^{e^2} x\,(\ln x)\,dx.$$

We compute the integral using integration by parts. Let

$$\begin{aligned}
u &= \ln x \quad \Rightarrow \quad du = \frac{1}{x}dx \\
dv &= x\,dx \quad \Rightarrow \quad v = \frac{x^2}{2}
\end{aligned}$$

Then,

$$\begin{aligned}
\int x \ln x\,dx &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx \\
&= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx \\
&= \frac{x^2}{2}\ln x - \frac{x^2}{4}.
\end{aligned}$$

Evaluating from $x = 1$ to $x = e^2$:

$$\begin{aligned}
&\left.\frac{x^2}{2}\ln x - \frac{x^2}{4}\right|_{x=1}^{e^2} \\
&= \left[\frac{(e^2)^2}{2}\ln(e^2) - \frac{(e^2)^2}{4}\right] - \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] \\
&= \left[\frac{e^4}{2}\cdot 2 - \frac{e^4}{4}\right] - \left[0 - \frac{1}{4}\right] \\
&= \left[e^4 - \frac{e^4}{4}\right] + \frac{1}{4} \\
&= \frac{4e^4 - e^4 + 1}{4} \\
&= \frac{3e^4 + 1}{4}.
\end{aligned}$$

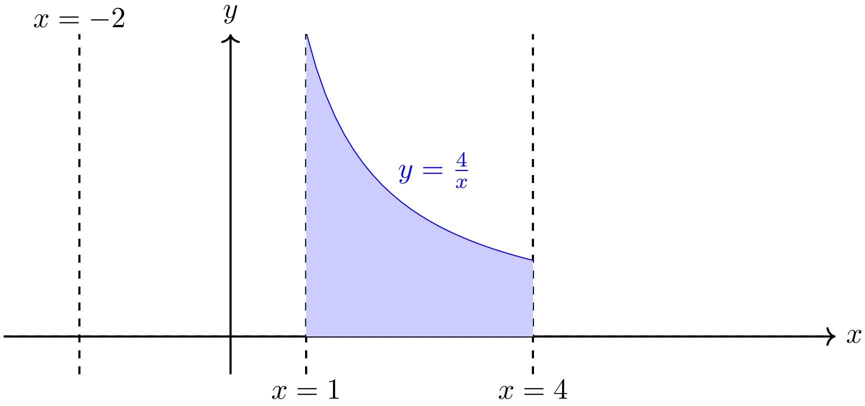
Thus,

$$\begin{aligned}
V_{\text{shells}} &= 2\pi \cdot \frac{3e^4 + 1}{4} \\
&= \frac{\pi(3e^4 + 1)}{2}.
\end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \frac{4}{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the line $x = -2$.



Solution Using the Shell Method:

A vertical slice at position x (with $1 \leq x \leq 4$) has height

$$h(x) = \frac{4}{x},$$

and its distance (radius) from the axis $x = -2$ is

$$r(x) = x - (-2) = x + 2.$$

Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 r(x)\, h(x)\, dx \\ &= 2\pi \int_1^4 (x+2)\left(\frac{4}{x}\right) dx \\ &= 2\pi \int_1^4 \frac{4(x+2)}{x}\, dx \\ &= 8\pi \int_1^4 \left(1 + \frac{2}{x}\right) dx \\ &= 8\pi \left[\int_1^4 1\, dx + 2\int_1^4 \frac{1}{x}\, dx\right] \\ &= 8\pi \left[(4-1) + 2 \ln x \Big|_1^4\right] \\ &= 8\pi\, [3 + 2\ln(4)] \end{aligned}$$

6.4 Work (Solutions)

Spring Problems (Solutions)

1. A spring has a natural length of 20 m. A force of 12 N is required to stretch the spring to 25 m. Determine the work required to stretch the spring from 20 m to 30 m.

   - By Hooke's Law, the force required to stretch a spring is:

     $$F = kx,$$

     where k is the spring constant and x is the displacement from the natural length.
   - Given that a force of 12 N is required to stretch the spring to 25 m:

     $$12 = k(5).$$

     Solving for k:

     $$k = \frac{12}{5} = 2.4.$$
   - The work done to stretch the spring from $x = a$ to $x = b$ is given by:

     $$W = \int_a^b kx\,dx.$$
   - Here, we compute the work to stretch from 20 m to 30 m, which corresponds to $x = 0$ to $x = 10$ m:

     $$W = \int_0^{10} 2.4x\,dx.$$
   - Computing the integral:

     $$\begin{aligned} W &= 2.4 \int_0^{10} x\,dx \\ &= 2.4 \left[\frac{x^2}{2}\right]_0^{10} \\ &= 2.4 \left(\frac{100}{2} - \frac{0}{2}\right) \\ &= 2.4 \times 50 \\ &= 120 \text{ J}. \end{aligned}$$

2. A spring has a natural length of 15 m. A force of 10 N is required to stretch the spring to 18 m. Determine the work required to stretch the spring from 16 m to 22 m.

   - Hooke's Law states that $F = kx$, where k is the spring constant and x is the displacement from the natural length.
   - Given that a force of 10 N stretches the spring to 18 m, we find k:

     $$\begin{aligned} 10 &= k(18 - 15) \\ 10 &= k(3) \\ k &= \frac{10}{3} \text{ N/m} \end{aligned}$$
   - The work done to stretch the spring from $x_1 = 16$ m to $x_2 = 22$ m is given by:

     $$\begin{aligned} W &= \int_{x_1}^{x_2} kx\,dx \\ &= \int_1^7 \frac{10}{3} x\,dx \\ &= \frac{10}{3} \left[\frac{x^2}{2}\right]_1^7 \\ &= \frac{10}{3} \left(\frac{49}{2} - \frac{1}{2}\right) \\ &= \frac{10}{3} \times \frac{48}{2} \\ &= \frac{10}{3} \times 24 \\ &= 80 \text{ J} \end{aligned}$$

3. A spring has a natural length of 30 m. A force of 8 N is required to stretch the spring to 35 m. Determine the work required to compress the spring from 30 m to 20 m.

    - Using Hooke's Law, we first determine the spring constant k:

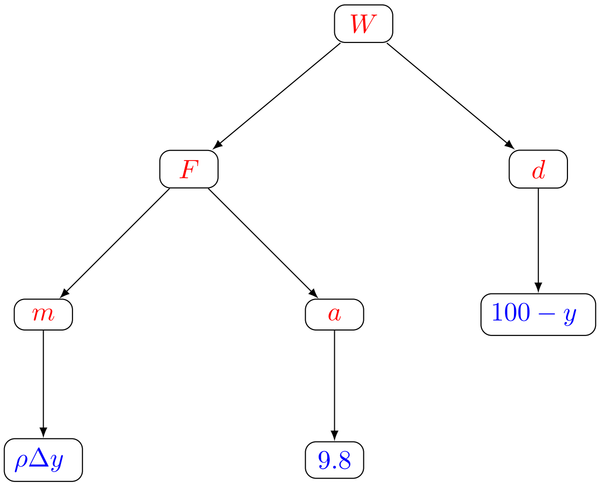
$$\begin{aligned} 8 &= k(35-30) \\ 8 &= k(5) \\ k &= \frac{8}{5}\ \text{N/m} \end{aligned}$$

    - The work required to compress the spring from $x_1 = 30$ m to $x_2 = 20$ m is:

$$\begin{aligned} W &= \int_{x_1}^{x_2} kx\,dx \\ &= \int_{0}^{-10} \frac{8}{5}x\,dx \\ &= \frac{8}{5}\left[\frac{x^2}{2}\right]_0^{-10} \\ &= \frac{8}{5}\left(\frac{100}{2}-0\right) \\ &= \frac{8}{5}\times 50 \\ &= 80\ \text{J} \end{aligned}$$

Cable Problems (Solutions)

1. A **100-meter-long** cable with a **linear density of 5 kg/m** is hanging from a winch at the top of a well. The cable is initially fully extended into the well and is lifted **to the top**. Compute the work required to lift the entire cable.



Step 1: Define Variables and Divide the Cable into Slices

- Let y be the height above the **bottom of the well**, with $y = 0$ at the bottom and $y = 100$ at the top.
- Partition the interval $[0, 100]$ into n slices of equal height Δy.
- The mass of a slice of cable at height y_i^* is:

$$m_i = \rho\Delta y = 5\Delta y.$$

- The force due to gravity acting on the slice is:

$$F_i = g \cdot m_i = 9.8 \cdot (5\Delta y).$$

Step 2: Compute the Work on Each Slice

- Each slice at height y_i^* must be lifted from its original position y_i^* to the top ($y = 100$).
- The lifting distance for the slice is:

$$d_i = 100 - y_i^*.$$

- The work done to lift the ith slice is:

$$W_i = F_i \cdot d_i = (9.8 \cdot 5\Delta y) \cdot (100 - y_i^*).$$

Step 3: Express Work as an Integral

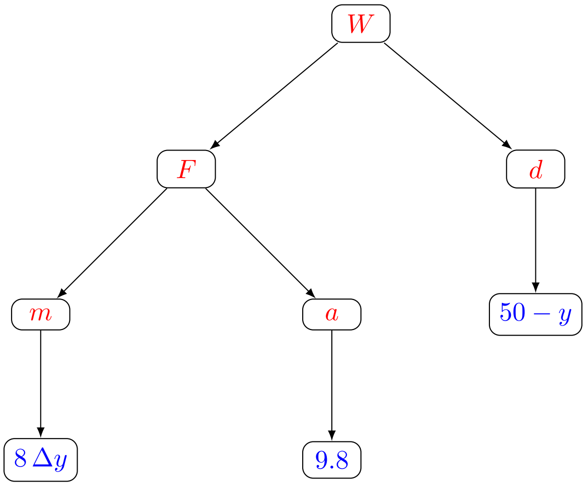
- Summing over all slices and taking the limit as $n \to \infty$, the total work is:

$$W = \int_0^{100} 9.8 \cdot 5 \cdot (100 - y)\, dy.$$

- Evaluating the integral:

$$\begin{aligned} W &= 49 \int_0^{100} (100 - y)\, dy \\ &= 49 \left[100y - \frac{y^2}{2}\right]_0^{100} \\ &= 49 \left(100(100) - \frac{100^2}{2}\right) \\ &= 49\,(10000 - 5000) \\ &= 49 \times 5000 \\ &= 245000 \text{ J.} \end{aligned}$$

2. A **50-meter-long** chain with a **linear density of 8 kg/m** is hanging from a pulley at the top of a mine shaft. The chain is initially fully extended into the shaft and is lifted **to the top**. Compute the work required to lift the entire chain.



Step 1: Define Variables and Divide the Chain into Slices

- Let y denote the height above the **bottom of the shaft** (with $y = 0$ at the bottom and $y = 50$ at the top).
- Divide the interval $[0, 50]$ into n slices of equal thickness Δy.
- The mass of a slice at height y_i^* is:

$$m_i = \rho\,\Delta y = 8\,\Delta y.$$

- The weight (force due to gravity) on the slice is:

$$F_i = m_i\,g = 8(9.8)\,\Delta y = 78.4\,\Delta y.$$

Step 2: Compute the Work on Each Slice

- A slice at height y_i^* is lifted to the top, a distance of:

$$d_i = 50 - y_i^*.$$

- The work done on this slice is approximately:

$$W_i = F_i \cdot d_i = 78.4\,(50 - y_i^*)\,\Delta y.$$

Step 3: Express Work as an Integral

- Summing over all slices and taking the limit as $n \to \infty$ gives:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 78.4\,(50 - y_i^*)\,\Delta y \\ &= 78.4 \int_0^{50} (50 - y)\,dy. \end{aligned}$$

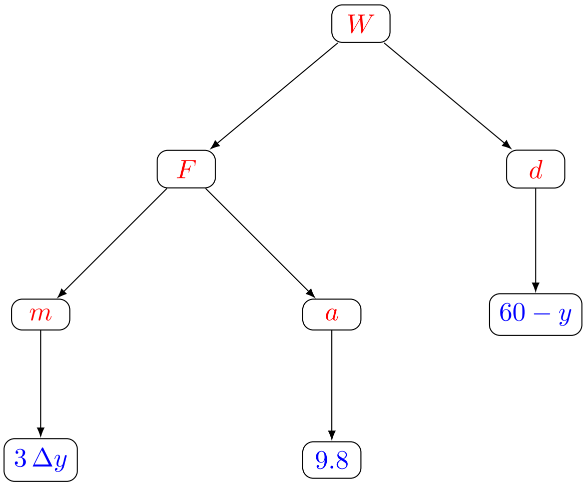
- Evaluate the integral:

$$\begin{aligned} \int_0^{50} (50 - y)\,dy &= \left[50y - \frac{y^2}{2}\right]_0^{50} \\ &= 50(50) - \frac{50^2}{2} \\ &= 2500 - 1250 \\ &= 1250. \end{aligned}$$

- Thus, the total work is:

$$W = 78.4 \times 1250 = 98\,000 \text{ J}.$$

3. A **60-meter-long** rope with a **linear density of 3 kg/m** is hanging over the edge of a cliff, with one end secured at the top and the other end dangling freely. The rope is slowly lifted until it is fully coiled at the top of the cliff. Compute the work required to lift the rope.



Step 1: Define Variables and Divide the Rope into Slices

- Let y denote the height above the **bottom of the rope** (with $y = 0$ at the bottom and $y = 60$ at the top).
- Divide the interval $[0, 60]$ into n slices of equal length Δy.
- The mass of a slice is:

$$m_i = \rho\,\Delta y = 3\,\Delta y.$$

- The weight on the slice is:

$$F_i = m_i\,g = 3(9.8)\,\Delta y = 29.4\,\Delta y.$$

Step 2: Compute the Work on Each Slice

- Each slice at height y_i^* is lifted a distance:

$$d_i = 60 - y_i^*.$$

- The work done on the slice is:

$$W_i = F_i \cdot d_i = 29.4\,(60 - y_i^*)\,\Delta y.$$

Step 3: Express Work as an Integral

- The total work is:

$$W = 29.4 \int_0^{60} (60 - y)\,dy.$$

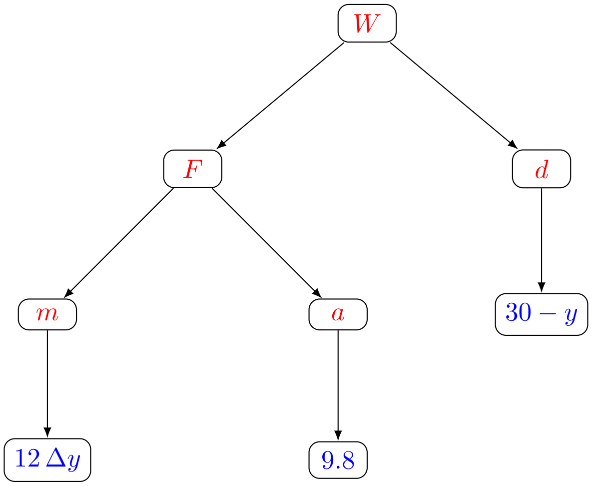
- Evaluate the integral:

$$\begin{aligned}\int_0^{60} (60 - y)\,dy &= \left[60y - \frac{y^2}{2}\right]_0^{60} \\ &= 3600 - 1800 \\ &= 1800.\end{aligned}$$

- Hence, the work is:

$$W = 29.4 \times 1800 = 52\,920 \text{ J}.$$

4. A **30-meter-long** anchor chain with a **linear density of 12 kg/m** is hanging from the side of a ship, with one end attached to the ship and the other submerged in the water. The chain is hoisted onto the deck of the ship. Compute the work required to lift the entire chain onto the ship.



Step 1: Define Variables and Divide the Chain into Slices

- Let y denote the height above the **bottom of the chain** (with $y = 0$ at the submerged end and $y = 30$ at the deck).
- Divide the interval $[0, 30]$ into n slices of thickness Δy.
- The mass of a slice is:

$$m_i = \rho\,\Delta y = 12\,\Delta y.$$

- The weight on the slice is:

$$F_i = m_i\,g = 12(9.8)\,\Delta y = 117.6\,\Delta y.$$

Step 2: Compute the Work on Each Slice

- A slice at height y_i^* is lifted a distance:

$$d_i = 30 - y_i^*.$$

- The work done on the slice is:

$$W_i = F_i \cdot d_i = 117.6\,(30 - y_i^*)\,\Delta y.$$

Step 3: Express Work as an Integral

- The total work is:

$$W = 117.6 \int_0^{30} (30 - y)\,dy.$$

- Evaluate the integral:

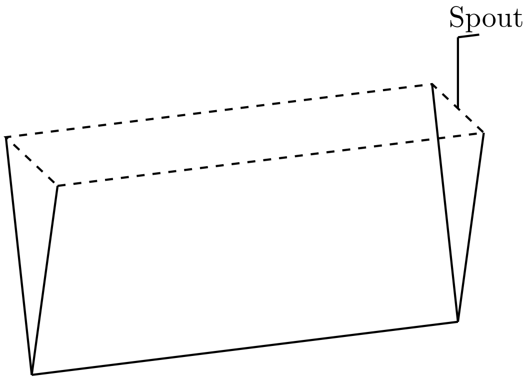
$$\begin{aligned}\int_0^{30} (30 - y)\,dy &= \left[30y - \frac{y^2}{2}\right]_0^{30}\\ &= 900 - 450\\ &= 450.\end{aligned}$$

- Thus, the work is:

$$W = 117.6 \times 450 = 52\,920 \text{ J}.$$

Tank Problems (Solutions)

1. **Rectangular Tank with Triangular Ends**. A tank is 6 m long (into the page) and its end view is an isosceles triangle with a base of 2 m and a height of 3 m. The tank is filled with water, and the water is pumped out through a spout located 0.5 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the bottom (vertex) and $y = 3$ at the top.

Partition the interval $[0, 3]$ into n subintervals of equal width Δy. For the ith subinterval, choose a representative point y_i^*.

By similar triangles, the width of the tank at y_i^* is

$$w_i = \frac{2}{3}\, y_i^*.$$

Since the tank is 6 m long, the cross-sectional area of a slice is

$$A_i = 6 \cdot w_i = 6\left(\frac{2}{3}\, y_i^*\right) = 4\, y_i^*.$$

Thus, the volume of the ith slice is given by

$$V_i = A_i\, \Delta y = 4\, y_i^*\, \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is obtained by multiplying the mass by gravitational acceleration:

$$F_i = \rho\, g\, V_i = 1000 \cdot 9.8 \cdot (4\, y_i^*\, \Delta y).$$

Each slice must be lifted to the spout, which is at a height of $3 + 0.5 = 3.5$ m. Hence, the lifting distance for the ith slice is

$$d_i = 3.5 - y_i^*.$$

Thus, the work done on the ith slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot 4\, y_i^*\, (3.5 - y_i^*)\, \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

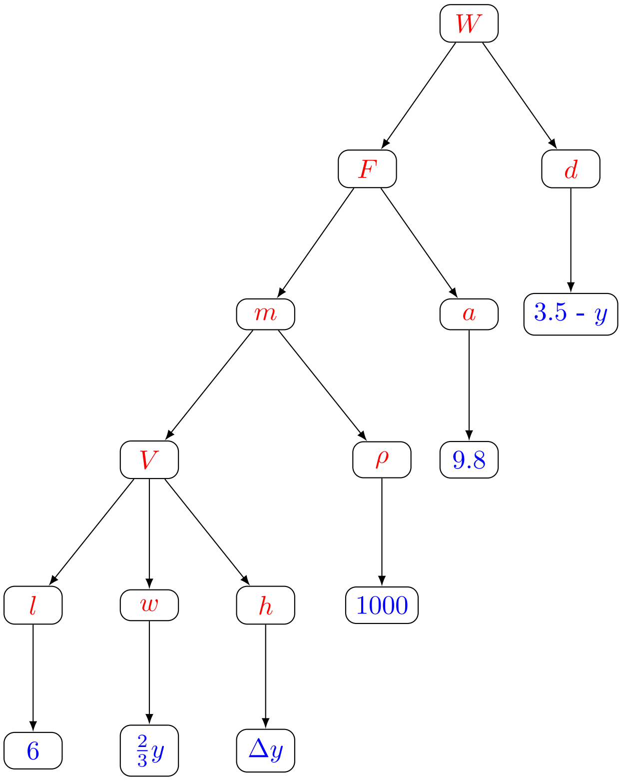
The total work required to pump the water is given by the Riemann sum

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1000 \cdot 9.8 \cdot 4\, y_i^* (3.5 - y_i^*)\, \Delta y\right].$$

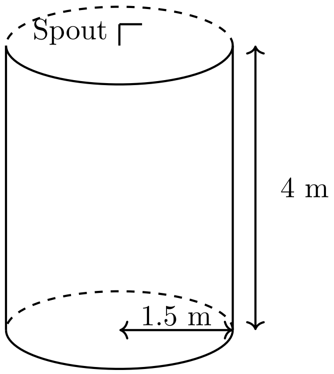
This is equivalent to the integral

$$W = \int_0^3 1000 \cdot 9.8 \cdot 4y\, (3.5 - y)\, dy.$$

Tree Diagram Representation:



2. **Cylindrical Tank with a Spout.** A vertical cylindrical tank is 4 m high with a radius of 1.5 m. The tank is completely full of water, and the water is pumped out through a spout located 0.3 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Let $y = 0$ at the bottom of the cylinder and $y = 4$ at the top. Partition $[0, 4]$ into n subintervals of equal width Δy. In the ith subinterval, pick a representative point y_i^*.

The cross-section at height y_i^* is a circle of radius 1.5. Hence, its area is

$$A = \pi \times (1.5)^2 = 2.25\pi.$$

The volume of the ith slice is

$$V_i = A\,\Delta y = 2.25\pi\,\Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is

$$F_i = \rho\,g\,V_i = 1000 \cdot 9.8 \cdot (2.25\pi\,\Delta y).$$

Each slice must be lifted to the spout, which is at $y = 4 + 0.3 = 4.3$. Hence, the lifting distance is

$$d_i = 4.3 - y_i^*.$$

Thus, the work for the ith slice is

$$W_i = F_i \cdot d_i \;=\; 1000 \cdot 9.8 \cdot 2.25\pi\,(4.3 - y_i^*)\,\Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

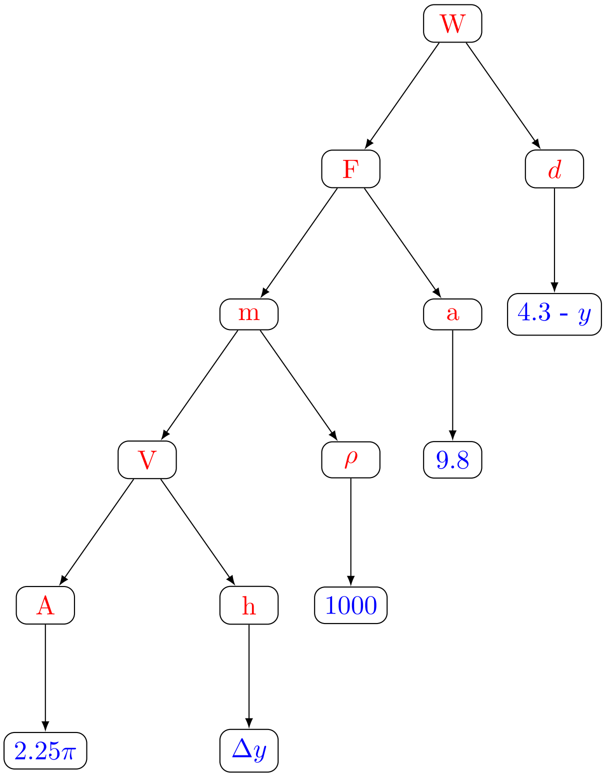
Summing over all slices,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \big[\,1000 \cdot 9.8 \cdot 2.25\pi\,(4.3 - y_i^*)\,\Delta y\big].$$

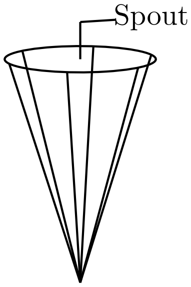
Equivalently, in integral form,

$$W = \int_0^4 1000 \cdot 9.8 \cdot 2.25\pi\,(4.3 - y)\,dy.$$

Tree Diagram Representation:



3. **Inverted Conical Tank**. An inverted conical tank has a height of 3 m and a top (open) radius of 1 m. The tank is completely full of water, and the water is pumped out through a spout located 0.2 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the tip (bottom) and $y = 3$ at the top. Partition the interval $[0, 3]$ into n equal subintervals of width Δy; in the ith subinterval choose a representative point y_i^*.

By similar triangles, the radius at height y_i^* is

$$r_i = \frac{1}{3}\, y_i^*,$$

so the cross-sectional area is

$$A_i = \pi \big(r_i\big)^2 = \pi \left(\frac{y_i^*}{3}\right)^2 = \frac{\pi (y_i^*)^2}{9}.$$

Thus, the volume of the ith slice is

$$V_i = A_i\, \Delta y = \frac{\pi (y_i^*)^2}{9}\, \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is given by

$$F_i = \rho\, g\, V_i = 1000 \cdot 9.8 \cdot \frac{\pi (y_i^*)^2}{9}\, \Delta y.$$

The water must be pumped up to the spout, which is located 0.2 m above the top. Since the top is at $y = 3$, the spout is at $y = 3.2$. Therefore, the lifting distance is

$$d_i = 3.2 - y_i^*.$$

Thus, the work done on the ith slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot \frac{\pi (y_i^*)^2}{9}\, (3.2 - y_i^*)\, \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

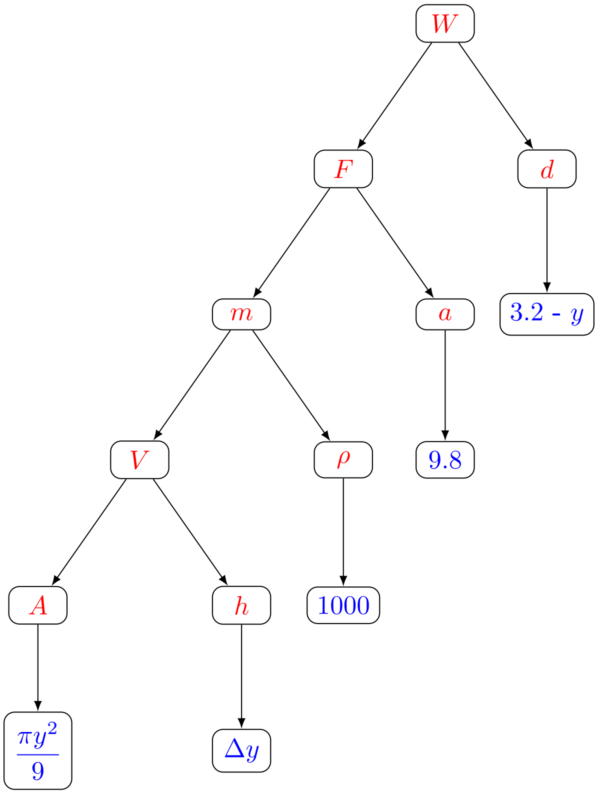
Summing over all slices, the total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1000 \cdot 9.8 \cdot \frac{\pi (y_i^*)^2}{9}\, (3.2 - y_i^*)\, \Delta y\right],$$

which is equivalent to the integral

$$W = \int_0^3 1000 \cdot 9.8 \cdot \frac{\pi y^2}{9}\, (3.2 - y)\, dy.$$

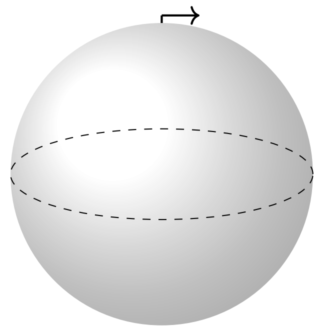
Tree Diagram Representation:



Thus, the work required to pump the water is

$$W = \int_0^3 1000 \cdot 9.8 \cdot \frac{\pi y^2}{9}\, (3.2 - y)\, dy.$$

4. **Spherical Tank**. A spherical tank of radius 2 m is completely filled with water. A spout is located 0.1 m above the top of the sphere.



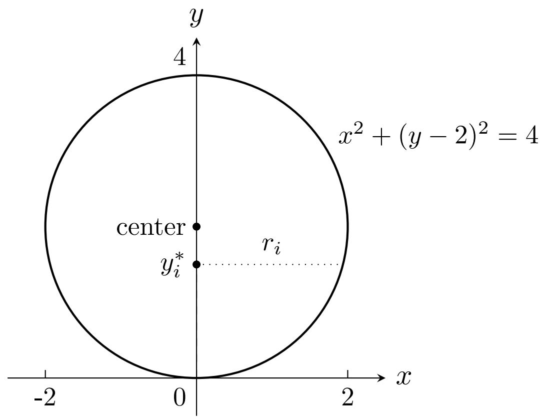
Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the bottom of the sphere, and $y = 4$ at the top. Partition the interval $[0, 4]$ into n subintervals of equal width Δy; in the ith subinterval, choose a representative point y_i^*.

At height y_i^*, the horizontal cross section of the sphere is a circle whose radius is given by

$$r_i = \sqrt{2^2 - (y_i^* - 2)^2} = \sqrt{4 - (y_i^* - 2)^2}.$$

To see this, consider the side view of the spherical tank:



Then at height y_i^*, the radius of the cross section is the corresponding x-coordinate on the graph of $x^2 + (y - 2)^2 = 4$. Plugging in y_i^* for y, we solve

$$r_i = x = \sqrt{4 - (y_i^* - 2)^2}$$

Thus, the area of the slice is

$$A_i = \pi \left(r_i\right)^2 = \pi\,[4 - (y_i^* - 2)^2].$$

The volume of the ith slice is then

$$V_i = A_i\,\Delta y = \pi\,[4 - (y_i^* - 2)^2]\,\Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is

$$F_i = \rho\, g\, V_i = 1000 \cdot 9.8 \cdot \pi\,[4 - (y_i^* - 2)^2]\,\Delta y.$$

The spout is 0.1 m above the top of the sphere (top is at $y = 4$), so the spout is at $y = 4.1$. Hence, the lifting distance for the ith slice is

$$d_i = 4.1 - y_i^*.$$

Thus, the work done on the ith slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot \pi\,[4 - (y_i^* - 2)^2]\,(4.1 - y_i^*)\,\Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

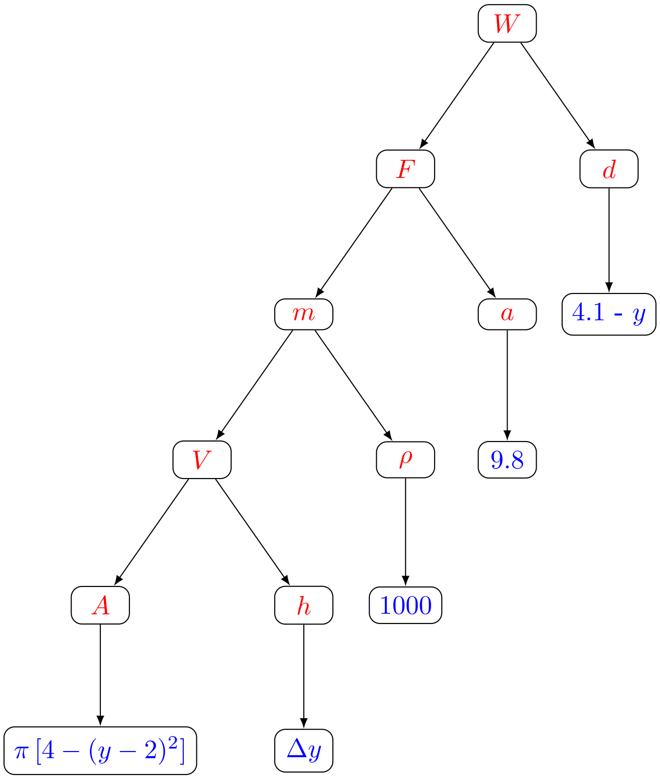
The total work required to pump the water is given by the Riemann sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[1000 \cdot 9.8 \cdot \pi\,[4 - (y_i^* - 2)^2]\,(4.1 - y_i^*)\,\Delta y\right],$$

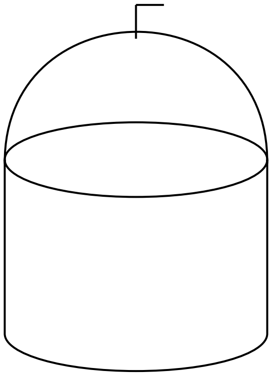
which is equivalent to the integral

$$W = \int_0^4 1000 \cdot 9.8 \cdot \pi\,[4 - (y - 2)^2]\,(4.1 - y)\,dy.$$

Tree Diagram Representation:



5. **Composite Tank – Cylinder with Hemispherical Top**. A tank consists of a cylindrical section 3 m high with a circular cross section of radius 1 m, topped by a hemispherical dome of radius 1 m. Water is pumped out through a spout located 0.15 m above the dome.



Step 1: Divide the Tank into Slices.

(a) Cylindrical Section (for $0 \le y \le 3$):

- Partition $[0,3]$ into n subintervals of width Δy; choose a representative point y_i^* in each.
- The cross-sectional area is constant:

$$A_{\text{cyl}} = \pi(1)^2 = \pi.$$

- The volume of the ith slice is

$$V_i^{\text{cyl}} = A_{\text{cyl}}\,\Delta y = \pi\,\Delta y.$$

(b) Hemispherical Dome (for $3 \le y \le 4$):

- Partition $[3,4]$ into subintervals of width Δy; choose a representative point y_i^* in each.
- The hemisphere is the upper half of a sphere of radius 1.
- At height y_i^*, the horizontal cross-sectional radius is

$$r_i = \sqrt{1-(y_i^*-3)^2},$$

(see the previous problem for an explanation), so the area is

$$A_i^{\text{hemi}} = \pi\,(r_i)^2 = \pi\Big[1-(y_i^*-3)^2\Big].$$

- The volume of the ith slice is then

$$V_i^{\text{hemi}} = A_i^{\text{hemi}}\,\Delta y.$$

Step 2: Compute the Work on Each Slice.

Let the density $\rho = 1000$ and gravitational acceleration $g = 9.8$.

(a) Cylindrical Section:

$$F_i^{\text{cyl}} = \rho\, g\, V_i^{\text{cyl}} = 1000 \cdot 9.8 \cdot \pi\,\Delta y.$$

Each slice must be lifted to the spout at $y = 4.15$; therefore, the lifting distance is

$$d_i^{\text{cyl}} = 4.15 - y_i^*.$$

Thus, the work on the ith cylindrical slice is

$$W_i^{\text{cyl}} = F_i^{\text{cyl}} \cdot d_i^{\text{cyl}} = 1000 \cdot 9.8 \cdot \pi\,(4.15 - y_i^*)\,\Delta y.$$

(b) Hemispherical Dome:

$$F_i^{\text{hemi}} = \rho\, g\, V_i^{\text{hemi}} = 1000\cdot 9.8\cdot\pi\Big[1-(y_i^*-3)^2\Big]\,\Delta y.$$

The lifting distance is the same:

$$d_i^{\text{hemi}} = 4.15 - y_i^*.$$

Thus, the work on the ith hemispherical slice is

$$\begin{aligned} W_i^{\text{hemi}} &= F_i^{\text{hemi}} \cdot d_i^{\text{hemi}} \\ &= 1000 \cdot 9.8 \cdot \pi\Big[1-(y_i^*-3)^2\Big]\,(4.15 - y_i^*)\,\Delta y. \end{aligned}$$

Step 3: Write the Total Work as a Riemann Sum.

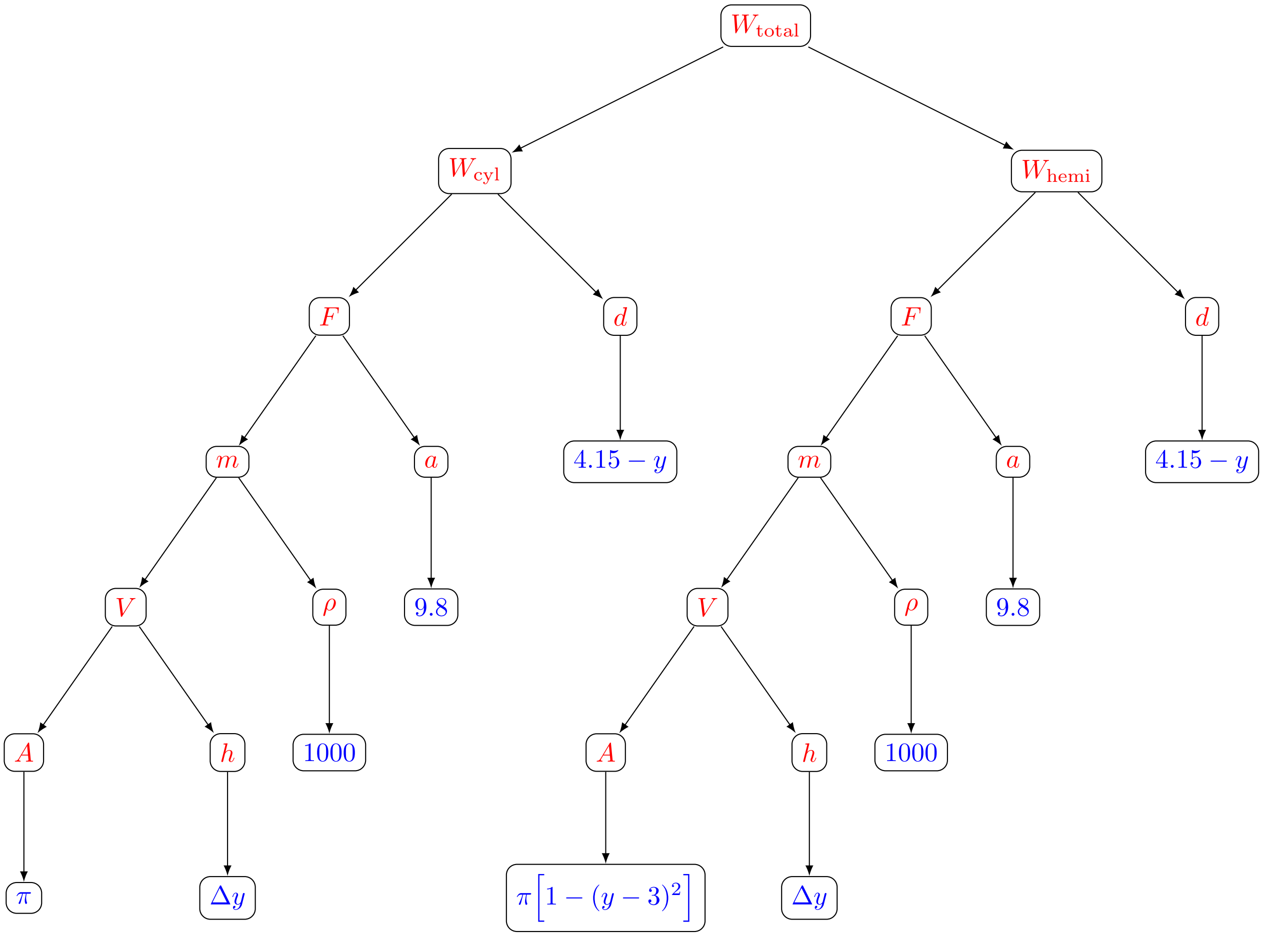
The total work is the sum of the work on the cylindrical section and the hemispherical dome:

$$W = \lim_{n\to\infty}\left[\sum_{i=1}^{n^{\text{cyl}}} W_i^{\text{cyl}} + \sum_{i=1}^{n^{\text{hemi}}} W_i^{\text{hemi}}\right],$$

which is equivalent to

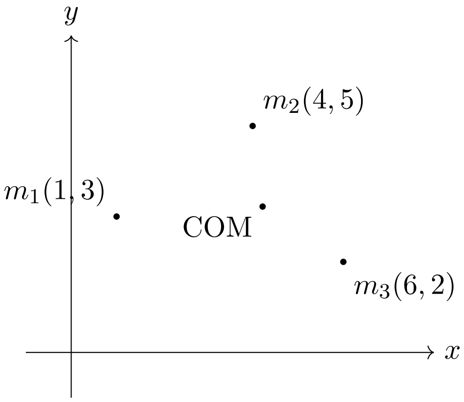
$$\begin{aligned} W = &\int_0^3 1000 \cdot 9.8 \cdot \pi\,(4.15 - y)\,dy \\ &+ \int_3^4 1000 \cdot 9.8 \cdot \pi\Big[1-(y-3)^2\Big]\,(4.15 - y)\,dy. \end{aligned}$$

Tree Diagram Representation:



8.3 Center of Mass (Solutions)

1. **Three Point Masses.**



- **Total mass:**

$$M = 2 + 3 + 4 = 9\,\text{kg}.$$

- **Compute the moments:**

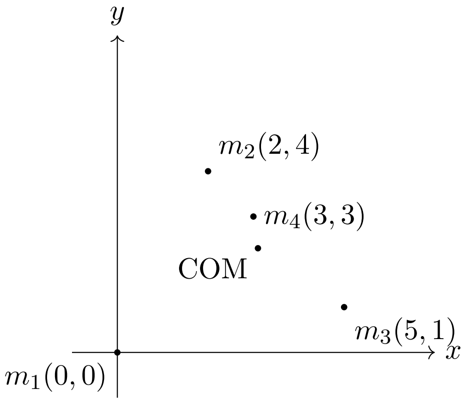
$$\begin{aligned} M_x &= 2(1) + 3(4) + 4(6) \\ &= 2 + 12 + 24 = 38 \end{aligned}$$

$$\begin{aligned} M_y &= 2(3) + 3(5) + 4(2) \\ &= 6 + 15 + 8 = 29. \end{aligned}$$

- **Center of Mass:**

$$\begin{aligned} \bar{x} &= \frac{38}{9}, \\ \bar{y} &= \frac{29}{9}. \end{aligned}$$

2. **Four Point Masses.**



- **Total mass:**

$$M = 1 + 2 + 3 + 4 = 10\,\text{kg}.$$

- **Compute the moments:**

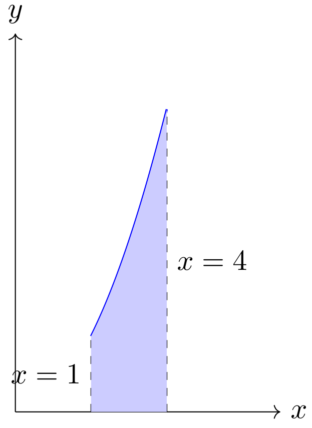
$$\begin{aligned} M_x &= 1(0) + 2(2) + 3(5) + 4(3) \\ &= 0 + 4 + 15 + 12 = 31 \end{aligned}$$

$$\begin{aligned} M_y &= 1(0) + 2(4) + 3(1) + 4(3) \\ &= 0 + 8 + 3 + 12 = 23. \end{aligned}$$

- **Center of Mass:**

$$\begin{aligned} \bar{x} &= \frac{31}{10} = 3.1, \\ \bar{y} &= \frac{23}{10} = 2.3. \end{aligned}$$

3. **Region Bounded by $y = x^2$, $y = 0$, $x = 1$, and $x = 2$.**



- **Area:**

$$\begin{aligned} A &= \int_1^2 \left(x^2 - 0\right) dx = \int_1^2 x^2\, dx \\ &= \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3 - 1^3}{3} = \frac{8-1}{3} = \frac{7}{3}. \end{aligned}$$

- **x-coordinate of the centroid:**

$$\begin{aligned} \bar{x} &= \frac{1}{A}\int_1^2 x\left(x^2\right) dx = \frac{1}{A}\int_1^2 x^3\, dx \\ &= \frac{1}{7/3}\left[\frac{x^4}{4}\right]_1^2 = \frac{3}{7}\left(\frac{16-1}{4}\right) \\ &= \frac{3}{7}\cdot\frac{15}{4} = \frac{45}{28}. \end{aligned}$$

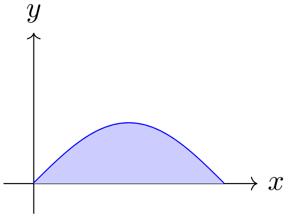
- **y-coordinate of the centroid:**

$$\begin{aligned} \bar{y} &= \frac{1}{A}\int_1^2 \frac{(x^2)^2}{2}\, dx = \frac{1}{A}\int_1^2 \frac{x^4}{2}\, dx \\ &= \frac{1}{7/3}\cdot\frac{1}{2}\left[\frac{x^5}{5}\right]_1^2 = \frac{3}{7}\cdot\frac{1}{2}\left(\frac{32-1}{5}\right) \\ &= \frac{3}{7}\cdot\frac{31}{10} = \frac{93}{70}. \end{aligned}$$

Thus, the center of mass is

$$\left(\frac{45}{28}, \frac{93}{70}\right).$$

4. **Region Bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.**



- **Area:**

$$\begin{aligned} A &= \int_0^\pi \sin x\, dx = [-\cos x]_0^\pi \\ &= [-\cos\pi] - [-\cos 0] = (1) - (-1) = 2. \end{aligned}$$

- **x-coordinate of the centroid:**

$$\bar{x} = \frac{1}{A}\int_0^\pi x\sin x\, dx.$$

Using integration by parts (with $u = x,\ dv = \sin x\, dx$):

$$\begin{aligned} \int_0^\pi x\sin x\, dx &= [-x\cos x]_0^\pi + \int_0^\pi \cos x\, dx \\ &= [-\pi\cos\pi + 0] + [\sin x]_0^\pi \\ &= -\pi(-1) + (0 - 0) = \pi. \end{aligned}$$

Thus,

$$\bar{x} = \frac{\pi}{2}.$$

- **y-coordinate of the centroid:**

$$\bar{y} = \frac{1}{A}\int_0^\pi \frac{[\sin x]^2}{2}\, dx = \frac{1}{2A}\int_0^\pi \sin^2 x\, dx.$$

Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$\begin{aligned} \int_0^\pi \sin^2 x\, dx &= \int_0^\pi \frac{1-\cos 2x}{2}\, dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi \\ &= \frac{1}{2}(\pi - 0) = \frac{\pi}{2}. \end{aligned}$$

Therefore,

$$\bar{y} = \frac{1}{2\cdot 2}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

Thus, the center of mass is

$$\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

5. **Uniform Semicircular Lamina.**

A uniform semicircular lamina of radius R (with the flat side along the x-axis) is symmetric about the y-axis. Therefore,

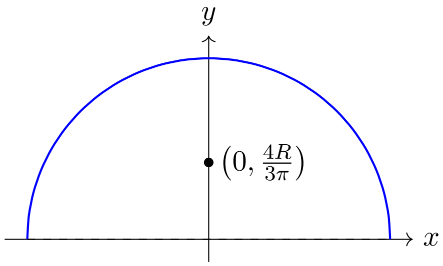
$$\bar{x} = 0.$$

It is a well-known result that the y-coordinate of the centroid is given by:

$$\bar{y} = \frac{4R}{3\pi}.$$

Thus, the center of mass is

$$\left(0, \frac{4R}{3\pi}\right).$$



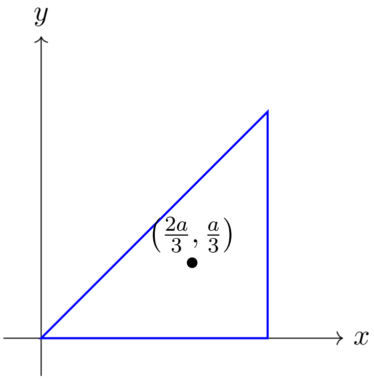
6. **Uniform Triangular Lamina.**

Consider the triangle with vertices $(0,0)$, $(a,0)$, and (a,a). The centroid (center of mass) of a triangle is the average of its vertices:

$$\bar{x} = \frac{0+a+a}{3} = \frac{2a}{3},$$
$$\bar{y} = \frac{0+0+a}{3} = \frac{a}{3}.$$

Thus, the center of mass is

$$\left(\frac{2a}{3}, \frac{a}{3}\right).$$



10.1 Parametric Curves (Solutions)

1. Sketch and identify the curve.

$$x = t^2 - 4t, \qquad y = t + 2, \qquad 0 \le t \le 5.$$

Solution. Eliminate the parameter. From $y = t + 2$, we have $t = y - 2$. Substitute into x:

$$x = (y - 2)^2 - 4(y - 2) = (y - 4)^2 - 4.$$

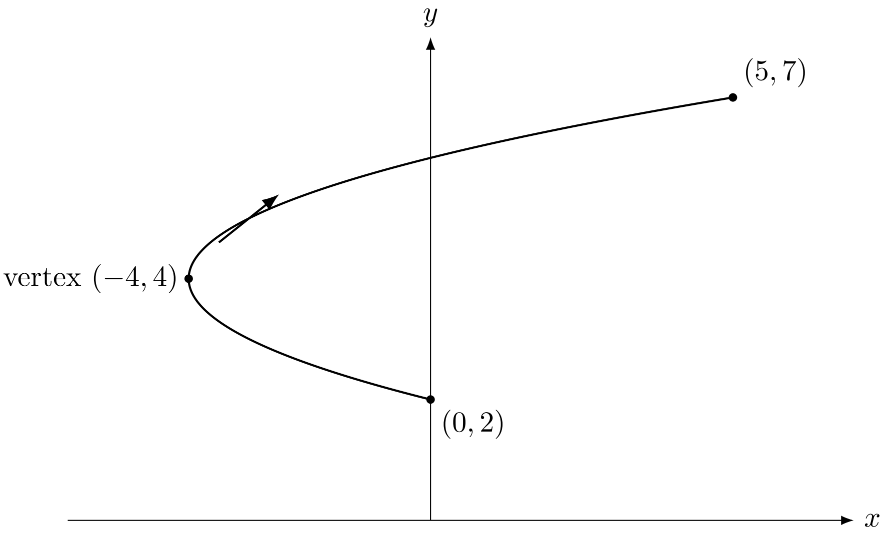
So the curve is the parabola

$$x = (y - 4)^2 - 4,$$

which opens to the *right* with vertex at $(-4, 4)$. Because $0 \le t \le 5$, we have $2 \le y = t + 2 \le 7$, so we only trace the portion with $2 \le y \le 7$. Key points:

$$t = 0 : \ (x, y) = (0, 2), \qquad t = 2 : \ (x, y) = (-4, 4), \qquad t = 5 : \ (x, y) = (5, 7).$$

As t increases, y increases, so the motion is upward along the parabola (from $(0, 2)$ toward the vertex and then out to $(5, 7)$).



2. Sketch and identify the curve.

$$x = 2\cos t - 1, \qquad y = 2\sin t + 3, \qquad 0 \le t \le 2\pi.$$

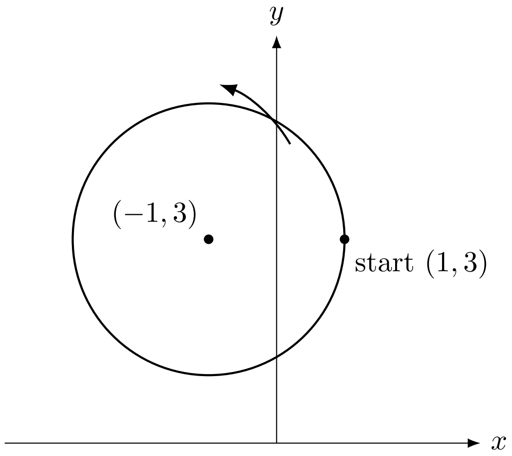
Solution. Rewrite as

$$x + 1 = 2\cos t, \qquad y - 3 = 2\sin t.$$

Square and add:

$$(x+1)^2 + (y-3)^2 = 4(\cos^2 t + \sin^2 t) = 4.$$

So the curve is a circle centered at $(-1, 3)$ with radius 2. At $t = 0$, the point is $(x, y) = (1, 3)$ (the rightmost point), and as t increases the motion is counterclockwise.



3. Sketch and identify the curve.

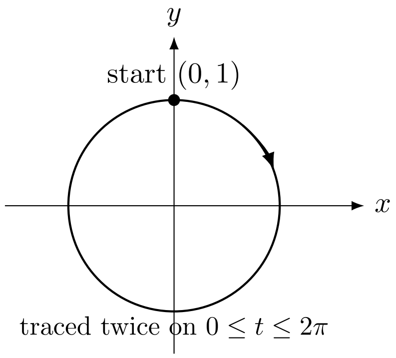
$$x = \sin(2t), \qquad y = \cos(2t), \qquad 0 \le t \le 2\pi.$$

Solution. Eliminate the parameter:

$$x^2 + y^2 = \sin^2(2t) + \cos^2(2t) = 1,$$

so the curve is the unit circle centered at the origin.

Let $\theta = 2t$. As t runs from 0 to 2π, θ runs from 0 to 4π, so the circle is traced *twice*. At $t = 0$, $(x, y) = (0, 1)$. For small $t > 0$, $x = \sin(2t) > 0$ and $y = \cos(2t) < 1$, so the motion initially goes to the right from the top point, meaning the direction is *clockwise*. Thus the unit circle is traced twice clockwise.



4. Sketch and identify the curve.

$$x = t^3, \qquad y = t, \qquad -2 \le t \le 2.$$

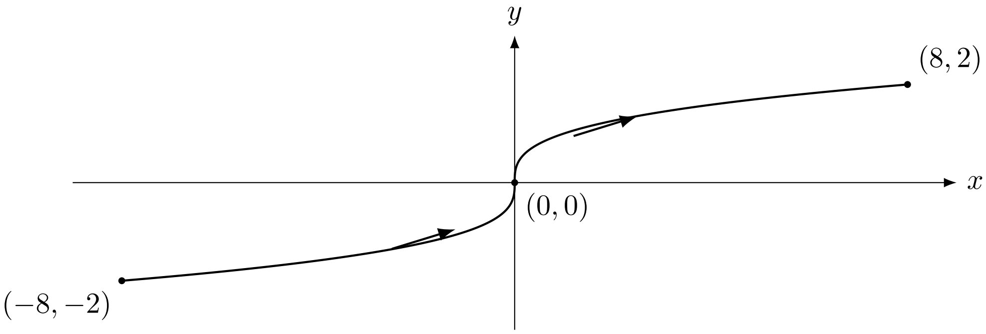
Solution. Since $y = t$, substitute $t = y$ into $x = t^3$:

$$x = y^3.$$

So the curve is the cubic $x = y^3$ (equivalently $y = \sqrt[3]{x}$). The parameter interval $-2 \le t \le 2$ gives

$$t = -2: \ (x, y) = (-8, -2), \qquad t = 0: \ (0, 0), \qquad t = 2: \ (8, 2).$$

As t increases, the motion goes from $(-8, -2)$ through $(0, 0)$ to $(8, 2)$.



5. Sketch and identify the curve.

$$x = \sin t, \qquad y = \sin^2 t, \qquad 0 \le t \le 2\pi.$$

Solution. Eliminate the parameter:

$$y = \sin^2 t = (\sin t)^2 = x^2.$$

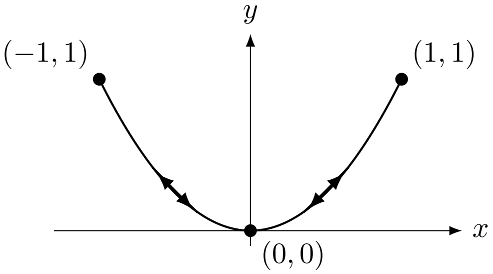
So the curve lies on the parabola $y = x^2$. Since $x = \sin t$, we have $-1 \le x \le 1$. Therefore the curve is

$$y = x^2, \qquad -1 \le x \le 1.$$

Direction/motion (key times):

$$t = 0 : (0,0), \quad t = \frac{\pi}{2} : (1,1), \quad t = \pi : (0,0), \quad t = \frac{3\pi}{2} : (-1,1), \quad t = 2\pi : (0,0).$$

So the particle goes from $(0,0)$ up the *right* branch to $(1,1)$, returns to $(0,0)$, then goes up the *left* branch to $(-1,1)$, and returns again to $(0,0)$.



6. Find a parametrization $\mathbf{r}(t) = (x(t), y(t))$ of the line segment from

$$P_0 = (-2, 1) \quad \text{to} \quad P_1 = (4, -3),$$

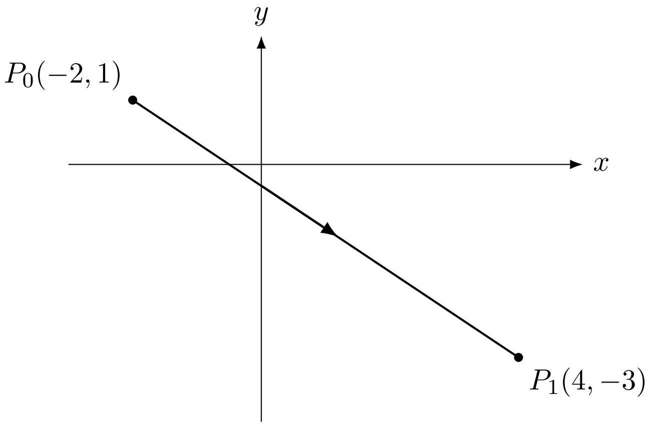
using $0 \le t \le 1$.

Solution. A standard line-segment parametrization is

$$\mathbf{r}(t) = (1-t)P_0 + tP_1, \qquad 0 \le t \le 1.$$

Thus

$$\mathbf{r}(t) = (1-t)(-2,1) + t(4,-3) = (-2+6t,\ 1-4t), \qquad 0 \le t \le 1.$$



7. Give parametric equations for the circle of radius 3 centered at $(2, -1)$ that

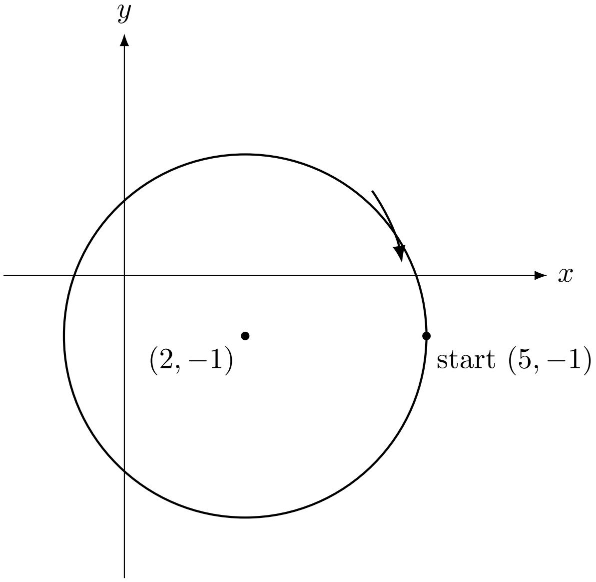
   - starts at the *rightmost* point of the circle, and
   - travels *clockwise* exactly once.

   Solution. A circle of radius 3 centered at $(2, -1)$ can be written

   $$x = 2 + 3\cos t, \qquad y = -1 + 3\sin t.$$

   This starts at the rightmost point $(5, -1)$ when $t = 0$ and goes counterclockwise. To make it go *clockwise*, negate the sine term:

   $$\boxed{x = 2 + 3\cos t, \qquad y = -1 - 3\sin t, \qquad 0 \le t \le 2\pi.}$$



8. Give a parametrization of the upper semicircle

$$x^2 + y^2 = 9, \qquad y \geq 0,$$

traced from $(-3, 0)$ to $(3, 0)$ (left to right). State an interval for t.

Solution. The circle $x^2 + y^2 = 9$ has radius 3. A standard parametrization is

$$x = 3\cos t, \qquad y = 3\sin t.$$

This traces the *upper* semicircle when $0 \leq t \leq \pi$, but it goes from $(3, 0)$ to $(-3, 0)$ (right to left). To go left to right, use

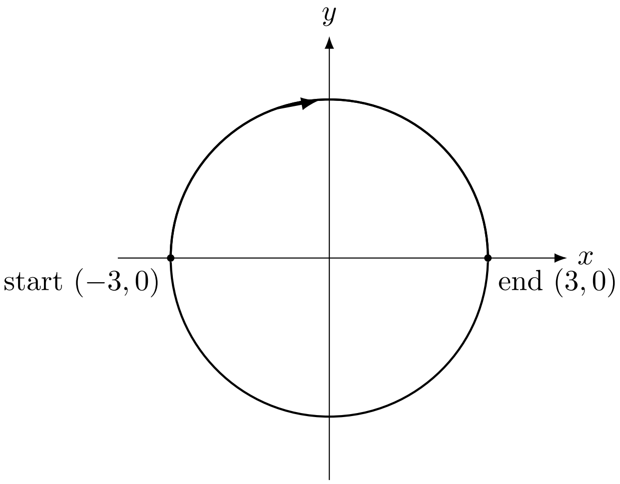
$$x = -3\cos t, \qquad y = 3\sin t.$$

Then

$$t = 0 : \ (-3, 0), \qquad t = \pi : \ (3, 0),$$

and $y \geq 0$ for $0 \leq t \leq \pi$. So

$$\boxed{x = -3\cos t, \qquad y = 3\sin t, \qquad 0 \leq t \leq \pi.}$$



10.2 Tangents to Parametric Curves (Solutions)

1. Consider the parametric equations given by

$$x = t^2 + 1, \quad y = 2t - 3.$$

The slope of the curve is determined by the ratio of the derivatives with respect to t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Differentiating the component functions yields

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 2.$$

Substituting these into the slope formula, we obtain

$$\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}.$$

To find the slope at the specific parameter value $t = 1$, we substitute directly:

$$\frac{dy}{dx} = \frac{1}{1} = 1.$$

2. For the curve defined by

$$x = \sin t, \quad y = \cos t,$$

horizontal tangents occur where the derivative $\frac{dy}{dt}$ is zero, provided that $\frac{dx}{dt}$ is non-zero. First, we find the values of t for which $\frac{dy}{dt} = 0$:

$$\frac{dy}{dt} = -\sin t = 0 \implies t = 0, \pi, 2\pi.$$

Next, we verify that $\frac{dx}{dt} \neq 0$ at these values. Calculating $\frac{dx}{dt} = \cos t$:

$$\begin{aligned} &\text{At } t = 0, 2\pi, \quad \cos t = 1 \neq 0, \\ &\text{At } t = \pi, \quad \cos t = -1 \neq 0. \end{aligned}$$

The points of horizontal tangency correspond to these t values:

$$(0, 1) \quad \text{and} \quad (0, -1).$$

3. Let the curve be given by

$$x = e^t, \quad y = e^{-t}.$$

The slope of the tangent line is derived as follows:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^{-t}}{e^t} = -e^{-2t}.$$

At $t = 0$, the slope m is

$$m = -e^{-2(0)} = -1.$$

To find the equation of the tangent line, we determine the point of tangency at $t = 0$:

$$x = e^0 = 1, \quad y = e^{-0} = 1.$$

Using the point-slope form $y - y_1 = m(x - x_1)$ with the point $(1, 1)$ and slope -1:

$$y - 1 = -1(x - 1) \implies y = -x + 2.$$

4. Consider the curve

$$x = t - \sin t, \quad y = 1 - \cos t.$$

The slope is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}.$$

Substituting $t = \frac{\pi}{4}$ into the expression yields

$$\frac{dy}{dx} = \frac{\sin(\pi/4)}{1 - \cos(\pi/4)} = \frac{\frac{\sqrt{2}}{2}}{1 - \frac{\sqrt{2}}{2}}.$$

Multiplying the numerator and denominator by 2 simplifies the fraction to

$$\frac{\sqrt{2}}{2 - \sqrt{2}}.$$

Rationalizing the denominator gives the final slope:

$$\frac{\sqrt{2}(2 + \sqrt{2})}{4 - 2} = \frac{2\sqrt{2} + 2}{2} = \sqrt{2} + 1.$$

5. For the parametric equations

$$x = \ln t, \quad y = t^2,$$

the derivative is calculated as

$$\frac{dy}{dx} = \frac{2t}{1/t} = 2t^2.$$

When $t = 1$, the slope is $m = 2(1)^2 = 2$. The corresponding point on the curve is

$$x = \ln 1 = 0, \quad y = 1^2 = 1.$$

The equation of the tangent line passing through $(0, 1)$ with slope 2 is

$$y - 1 = 2(x - 0) \implies y = 2x + 1.$$

6. Given the curve

$$x = 3t^2 + 2, \quad y = 4t^3 - 5,$$

we compute the general slope formula:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t^2}{6t} = 2t \quad (\text{assuming } t \neq 0).$$

Evaluating this at $t = -1$ gives

$$\frac{dy}{dx} = 2(-1) = -2.$$

7. Consider the curve defined by

$$x = t^2 + 2t, \quad y = 3t - 1.$$

The slope is given by

$$\frac{dy}{dx} = \frac{3}{2t+2}.$$

At $t = 2$, the slope becomes

$$m = \frac{3}{2(2)+2} = \frac{3}{6} = \frac{1}{2}.$$

The coordinate of the point of tangency at $t = 2$ is

$$x = 2^2 + 2(2) = 8, \quad y = 3(2) - 1 = 5.$$

Finally, the equation of the tangent line at $(8, 5)$ is

$$y - 5 = \frac{1}{2}(x - 8) \implies y = \frac{1}{2}x + 1.$$

8.1, 10.2 Arc Length (Solutions)

1. The arc length formula is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

For $f(x) = \ln(\cos x)$:

- Compute the derivative:

$$f'(x) = \frac{d}{dx} \ln(\cos x) = -\tan x.$$

- Compute $1 + (f'(x))^2$:

$$1 + \tan^2 x = \sec^2 x.$$

- Compute the arc length integral:

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx \\ &= \int_0^{\frac{\pi}{4}} \sec x\, dx. \end{aligned}$$

- Use the integral result:

$$\int \sec x\, dx = \ln|\sec x + \tan x|.$$

- Evaluate from 0 to $\frac{\pi}{4}$:

$$\begin{aligned} L &= \ln|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}| - \ln|\sec 0 + \tan 0| \\ &= \ln(\sqrt{2} + 1) - \ln 1 \\ &= \ln(\sqrt{2} + 1). \end{aligned}$$

2. To find the arc length of the function $f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$ over the interval $[0, 2]$, we use the standard arc length formula:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

First, we determine the derivative of the function:

$$f'(x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2}.$$

Next, we simplify the expression under the square root, $1 + (f'(x))^2$:

$$1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4 + e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4}.$$

Recognizing that the numerator is a perfect square, $(e^x + e^{-x})^2$, the expression simplifies to:

$$1 + (f'(x))^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2.$$

Substituting this back into the integral yields:

$$L = \int_0^2 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx = \int_0^2 \frac{e^x + e^{-x}}{2}\, dx.$$

We can split the integral for easier evaluation:

$$L = \frac{1}{2}\int_0^2 e^x\, dx + \frac{1}{2}\int_0^2 e^{-x}\, dx.$$

Evaluating the antiderivatives from 0 to 2:

$$\begin{aligned}
L &= \frac{1}{2}\left[e^x\right]_0^2 + \frac{1}{2}\left[-e^{-x}\right]_0^2 \\
&= \frac{1}{2}(e^2 - 1) + \frac{1}{2}(-e^{-2} - (-1)) \\
&= \frac{e^2 - 1 - e^{-2} + 1}{2} \\
&= \frac{e^2 - e^{-2}}{2}.
\end{aligned}$$

3. Consider the curve defined by $x = g(y) = \frac{1}{3}y^{3/2} - y^{1/2}$ on the interval $[1, 4]$. The arc length is given by:

$$L = \int_a^b \sqrt{1 + (g'(y))^2}\, dy.$$

We begin by computing the derivative with respect to y:

$$g'(y) = \frac{d}{dy}\left(\frac{1}{3}y^{3/2} - y^{1/2}\right) = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}.$$

Next, we simplify the integrand $1 + (g'(y))^2$:

$$\begin{aligned} 1 + \left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2 &= 1 + \frac{1}{4}\left(y - 2 + y^{-1}\right) \\ &= \frac{4 + y - 2 + y^{-1}}{4} \\ &= \frac{y + 2 + y^{-1}}{4}. \end{aligned}$$

This expression forms a perfect square:

$$\frac{(y^{1/2} + y^{-1/2})^2}{4} = \left(\frac{y^{1/2} + y^{-1/2}}{2}\right)^2.$$

The arc length integral therefore becomes:

$$L = \int_1^4 \frac{y^{1/2} + y^{-1/2}}{2}\, dy = \frac{1}{2}\int_1^4 y^{1/2}\, dy + \frac{1}{2}\int_1^4 y^{-1/2}\, dy.$$

Computing the antiderivatives:

$$\int y^{1/2}\, dy = \frac{2}{3}y^{3/2} \quad \text{and} \quad \int y^{-1/2}\, dy = 2y^{1/2}.$$

Evaluating from 1 to 4:

$$\begin{aligned} L &= \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_1^4 + \frac{1}{2}\left[2y^{1/2}\right]_1^4 \\ &= \frac{1}{2}\left(\frac{2}{3}(8) - \frac{2}{3}(1)\right) + \frac{1}{2}\left(2(2) - 2(1)\right) \\ &= \frac{1}{2}\left(\frac{14}{3}\right) + \frac{1}{2}(2) \\ &= \frac{7}{3} + 1 = \frac{10}{3}. \end{aligned}$$

4. We wish to find the arc length of the curve $x = \frac{2}{3}y^{3/2}$ over the interval $[0, 4]$. First, compute the derivative $g'(y)$:

$$g'(y) = \frac{d}{dy}\left(\frac{2}{3}y^{3/2}\right) = y^{1/2}.$$

Substituting this into the arc length formula:

$$L = \int_0^4 \sqrt{1 + (g'(y))^2}\, dy = \int_0^4 \sqrt{1 + y}\, dy.$$

To evaluate this integral, we use the substitution $u = 1 + y$, which implies $du = dy$. The limits of integration change as follows:

- When $y = 0$, $u = 1$.
- When $y = 4$, $u = 5$.

The integral becomes:

$$L = \int_1^5 u^{1/2}\, du.$$

Evaluating the antiderivative:

$$L = \left[\frac{2}{3}u^{3/2}\right]_1^5 = \frac{2}{3}\left(5^{3/2} - 1^{3/2}\right) = \frac{2}{3}\left(5\sqrt{5} - 1\right).$$

Thus, the arc length is:

$$L = \frac{10\sqrt{5} - 2}{3}.$$

5. Consider the parametric curve given by $x = t^2$ and $y = t^3$ for $t \in [0,1]$. The derivatives with respect to t are:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2.$$

The arc length integral is:

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt.$$

We can factor t^2 out from under the radical:

$$L = \int_0^1 \sqrt{t^2(4 + 9t^2)}\, dt = \int_0^1 t\sqrt{4 + 9t^2}\, dt.$$

We use the substitution $u = 4 + 9t^2$, so $du = 18t\, dt$, or $t\, dt = \frac{1}{18}du$. The limits change from $t \in [0,1]$ to $u \in [4, 13]$.

$$L = \frac{1}{18}\int_4^{13} u^{1/2}\, du.$$

Evaluating the integral:

$$L = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) = \frac{1}{27}\left(13\sqrt{13} - 8\right).$$

6. We determine the arc length of one arch of the cycloid defined by $x = t - \sin t$ and $y = 1 - \cos t$ for $t \in [0, 2\pi]$.

   First, we compute the derivatives:
   $$\frac{dx}{dt} = 1 - \cos t, \quad \frac{dy}{dt} = \sin t.$$

   The integrand for the arc length is:
   $$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(1 - \cos t)^2 + \sin^2 t}.$$

   Expanding and simplifying the terms inside the square root:
   $$\begin{aligned}(1 - \cos t)^2 + \sin^2 t &= 1 - 2\cos t + \cos^2 t + \sin^2 t \\ &= 1 - 2\cos t + 1 \\ &= 2 - 2\cos t.\end{aligned}$$

   Thus, the integral becomes:
   $$L = \int_0^{2\pi} \sqrt{2(1 - \cos t)}\, dt.$$

   Using the half-angle identity $1 - \cos t = 2\sin^2 \frac{t}{2}$, we substitute:
   $$L = \int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt = \int_0^{2\pi} 2\left|\sin \frac{t}{2}\right|\, dt.$$

   Since $\sin \frac{t}{2} \geq 0$ for $t \in [0, 2\pi]$, we can remove the absolute value bars:
   $$L = 2\int_0^{2\pi} \sin \frac{t}{2}\, dt.$$

   Let $u = \frac{t}{2}$, so $du = \frac{1}{2}dt \implies dt = 2du$. The limits become 0 to π:
   $$L = 2\int_0^{\pi} \sin u\,(2du) = 4\int_0^{\pi} \sin u\, du.$$

   Finally, evaluating the definite integral:
   $$L = 4\left[-\cos u\right]_0^{\pi} = 4\left(-\cos \pi - (-\cos 0)\right) = 4(1 + 1) = 8.$$

10.3 Intro to Polar Coordinates (Solutions)

1. A
2. C
3. D
4. B
5. B
6. A
7. B
8. D

10.4 Calculus in Polar Coordinates (Solutions)

Areas Between Polar Curves

1. **Set up an integral to find the area enclosed by the cardioid:**

$$r = 2(1 + \cos\theta).$$

Solution:

- The formula for the area enclosed by a polar curve is:

$$A = \frac{1}{2}\int_a^b r^2 \, d\theta.$$

- Substituting $r = 2(1 + \cos\theta)$:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} [2(1+\cos\theta)]^2 \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left[4(1 + 2\cos\theta + \cos^2\theta)\right] d\theta. \end{aligned}$$

- Expanding the integral:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} 4 + 8\cos\theta + 4\cos^2\theta \, d\theta \\ &= \frac{1}{2}\left[\int_0^{2\pi} 4\, d\theta + \int_0^{2\pi} 8\cos\theta \, d\theta + \int_0^{2\pi} 4\cos^2\theta \, d\theta\right]. \end{aligned}$$

- Evaluating each integral:
  - $\int_0^{2\pi} 4\, d\theta = 4(2\pi) = 8\pi$.
  - $\int_0^{2\pi} 8\cos\theta \, d\theta = 0$.
  - Using the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$:

$$\int_0^{2\pi} 4\cos^2\theta \, d\theta = 4\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta.$$

  - Splitting the integral:

$$\begin{aligned} \int_0^{2\pi} 4\cos^2\theta \, d\theta &= 4\left[\frac{1}{2}\int_0^{2\pi} 1\, d\theta + \frac{1}{2}\int_0^{2\pi} \cos 2\theta \, d\theta\right] \\ &= 4\left[\frac{1}{2}(2\pi) + \frac{1}{2}(0)\right] = 4[\pi] = 4\pi. \end{aligned}$$

- Summing the results:

$$A = \frac{1}{2}[8\pi + 0 + 4\pi] = \frac{1}{2}(12\pi) = 6\pi.$$

- **Final Answer:**

$$\boxed{6\pi}$$

2. **Find the area common to both polar curves:**

$$r = 3 + \cos\theta, \quad r = 3 - \cos\theta.$$

Solution:

- To find the intersection points, set $r_1 = r_2$:

$$3 + \cos\theta = 3 - \cos\theta.$$

- Solving for θ:

$$\cos\theta = -\cos\theta \quad \Rightarrow \quad \cos\theta = 0.$$

- This occurs at:

$$\theta = \frac{\pi}{2}, \quad \theta = \frac{3\pi}{2}.$$

3. **Find the area enclosed by the four-leaved rose:**

$$r = 3\cos(2\theta).$$

Solution:

- **Determine Limits for One Petal:**

$$3\cos(2\theta) = 0 \quad \Rightarrow \quad \cos(2\theta) = 0.$$

This yields

$$2\theta = \pm\frac{\pi}{2} \quad \Rightarrow \quad \theta = \pm\frac{\pi}{4},$$

so one petal is traced when

$$\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right].$$

- **Area of One Petal:**

$$\begin{aligned} A_{\text{petal}} &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \Big(3\cos(2\theta)\Big)^2 \, d\theta \\ &= \frac{9}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta. \end{aligned}$$

- **Substitute $u = 2\theta$:**

$$d\theta = \frac{du}{2}, \quad \theta = -\frac{\pi}{4} \Rightarrow u = -\frac{\pi}{2}, \quad \theta = \frac{\pi}{4} \Rightarrow u = \frac{\pi}{2}.$$

Hence,

$$\begin{aligned} A_{\text{petal}} &= \frac{9}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta \\ &= \frac{9}{2}\int_{-\pi/2}^{\pi/2} \cos^2(u)\, \frac{du}{2} \\ &= \frac{9}{4}\int_{-\pi/2}^{\pi/2} \cos^2(u)\, du \\ &= \frac{9}{4}\int_{-\pi/2}^{\pi/2} \frac{1+\cos(2u)}{2}\, du \\ &= \frac{9}{8}\int_{-\pi/2}^{\pi/2} (1+\cos(2u))\, du \\ &= \frac{9}{8}\left[\left(u + \frac{\sin(2u)}{2}\right)\right]_{-\pi/2}^{\pi/2} \\ &= \frac{9}{8}\cdot\pi \end{aligned}$$

- **Total Area:**

$$A_{\text{total}} = 4\cdot A_{\text{petal}} = 4\cdot\frac{9\pi}{8} = \frac{9\pi}{2}.$$

4. **Compute the area inside one petal of the rose curve:**

$$r = 2\sin(3\theta).$$

Solution:

- The formula for the area enclosed by a polar curve is:

$$A = \frac{1}{2}\int_a^b r^2 \, d\theta.$$

- A three-petal rose $r = 2\sin(3\theta)$ has three identical petals.
- One petal is traced when θ runs from 0 to $\frac{\pi}{3}$.
- The area of one petal is:

$$A = \frac{1}{2}\int_0^{\pi/3} (2\sin(3\theta))^2 \, d\theta.$$

- Expanding the square:

$$A = \frac{1}{2}\int_0^{\pi/3} 4\sin^2(3\theta) \, d\theta.$$

- Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$A = \frac{1}{2}\int_0^{\pi/3} 4 \times \frac{1-\cos 6\theta}{2} \, d\theta.$$

- Splitting the integral:

$$A = \frac{1}{2}\left[\int_0^{\pi/3} 2 \, d\theta - \int_0^{\pi/3} 2\cos 6\theta \, d\theta\right].$$

- Evaluating:
  - $\int_0^{\pi/3} 2 \, d\theta = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$.
  - $\int_0^{\pi/3} 2\cos 6\theta \, d\theta = 2 \times \frac{\sin 6\theta}{6}\Big|_0^{\pi/3} = 0$.
- So the final area is:

$$A = \frac{1}{2} \times \frac{2\pi}{3} = \frac{\pi}{3}.$$

- **Final Answer:**

$$\boxed{\frac{\pi}{3}}$$

5. **Find the area inside $r = 2 + \cos\theta$ and outside $r = 1$.**

Solution:

The area enclosed between two polar curves is given by:

$$A = \frac{1}{2}\int_a^b \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta.$$

Here, $r_{\text{outer}} = 2 + \cos\theta$ and $r_{\text{inner}} = 1$.

Step 1: Find the Limits of Integration

The curves intersect when:

$$2 + \cos\theta = 1.$$

Solving for θ: $\cos\theta = -1 \quad \Rightarrow \quad \theta = \pi$. Since we are computing the total enclosed area, we integrate from $\theta = 0$ to $\theta = 2\pi$.

Step 2: Integrate

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} \left((2+\cos\theta)^2 - 1^2\right) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(4 + 4\cos\theta + \cos^2\theta - 1\right) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(3 + 4\cos\theta + \cos^2\theta\right) d\theta \\
&= \frac{1}{2}\left[\int_0^{2\pi} 3\,d\theta + \int_0^{2\pi} 4\cos\theta\,d\theta + \int_0^{2\pi} \cos^2\theta\,d\theta\right].
\end{aligned}$$

Computing each integral:

$$\begin{aligned}
&\int_0^{2\pi} 3\,d\theta = 3(2\pi) = 6\pi, \\
&\int_0^{2\pi} 4\cos\theta\,d\theta = 0.
\end{aligned}$$

Using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, we rewrite:

$$\int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta.$$

Splitting:

$$\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\int_0^{2\pi} 1\,d\theta + \frac{1}{2}\int_0^{2\pi} \cos 2\theta\,d\theta.$$

Since $\int_0^{2\pi} \cos 2\theta\,d\theta = 0$, we obtain:

$$\frac{1}{2}\cdot 2\pi = \pi.$$

Step 3: Compute the Final Result

$$\begin{aligned}
A &= \frac{1}{2}(6\pi + 0 + \pi) \\
&= \frac{7\pi}{2}.
\end{aligned}$$

6. **Find the area inside $r = 6\sin\theta$ and outside $r = 3$.**

Solution:

The area enclosed between two polar curves is given by:

$$A = \frac{1}{2}\int_a^b \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta.$$

Here, $r_{\text{outer}} = 6\sin\theta$ and $r_{\text{inner}} = 3$.

Step 1: Find the Limits of Integration

The curves intersect when:

$$6\sin\theta = 3 \quad \Rightarrow \quad \sin\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}.$$

We integrate from $\theta = \frac{\pi}{6}$ to $\theta = \frac{5\pi}{6}$.

Step 2: Set Up the Integral

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((6\sin\theta)^2 - 3^2\right) d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36\sin^2\theta - 9\right) d\theta. \end{aligned}$$

Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, we substitute:

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36 \cdot \frac{1-\cos 2\theta}{2} - 9\right) d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (9 - 18\cos 2\theta)\, d\theta. \end{aligned}$$

Step 3: Evaluate the Integral

Computing each term:

$$\begin{aligned} \int_{\pi/6}^{5\pi/6} 9\, d\theta &= 9\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = 9 \cdot \frac{4\pi}{6} = 6\pi, \\ \int_{\pi/6}^{5\pi/6} 18\cos 2\theta\, d\theta &= 18 \cdot \frac{\sin 2\theta}{2}\Big|_{\pi/6}^{5\pi/6} \\ &= 9\left(\sin\frac{5\pi}{3} - \sin\frac{\pi}{3}\right) \\ &= -9\sqrt{3} \end{aligned}$$

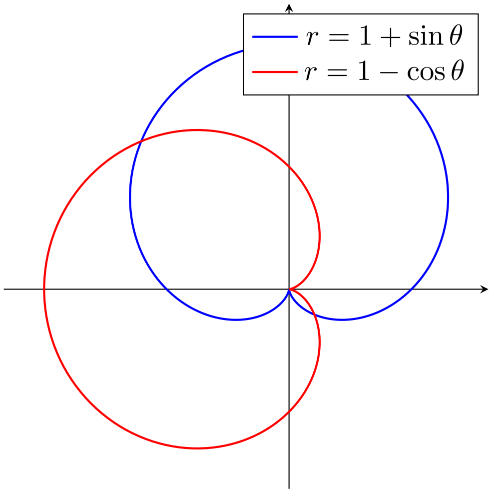
Step 4: Compute the Final Result

$$\begin{aligned} A &= \frac{1}{2}\left(6\pi + 9\sqrt{3}\right) \\ &= 3\pi + \frac{9\sqrt{3}}{2}. \end{aligned}$$

Tangent Lines and Arc Length

1. Find all points of intersection of the curves:

$$r = 1 + \sin\theta, \quad r = 1 - \cos\theta.$$



Solution:

Step 1: Justifying the Range $[0, 2\pi]$

While the equation $r = 1 + \sin\theta$ completes one full trace over $0 \le \theta \le \pi$, and $r = 1 - \cos\theta$ also completes its trace over $[0, \pi]$, we must consider θ in the full range $[0, 2\pi]$.

The reason is that in polar coordinates, a curve may intersect itself or intersect another curve at different angles, sometimes occurring at θ values outside $[0, \pi]$. Additionally, the same point can have multiple representations in polar form, meaning that intersections might not be obvious just by looking at $[0, \pi]$.

Step 2: Finding Intersection Points

To find intersections, we set the equations equal to each other:

$$1 + \sin\theta = 1 - \cos\theta.$$

Simplifying:

$$\sin\theta + \cos\theta = 0.$$

Dividing by $\cos\theta$ (where valid):

$$\tan\theta = -1.$$

The general solutions to $\tan\theta = -1$ are:

$$\theta = \frac{3\pi}{4} + k\pi, \quad k \in \mathbb{Z}.$$

Restricting to $[0, 2\pi]$, the valid solutions are:

$$\theta = \frac{3\pi}{4}, \quad \theta = \frac{7\pi}{4}.$$

Step 3: Compute r at These Intersections

Substituting into $r = 1 + \sin\theta$:

$$r = 1 + \sin\frac{3\pi}{4} = 1 + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2}.$$

$$r = 1 + \sin\frac{7\pi}{4} = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}.$$

Thus, two intersection points are:

$$\left(\frac{2+\sqrt{2}}{2}, \frac{3\pi}{4}\right), \quad \left(\frac{2-\sqrt{2}}{2}, \frac{7\pi}{4}\right).$$

Step 4: Checking for the Origin as an Intersection

A point is at the origin if $r = 0$ for some θ. Setting $r = 0$ in both equations:

- From $r = 1 + \sin\theta$, setting $1 + \sin\theta = 0$ gives:

$$\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}.$$

- From $r = 1 - \cos\theta$, setting $1 - \cos\theta = 0$ gives:

$$\cos\theta = 1 \Rightarrow \theta = 0, 2\pi.$$

Since both equations independently achieve $r = 0$ at different values of θ, they both pass through the origin, meaning $(0, 0)$ must be included as an intersection.

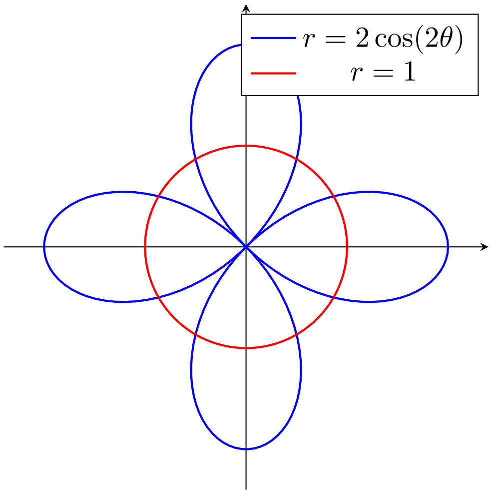
Step 5: Final Answer Including the Origin

The full set of intersection points is:

$$\left(\frac{2+\sqrt{2}}{2}, \frac{3\pi}{4}\right), \quad \left(\frac{2-\sqrt{2}}{2}, \frac{7\pi}{4}\right), \quad (0, 0).$$

2. Find all points of intersection of the curves:

$$r = 2\cos 2\theta, \quad r = 1.$$



Solution:

Step 1: Solve for Intersections with $r = 1$

To find where the curves intersect, we first solve:

$$2\cos 2\theta = 1.$$

Solving for θ:

$$\cos 2\theta = \frac{1}{2}.$$

The general solutions for $\cos x = \frac{1}{2}$ are:

$$2\theta = \pm\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}.$$

Dividing by 2:

$$\theta = \pm\frac{\pi}{6} + k\pi.$$

Restricting to $0 \le \theta < 2\pi$, the valid solutions are:

$$\theta = \frac{\pi}{6}, \quad \theta = \frac{5\pi}{6}, \quad \theta = \frac{7\pi}{6}, \quad \theta = \frac{11\pi}{6}.$$

Since we set $r = 1$, the intersection points are:

$$(1, \frac{\pi}{6}), \quad (1, \frac{5\pi}{6}), \quad (1, \frac{7\pi}{6}), \quad (1, \frac{11\pi}{6}).$$

Step 2: Solve for Intersections with $r = -1$

A point (r, θ) is equivalent to $(-r, \theta + \pi)$, so we solve:

$$2\cos 2\theta = -1.$$

Dividing by 2:

$$\cos 2\theta = -\frac{1}{2}.$$

The general solutions for $\cos x = -\frac{1}{2}$ are:

$$2\theta = \pm\frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}.$$

Dividing by 2:

$$\theta = \pm\frac{\pi}{3} + k\pi.$$

Restricting to $0 \leq \theta < 2\pi$, the valid solutions are:

$$\theta = \frac{\pi}{3}, \quad \theta = \frac{2\pi}{3}, \quad \theta = \frac{4\pi}{3}, \quad \theta = \frac{5\pi}{3}.$$

Since these correspond to $r = -1$, we convert to equivalent positive r representations:

$$(-1, \frac{\pi}{3}) = (1, \frac{4\pi}{3}),$$

$$(-1, \frac{2\pi}{3}) = (1, \frac{5\pi}{3}),$$

$$(-1, \frac{4\pi}{3}) = (1, \frac{\pi}{3}),$$

$$(-1, \frac{5\pi}{3}) = (1, \frac{2\pi}{3}).$$

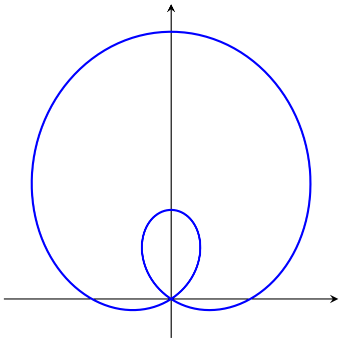
Final Answer: The full set of intersection points is:

$$(1, \frac{\pi}{6}), \quad (1, \frac{5\pi}{6}), \quad (1, \frac{7\pi}{6}), \quad (1, \frac{11\pi}{6}),$$

$$(1, \frac{\pi}{3}), \quad (1, \frac{2\pi}{3}), \quad (1, \frac{4\pi}{3}), \quad (1, \frac{5\pi}{3}).$$

3. Find the slope of the tangent line to the curve:

$$r = 1 + 2\sin\theta, \quad \theta = \frac{\pi}{6}.$$



Solution:

The slope of the tangent line to a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Step 1: Compute Cartesian Coordinates The Cartesian coordinates of a polar curve are:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Substituting $r = 1 + 2\sin\theta$:

$$x = (1 + 2\sin\theta)\cos\theta, \quad y = (1 + 2\sin\theta)\sin\theta.$$

Step 2: Compute Derivatives Differentiating x with respect to θ:

$$\frac{dx}{d\theta} = (2\cos\theta)\cos\theta - (1 + 2\sin\theta)\sin\theta.$$

Differentiating y with respect to θ:

$$\frac{dy}{d\theta} = (2\cos\theta)\sin\theta + (1 + 2\sin\theta)\cos\theta.$$

Step 3: Evaluate at $\theta = \frac{\pi}{6}$ First, compute necessary trigonometric values:

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Compute r at $\theta = \frac{\pi}{6}$:

$$r = 1 + 2\sin\frac{\pi}{6} = 1 + 2\cdot\frac{1}{2} = 2.$$

Compute $\frac{dx}{d\theta}$:

$$\begin{aligned}\frac{dx}{d\theta} &= (2\cos\frac{\pi}{6})\cos\frac{\pi}{6} - (1 + 2\sin\frac{\pi}{6})\sin\frac{\pi}{6}\\ &= \left(2\cdot\frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2} - (2)\frac{1}{2}\\ &= \left(\sqrt{3}\right)\frac{\sqrt{3}}{2} - 1\\ &= \frac{3}{2} - 1 = \frac{1}{2}.\end{aligned}$$

Compute $\frac{dy}{d\theta}$:

$$\begin{aligned}\frac{dy}{d\theta} &= (2\cos\frac{\pi}{6})\sin\frac{\pi}{6} + (1 + 2\sin\frac{\pi}{6})\cos\frac{\pi}{6}\\ &= \left(2\cdot\frac{\sqrt{3}}{2}\right)\frac{1}{2} + (2)\frac{\sqrt{3}}{2}\\ &= \left(\sqrt{3}\right)\frac{1}{2} + \sqrt{3}\\ &= \frac{\sqrt{3}}{2} + \sqrt{3} = \frac{3\sqrt{3}}{2}.\end{aligned}$$

Step 4: Compute the Slope

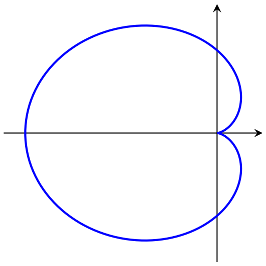
$$\frac{dy}{dx} = \frac{\frac{3\sqrt{3}}{2}}{\frac{1}{2}} = 3\sqrt{3}.$$

Final Answer:

$$\frac{dy}{dx} = 3\sqrt{3}.$$

4. Find points where the tangent line is horizontal or vertical for:

$$r = 2(1 - \cos\theta).$$



Solution:

The slope of the tangent line to a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Horizontal tangents occur where $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$. Vertical tangents occur where $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

Step 1: Compute First Derivatives

The Cartesian coordinates of the curve are:

$$x = r\cos\theta = 2(1 - \cos\theta)\cos\theta,$$
$$y = r\sin\theta = 2(1 - \cos\theta)\sin\theta.$$

Differentiating x with respect to θ:

$$\begin{aligned}\frac{dx}{d\theta} &= 2\left((1 - \cos\theta)(-\sin\theta) + \cos\theta\sin\theta\right)\\ &= 2\left(-\sin\theta + 2\cos\theta\sin\theta\right)\\ &= 2\sin\theta(2\cos\theta - 1).\end{aligned}$$

Differentiating y with respect to θ:

$$\begin{aligned}\frac{dy}{d\theta} &= 2\left((1 - \cos\theta)\cos\theta + \sin\theta\sin\theta\right)\\ &= 2\left(\cos\theta - \cos^2\theta + \sin^2\theta\right)\\ &= 2\left(\cos\theta - \cos 2\theta\right).\end{aligned}$$

Step 2: Solve $\frac{dy}{d\theta} = 0$

Setting $\frac{dy}{d\theta} = 0$:

$$\cos\theta - \cos 2\theta = 0.$$

Rearranging:

$$\cos\theta = \cos 2\theta.$$

The general solutions to $\cos A = \cos B$ are:

$$A = B + 2k\pi \quad \text{or} \quad A = -B + 2k\pi.$$

Applying $A = \theta$ and $B = 2\theta$, we get:

$$\theta = \frac{2k\pi}{3}.$$

Restricting to $0 \leq \theta \leq 2\pi$, the valid solutions are:

$$\theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 2\pi.$$

Step 3: Solve $\frac{dx}{d\theta} = 0$

Setting $\frac{dx}{d\theta} = 0$:

$$2\sin\theta(2\cos\theta - 1) = 0.$$

Solving each factor separately:

- $\sin\theta = 0$ gives:

$$\theta = 0, \pi, 2\pi.$$

- $2\cos\theta - 1 = 0$ gives:

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \quad \frac{5\pi}{3}.$$

Step 4: Apply L'Hôpital's Rule at $\theta = 0$ and $\theta = 2\pi$

At $\theta = 0$ and $\theta = 2\pi$, both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$, so we differentiate again:

$$\frac{d^2x}{d\theta^2} = 2(2\cos^2\theta - \cos\theta - 1 + \cos 2\theta).$$

$$\frac{d^2y}{d\theta^2} = 2(-\sin\theta + 2\sin 2\theta).$$

Evaluating at $\theta = 0$:

$$\frac{d^2x}{d\theta^2} = 2, \quad \frac{d^2y}{d\theta^2} = 0.$$

Since $\frac{d^2x}{d\theta^2} \neq 0$ and $\frac{d^2y}{d\theta^2} = 0$, $\theta = 0$ gives a horizontal tangent.

Evaluating at $\theta = 2\pi$:

$$\frac{d^2x}{d\theta^2} = 2, \quad \frac{d^2y}{d\theta^2} = 0.$$

Since $\frac{d^2x}{d\theta^2} \neq 0$ and $\frac{d^2y}{d\theta^2} = 0$, $\theta = 2\pi$ gives a horizontal tangent.

Final Answer:

The curve has horizontal tangents at:

$$\theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 2\pi.$$

The curve has vertical tangents at:

$$\theta = \pi, \quad \frac{\pi}{3}, \quad \frac{5\pi}{3}.$$

11.1 Sequences (Solutions)

1. We rewrite:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}.$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$, we get:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Thus, the sequence converges to 1.

2. The sequence oscillates between 1 and -1 and does not settle to a single value. Since it does not approach a single limit, the sequence diverges.

3. Since $\frac{1}{n^2} > 0$ for all n and $\frac{1}{n^2} \to 0$ as $n \to \infty$, the sequence converges to 0.

4. Dividing the numerator and denominator by n^2, we get:

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}}.$$

Since $\frac{1}{n^2} \to 0$, the limit is 1. Thus, the sequence converges to 1.

5. Applying L'Hôpital's Rule to the related function $f(x) = \frac{\ln x}{x}$, we differentiate:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Thus, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, meaning the sequence converges to 0.

6. Dividing by n:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}}.$$

Since $\frac{1}{n^2} \to 0$, we get:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1}} = 1.$$

Thus, the sequence converges to 1.

7. Since $|d_n| = \frac{1}{n} \to 0$, d_n converges to 0 as well.

8. Since $-1 \leq \cos(3n+1) \leq 1$, we have

$$-\frac{1}{n^2} \leq \frac{\cos(3n+1)}{n^2} \leq \frac{1}{n^2}.$$

Since $\frac{1}{n^2} \to 0$ as $n \to \infty$, by the Squeeze Theorem, the sequence converges to 0.

9. Dividing numerator and denominator by n^3:

$$c_n = \frac{\frac{n^2}{n^3} - \frac{3n}{n^3}}{\frac{n^3}{n^3} + \frac{5}{n^3}} = \frac{\frac{1}{n} - \frac{3}{n^2}}{1 + \frac{5}{n^3}}.$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} c_n = \frac{0-0}{1+0} = 0.$$

Thus, the sequence converges to 0.

10. Factor out n^4 inside the square root:

$$\sqrt{9n^4 + 4n^2} = \sqrt{n^4\left(9 + \frac{4}{n^2}\right)} = n^2\sqrt{9 + \frac{4}{n^2}}.$$

Thus,

$$a_n = \frac{n^2\sqrt{9 + \frac{4}{n^2}}}{n^3} = \frac{\sqrt{9 + \frac{4}{n^2}}}{n}.$$

As $n \to \infty$, the term $\sqrt{9 + \frac{4}{n^2}} \to 3$. Hence,

$$a_n \to \frac{3}{n} \to 0.$$

Therefore, the sequence $\{a_n\}$ converges and its limit is 0.

11. Dividing numerator and denominator by n:

$$b_n = \frac{5 + \frac{\sin(n)}{n}}{1 + \frac{10}{n}}.$$

Since $\frac{\sin(n)}{n} \to 0$ and $\frac{10}{n} \to 0$, we get:

$$\lim_{n\to\infty} b_n = \frac{5+0}{1+0} = 5.$$

Thus, the sequence converges to 5.

12. Dividing by n^3 in the numerator and n^3 inside the square root in the denominator:

$$c_n = \frac{1 + \frac{2}{n^3}}{\sqrt{1 + \frac{5}{n^4}}}.$$

Taking the limit:

$$\lim_{n\to\infty} c_n = \frac{1+0}{\sqrt{1+0}} = 1.$$

Thus, the sequence converges to 1.

13. Since $|\sin(5n)| \leq 1$, we have:

$$-\frac{1}{n^3} \leq \frac{\sin(5n)}{n^3} \leq \frac{1}{n^3}.$$

Since $\frac{1}{n^3} \to 0$, by the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$. The sequence converges to 0.

14. Dividing numerator and denominator by n^3:

$$b_n = \frac{\frac{3\sqrt{n}}{n^3} + 1}{1 + \frac{\sqrt{n}}{n^3}}.$$

Taking the limit as $n \to \infty$, the terms with $\frac{\sqrt{n}}{n^3} \to 0$, so:

$$\lim_{n\to\infty} b_n = \frac{0+1}{1+0} = 1.$$

Thus, the sequence converges to 1.

15. Dividing by n^3:

$$c_n = \frac{1 - \frac{2}{n^2}}{\sqrt{4 + \frac{7}{n^5}}}.$$

Taking the limit:

$$\lim_{n\to\infty} c_n = \frac{1-0}{\sqrt{4+0}} = \frac{1}{2}.$$

Thus, the sequence converges to $\frac{1}{2}$.

11.2 Series & Partial Sums

1. $S_n = 3 - \dfrac{1}{n}$.

$$\lim_{n\to\infty} S_n = 3 - \lim_{n\to\infty} \frac{1}{n} = 3.$$

2. $S_n = \dfrac{5n}{n+1}$.

$$S_n = \frac{5n}{n\left(1+\frac{1}{n}\right)} = \frac{5}{1+\frac{1}{n}} \longrightarrow 5.$$

3. $S_n = \dfrac{2n^2+1}{n^2+2n+1}$.

$$S_n = \frac{n^2\left(2+\frac{1}{n^2}\right)}{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} = \frac{2+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} \longrightarrow 2.$$

4. $S_n = \dfrac{4n^2}{n^2+1}$.

$$S_n = \frac{4}{1+\frac{1}{n^2}} \longrightarrow 4.$$

5. $S_n = 1 - \dfrac{2}{n^2+1}$.

$$\lim_{n\to\infty} S_n = 1 - 2\lim_{n\to\infty} \frac{1}{n^2+1} = 1.$$

6. $S_n = \dfrac{6n^2+3n+2}{2n^2+5n+1}$.

$$S_n = \frac{6+\frac{3}{n}+\frac{2}{n^2}}{2+\frac{5}{n}+\frac{1}{n^2}} \longrightarrow \frac{6}{2} = 3.$$

7. $S_n = \dfrac{5n}{\ln(n+1)+2n}$.

$$S_n = \frac{5}{\frac{\ln(n+1)}{n}+2}.$$

Since $\dfrac{\ln(n+1)}{n} \to 0$, it follows that

$$\lim_{n\to\infty} S_n = \frac{5}{2}.$$

8. $S_n = \dfrac{n+4}{\sqrt{n^2+9}}$.

$$S_n = \frac{n\left(1+\frac{4}{n}\right)}{n\sqrt{1+\frac{9}{n^2}}} = \frac{1+\frac{4}{n}}{\sqrt{1+\frac{9}{n^2}}} \longrightarrow 1.$$

9. $S_n = \dfrac{4n+\ln n}{3n+1}$.

$$S_n = \frac{4+\frac{\ln n}{n}}{3+\frac{1}{n}}.$$

Since $\dfrac{\ln n}{n} \to 0$, we obtain

$$\lim_{n\to\infty} S_n = \frac{4}{3}.$$

10. $S_n = \dfrac{n^2+1}{n\ln n+1}$.

$$S_n = \frac{n^2\left(1+\frac{1}{n^2}\right)}{n\ln n\left(1+\frac{1}{n\ln n}\right)} = \frac{n}{\ln n}\cdot\frac{1+\frac{1}{n^2}}{1+\frac{1}{n\ln n}}.$$

The second factor tends to 1, while $\dfrac{n}{\ln n}\to\infty$. Hence

$$\lim_{n\to\infty} S_n = \infty,$$

so the series diverges.