

Midterm 2 Study Guide

MATH2300 - Calculus II

Spring 2025

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Overview of Topics

6.1 - Areas Between Curves

- Definition of the area between two curves.
- Computing area using vertical and horizontal slices.
- Integrating with respect to x vs. integrating with respect to y .

6.2 - Volumes

- Computing volumes using known cross-sections.
- The Disk Method: $V = \pi \int_a^b [R(x)]^2 dx$.
- The Washer Method: $V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx$.
- Rotating around different axes.

6.3 - Cylindrical Shells

- The Shell Method formula: $V = 2\pi \int_a^b (\text{radius})(\text{height}) dx$.
- Choosing between the disk/washer method and the shell method.

6.5 - Average Value of a Function

- Definition: $f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$.
- Interpretation as the mean height of the function.

6.4 - Work

- Work as an integral: $W = \int_a^b F(x) dx$.
- Work done in stretching a spring (Hooke's Law).
- Work required to pump fluid.
- Work done by a variable force.

8.3 - Center of Mass

- Definition of center of mass for a system of particles.
- Center of mass for a continuous body.

10.1 - Parametric Curves

- Definition and examples of parametric equations.
- Eliminating the parameter.
- Graphing parametric curves.

10.2 - Calculus with Parametric Curves

- Derivatives: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
- Tangent lines to parametric curves.
- Finding horizontal and vertical tangents.

8.1/10.2 - Arc Length

- Arc length formula in Cartesian coordinates for functions $f(x)$ or $g(y)$.
- Arc length for parametric curves.

10.3 - Polar Coordinates

- Conversion between Cartesian and polar coordinates.
- Graphing polar curves.
- Common polar graphs: circles, spirals, limaçons, cardioids.

10.4 - Calculus in Polar Coordinates

- Derivatives in polar coordinates: $\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$.
- Area enclosed by a polar curve: $A = \frac{1}{2} \int_a^b r^2 d\theta$.
- Arc length: $L = \int_a^b \sqrt{r^2 + (r')^2} d\theta$.

11.1 - Sequences

- Definition of a sequence.
- Limits of sequences.
- Convergence and divergence of sequences.
- The Squeeze Theorem.
- Monotonic and bounded sequences.

Formulas

Trigonometric Identities

Pythagorean: $\sin^2 x + \cos^2 x = 1$

Tangent: $\tan^2 x + 1 = \sec^2 x$

Power-Reducing: $\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Product-to-Sum: $\sin x \cos x = \frac{1}{2} \sin 2x$

Double-Angle: $\sin 2x = 2 \sin x \cos x, \quad \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$

Areas Between Curves

$$A = \int_a^b [f(x) - g(x)] dx$$

Volumes

Cross Sections: $V = \int_a^b A(x) dx$

Disks/Washers: $V = \pi \int_a^b [R(x)^2 - r(x)^2] dx$

Shells: $V = 2\pi \int_a^b (\text{radius})(\text{height}) dx$

Average Value

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

Work

General: $W = \int_a^b F(x) dx$

Hooke's Law: $F = kx, \quad W = \int_0^x kx dx = \frac{1}{2}kx^2$

Center of Mass

Continuous: $\bar{x} = \frac{1}{A} \int_a^b xf(x) dx, \quad \bar{y} = \frac{1}{A} \int_a^b yf(x) dx$

Discrete: $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}$

Parametric Equations

Slope: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Tangent Line: $y - y(t_0) = \frac{dy/dt|_{t=t_0}}{dx/dt|_{t=t_0}}(x - x(t_0))$

Arc Length

$$y = f(x) : \quad L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

$$x = g(y) : \quad L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy$$

$$\text{Parametric:} \quad L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Polar Curves

$$\text{Conversion:} \quad x = r \cos \theta, \quad y = r \sin \theta$$

$$\text{Tangent Line:} \quad \frac{dy}{dx} = \frac{(dr/d\theta) \sin \theta + r \cos \theta}{(dr/d\theta) \cos \theta - r \sin \theta}$$

Polar Areas and Arc Length

$$\text{Area:} \quad A = \frac{1}{2} \int_{\alpha}^{\beta} [r(\theta)]^2 \, d\theta$$

$$\text{Arc Length:} \quad L = \int_{\alpha}^{\beta} \sqrt{[r(\theta)]^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Sequences

$$\text{Limit:} \quad \lim_{n \rightarrow \infty} a_n = L$$

$$\text{Arithmetic:} \quad a_n = a_1 + (n - 1)d$$

$$\text{Geometric:} \quad a_n = a_1 \cdot r^{n-1}$$

6.1 Areas Between Curves

Overview

Integrating with Respect to x

Definition (Area with Respect to x). Suppose a region S is bounded by the curves $y = f(x)$ and $y = g(x)$ and the horizontal lines $x = a$ and $x = b$, with

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b].$$

Then the area of S is given by

$$A = \int_a^b [f(x) - g(x)] \, dx.$$

Alternatively, if we denote by y_T the top boundary and by y_B the bottom boundary, then

$$A = \int_a^b [y_T - y_B] \, dy.$$

Theorem (Area Between Two Curves). Let f and g be continuous on $[a, b]$ with $f(x) \geq g(x)$ for all $x \in [a, b]$. Then the area A of the region between the curves is given by

$$A = \int_a^b [f(x) - g(x)] \, dx.$$

Remark. If $f(x)$ and $g(x)$ intersect in $[a, b]$ (i.e. the order of the functions changes), the region must be partitioned into subregions over which one function is consistently above the other. In such cases, the area is computed as

$$A = \int_a^b |f(x) - g(x)| \, dx,$$

with the integral split appropriately.

Integrating with Respect to y

Definition (Area with Respect to y). Suppose a region S is bounded by the curves $x = f(y)$ and $x = g(y)$ and the horizontal lines $y = c$ and $y = d$, with

$$f(y) \geq g(y) \quad \text{for all } y \in [c, d].$$

Then the area of S is given by

$$A = \int_c^d [f(y) - g(y)] \, dy.$$

Alternatively, if we denote by x_R the right boundary and by x_L the left boundary, then

$$A = \int_c^d [x_R - x_L] \, dy.$$

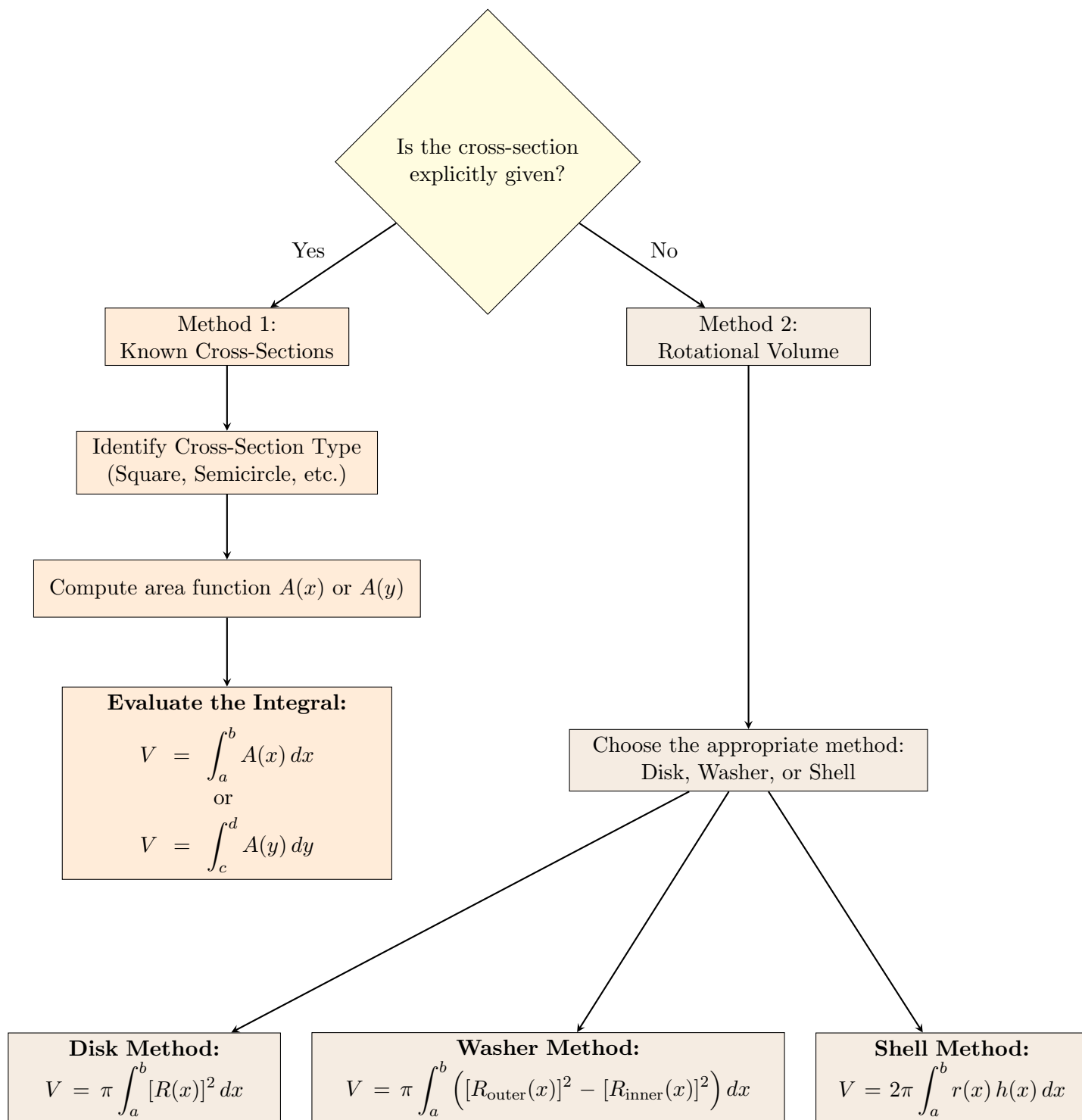
Areas Between Curves Problems

1. Find the area enclosed by the curves $y = x^2$ and $y = 4$.
2. Find the area between the curves $y = x^3$ and $y = x$.
3. Compute the area enclosed by the curves $y = \sin x$ and $y = \cos x$ over the interval $0 \leq x \leq \frac{\pi}{2}$.
4. Determine the area of the region bounded by $y = x^2 + 1$ and $y = 3$.
5. Find the area enclosed by the curves $y = e^x$ and $y = 4$ for $0 \leq x \leq \ln 4$.

6.2-6.3 Volume

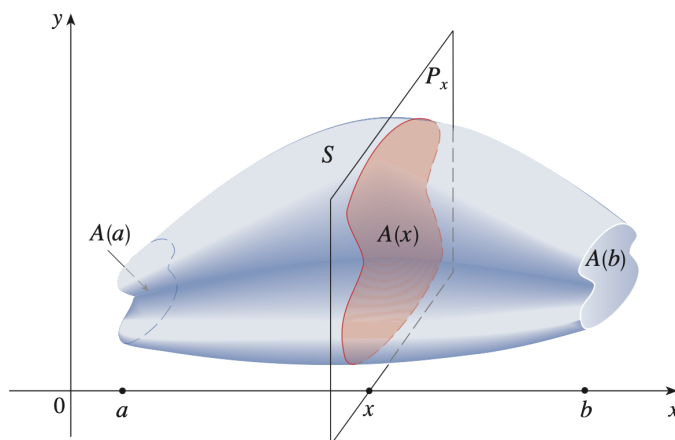
Volume: The Big Picture

Integration allows us to compute the total volume of a complex solid by summing up infinitely many small volume elements. By slicing the solid into simple shapes whose volumes we can calculate (e.g., disks, washers, or shells), integration provides a systematic way to add these pieces together.



Known Cross-Sections

When a solid has cross-sectional areas that are easy to compute, we can find its volume by “stacking” these slices.



Let S be a solid lying between $x = a$ and $x = b$. Suppose that for each x in $[a, b]$ the cross-sectional area perpendicular to the x -axis is given by $A(x)$. Then the volume V of S is defined by

$$V = \int_a^b A(x) dx.$$

Remark. This definition generalizes the familiar formula for a cylinder. For a cylinder with constant base area A and height h , we have $V = Ah$.

Common Cross-Section Shapes

1. Square Cross Sections. If the cross section perpendicular to the x -axis is a square with side length $s(x)$, then the area is

$$A(x) = [s(x)]^2.$$

2. Rectangular Cross Sections. For a rectangle with base $b(x)$ and height $h(x)$, the area is

$$A(x) = b(x) h(x).$$

3. Isosceles Right Triangle Cross Sections. If each cross section is an isosceles right triangle with leg length $s(x)$, then its area is given by

$$A(x) = \frac{1}{2} [s(x)]^2.$$

4. Equilateral Triangle Cross Sections. For an equilateral triangle with side length $s(x)$, the area is

$$A(x) = \frac{\sqrt{3}}{4} [s(x)]^2.$$

5. Circular Cross Sections (Disks). If the cross section is a circle (or disk) with radius $r(x)$, then

$$A(x) = \pi [r(x)]^2.$$

6. Semicircular Cross Sections. For a semicircular cross section with radius $r(x)$, the area is

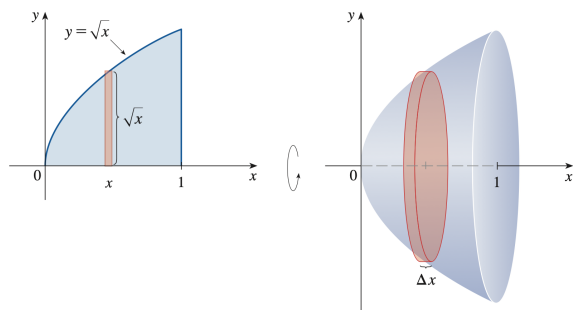
$$A(x) = \frac{1}{2} \pi [r(x)]^2.$$

Surfaces of Revolution

There are three primary slicing methods used to compute the volume of a solid of revolution:

Disk Method

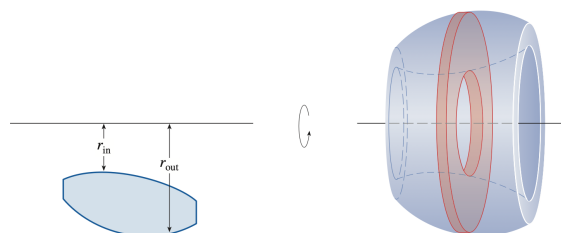
Slices are taken perpendicular to the axis of rotation. Each slice is a disk whose area is easy to compute.



$$V = \int_a^b \pi [R(x)]^2 dx$$

Washer Method

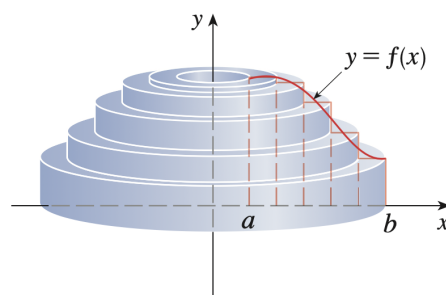
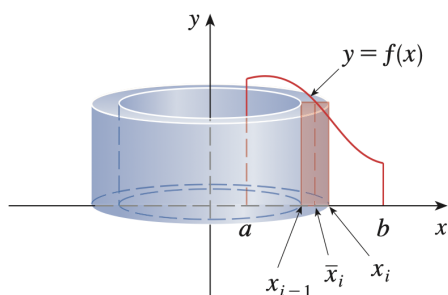
Slices are taken perpendicular to the axis of rotation. If there is a hole, each slice is washer whose area is easy to compute.



$$V = \int_a^b \pi ([R(x)]^2 - [r(x)]^2) dx$$

Shell Method

Slices are taken parallel to the axis of rotation. Each slice forms a cylindrical shell, and its volume is determined by its radius, height, and thickness.



$$V = 2\pi \int_a^b x f(x) dx$$

Volume Problems

Volumes by Known Cross-Sections

1. A solid has a circular base of radius 2 centered at the origin in the xy -plane. The cross-sections perpendicular to the x -axis (i.e., vertical slices) are squares. If the side of each square lies in the plane of the cross-section, find the volume of this solid.
2. A solid has a right isosceles triangular base in the xy -plane with vertices at $(0,0)$, $(4,0)$, and $(0,4)$. The cross-sections perpendicular to the x -axis are semicircles. Find the volume of this solid.
3. A solid has a square base in the xy -plane with corners at $(0,0)$, $(3,0)$, $(3,3)$, and $(0,3)$. The cross-sections perpendicular to the x -axis are equilateral triangles. Each triangle's base runs from $y = 0$ to $y = 3$. Find the volume of the solid.
4. A solid has a rectangular base in the xy -plane: $0 \leq x \leq 5$, $0 \leq y \leq 2$. Cross-sections perpendicular to the y -axis are right triangles whose legs lie along the base plane. Specifically, for each fixed y , the base of the right triangle is 5 (the full length in the x direction) and the height of the triangle is y . Find the volume of the solid.
5. A solid has a circular base of radius 1 (centered at the origin). Cross-sections perpendicular to the y -axis are equilateral triangles whose base is the chord of the circle at that y . Find the volume of this solid.
6. A solid has a right isosceles triangular base in the xy -plane with vertices at $(0,0)$, $(6,0)$, and $(0,6)$. For each fixed x in the interval $[0,6]$, the vertical line through the base (parallel to the y -axis) intersects the triangle from $y = 0$ up to $y = 6 - x$. At each such x , a rectangle is erected whose:
 - Base (lying in the xy -plane) is the segment from $y = 0$ to $y = 6 - x$, of length $6 - x$, and
 - Height (in the z -direction) is defined to be twice the length of that segment, i.e., $2(6 - x)$.

Find the volume of the solid.

7. A solid has an elliptical base given by $\frac{x^2}{9} + \frac{y^2}{4} \leq 1$. Cross-sections perpendicular to the y -axis are squares whose side length equals the x -span of the ellipse at that y . Find the volume.
8. A solid sits over a square base $[0,4] \times [0,4]$ in the xy -plane. The cross-sections perpendicular to the x -axis are isosceles trapezoids with:
 - One base along the y -range from 0 to 4 (i.e., length 4).
 - The other base has length 2, centered with respect to the first base.
 - The height (thickness in the z -direction) of each trapezoid is $h = 1$ (constant).

Find the volume of the solid.

9. A solid has a base in the xy -plane bounded by $y = 0$, $x = 2$, and $y = 4 - x^2$ (which intersects $x = 2$ at $y = 0$). The cross-sections perpendicular to the x -axis are rectangles. The width of each rectangle is from $y = 0$ to $y = 4 - x^2$, and its height (out of the plane) is twice the y -value at that slice. Find the volume.

Disk Method

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x -axis.

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x^2, \quad y = 0$$

about the x -axis.

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 0, \quad y = 3$$

about the y -axis.

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2x, \quad x = 0, \quad y = 4$$

about the y -axis.

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x-3}, \quad y = 0, \quad x = 7$$

about the vertical line $x = 7$.

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x+1}, \quad y = 0, \quad x = 3$$

about the vertical line $x = -1$.

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - \sqrt{x}, \quad y = 2, \quad x = 1$$

about the horizontal line $y = 2$.

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = -1 + \sqrt{x}, \quad y = -1, \quad x = 1$$

about the horizontal line $y = -1$.

Washer Method

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = x^2$$

about the x -axis.

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = x, \quad x = 0$$

about the x -axis.

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y, \quad x = y^2$$

about the y -axis.

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = 4 - x, \quad \text{and} \quad y = 0$$

about the y -axis.

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 5 - y^2, \quad y = -1, \quad \text{and} \quad y = 1$$

about the vertical line $x = 7$.

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x - 1, \quad y = 4 - x, \quad y = 0, \quad \text{and} \quad y = 1$$

about the vertical line $x = -1$.

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the horizontal line $y = 2$.

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -1$.

Cylindrical Shell Method

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x -axis.

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = 0, \quad x = 0$$

about the x -axis.

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 4, \quad x = 0$$

about the y -axis.

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 9$$

about the y -axis.

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the vertical line $x = -1$.

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x + 2, \quad y = 0, \quad x = -1, \quad x = 3$$

about the vertical line $x = 5$.

7. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4, \quad y = 0$$

about the horizontal line $y = 3$.

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -2$.

Additional Problems

1. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4$$

about the y -axis.

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x$$

about the vertical line $x = -2$.

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

about the line $y = 1$.

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the y -axis.

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = 1 - y^2, \quad x = 0$$

about the y -axis.

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x^2$$

about the line $y = -1$.

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln x, \quad y = 0, \quad x = 1, \quad x = e^2$$

about the y -axis.

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \frac{4}{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the line $x = -2$.

6.5 Average Value of a Function

Overview

In many practical applications (such as finding the average temperature over a day) we wish to define the average value of a function when its values are known at infinitely many points. The idea is analogous to averaging finitely many numbers.

Definition (Average Value of a Function). Let f be a continuous function on the interval $[a, b]$. The *average value* of f on $[a, b]$ is defined by

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Remark. This definition arises naturally by partitioning the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$ and considering the average of the function values:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}.$$

As $n \rightarrow \infty$, this sum becomes

$$f_{\text{avg}} = \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \frac{1}{b-a} \int_a^b f(x) dx.$$

The Mean Value Theorem for Integrals

Theorem (Mean Value Theorem for Integrals). If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx,$$

or equivalently,

$$f(c)(b-a) = \int_a^b f(x) dx.$$

In other words, c is the point at which f attains its average value.

Remark. Geometrically, for a positive function f , this theorem guarantees that there is a point c in $[a, b]$ such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b .

Average Value of a Function Problems

1. Let $f(x) = x^2 + 1$ on the interval $[0, 3]$.
 - Find the average value of f on $[0, 3]$.
 - Find all $c \in [0, 3]$ such that $f(c)$ equals the average value.
2. Let $f(x) = (x - 1)^2$ on the interval $[0, 2]$.
 - Find the average value of f on $[0, 2]$.
 - Find all $c \in [0, 2]$ such that $f(c)$ equals the average value.
3. Let $f(x) = 2 \cos x + 1$ on the interval $[0, \pi]$.
 - Find the average value of f on $[0, \pi]$.
 - Find all $c \in [0, \pi]$ such that $f(c)$ equals the average value.
4. Let $f(x) = e^x$ on the interval $[0, 1]$.
 - Find the average value of f on $[0, 1]$.
 - Find all $c \in [0, 1]$ such that $f(c)$ equals the average value.
5. Let $f(x) = x$ on the interval $[-2, 2]$.
 - Find the average value of f on $[-2, 2]$.
 - Find all $c \in [-2, 2]$ such that $f(c)$ equals the average value.

6.4 Work

Overview

Definition (Force and Newton’s Second Law). If an object of mass m moves along a straight line with position function $s(t)$, then its *acceleration* is

$$a(t) = \frac{d^2 s}{dt^2}.$$

By Newton’s Second Law of Motion, the *force* F on the object is

$$F = m a = m \frac{d^2 s}{dt^2}.$$

Definition (Work for Constant Force). If a constant force F acts on an object and moves it through a distance d in the direction of the force, then the *work* W done by the force is

$$W = F \cdot d \quad (\text{force} \times \text{distance}).$$

Definition (Units of Work).

- In the **SI (metric) system**, force is measured in *newtons* (N) and distance in *meters* (m). Hence work is measured in *newton-meters*, also called *joules* (J).

$$1 \text{ joule} = 1 \text{ newton} \cdot 1 \text{ meter}.$$

- In the **US Customary system**, force is measured in *pounds* (lb) and distance in *feet* (ft). Hence work is measured in *foot-pounds* (ft-lb).

$$1 \text{ ft-lb} \approx 1.36 \text{ joules}.$$

Definition (Weight vs. Mass).

- An object’s *mass* (in kilograms, kg) times the acceleration due to gravity ($\approx 9.8 \text{ m/s}^2$) gives its *weight* (in newtons).

$$\text{weight} = m \times 9.8 \quad (\text{in newtons}).$$

- In the US system, the word “pound” itself denotes a force (weight). Hence an object that weighs W pounds has mass $m = W/g$ (in “slugs”) if $g \approx 32 \text{ ft/s}^2$.

Theorem (Work for a Variable Force). Let an object move along the x -axis from $x = a$ to $x = b$, acted upon by a *continuous* force $f(x)$ that depends on the position x . The *work* W done by this force is given by the definite integral

$$W = \int_a^b f(x) dx.$$

Interpretation: We partition the interval $[a, b]$ into small segments, approximate the (nearly constant) force on each segment, multiply by the small distance, and then let the partition become finer. The limit of these Riemann sums is the above integral.

Definition (Hooke's Law for Springs). If a spring is stretched (or compressed) x units from its natural length (where x is not too large), the *force* F required to hold it there obeys

$$F(x) = kx,$$

where k is a positive constant called the *spring constant*. Consequently, the work required to stretch a spring from $x = a$ to $x = b$ (beyond its natural length) is

$$W = \int_a^b kx \, dx = \frac{k}{2} [b^2 - a^2].$$

Typical Work Applications

1. **Stretching Springs.** Using Hooke's Law, if a spring is stretched from $x = a$ to $x = b$ (measured from natural length), then

$$W = \int_a^b kx \, dx = \frac{k}{2} [b^2 - a^2].$$

2. **Lifting Objects.** If you lift an object of *constant weight* F through a distance d , the work is $W = Fd$. But for a *variable weight* distribution (e.g. a hanging rope, cable, or chain), one usually slices the rope into small segments, each with (approximately) constant weight and distance, then integrates.
3. **Pumping Liquids.** To find the work done in pumping water (or other fluid) from one level to another, one typically:
 - (a) Slices the fluid into thin horizontal layers.
 - (b) Finds the volume (or mass) of each slice.
 - (c) Multiplies by g (acceleration due to gravity) to get the weight of the slice.
 - (d) Multiplies by the distance each slice must be moved.
 - (e) Integrates over the entire depth of fluid.

Work Problems

Spring Problems

1. A spring has a natural length of 20 m. A force of 12 N is required to stretch the spring to 25 m. Determine the work required to stretch the spring from 20 m to 30 m.
2. A spring has a natural length of 15 m. A force of 10 N is required to stretch the spring to 18 m. Determine the work required to stretch the spring from 16 m to 22 m.
3. A spring has a natural length of 30 m. A force of 8 N is required to stretch the spring to 35 m. Determine the work required to compress the spring from 30 m to 20 m.

Cable Problems

1. A 100-meter-long cable with a linear density of 5 kg/m is hanging from a winch at the top of a well. The cable is initially fully extended into the well and is lifted to the top. Compute the work required to lift the entire cable.
2. A 50-meter-long chain with a linear density of 8 kg/m is hanging from a pulley at the top of a mine shaft. The chain is initially fully extended into the shaft and is lifted to the top. Compute the work required to lift the entire chain.
3. A 60-meter-long rope with a linear density of 3 kg/m is hanging over the edge of a cliff, with one end secured at the top and the other end dangling freely. The rope is slowly lifted until it is fully coiled at the top of the cliff. Compute the work required to lift the rope.
4. A 30-meter-long anchor chain with a linear density of 12 kg/m is hanging from the side of a ship, with one end attached to the ship and the other submerged in the water. The chain is hoisted onto the deck of the ship. Compute the work required to lift the entire chain onto the ship.

Tank Problems

For each problem, set up (but do not solve) the work integral for pumping water out of the tank. In all setups, use the density of water as $1000 \text{ (kg/m}^3\text{)}$ and gravity as $9.8 \text{ (m/s}^2\text{)}$.

1. **Rectangular Tank with Triangular Ends:** A tank is 6 m long, and its end view is an isosceles triangle with a base of 2 m and a height of 3 m. Water is pumped out through a spout located 0.5 m above the top of the tank.
2. **Cylindrical Tank with a Spout:** A vertical cylindrical tank is 4 m high with a circular cross section of radius 1.5 m. Water is pumped out through a spout that is 0.3 m above the top of the tank.
3. **Inverted Conical Tank:** An inverted conical tank has a height of 3 m and an open top with a radius of 1 m. Water is pumped out to a spout 0.2 m above the top.
4. **Spherical Tank:** A spherical tank of radius 2 m is completely filled with water. Water is pumped out through a spout located 0.1 m above the top of the sphere.
5. **Composite Tank – Cylinder with Hemispherical Top:** The tank consists of a cylindrical section 3 m high with a circular cross section of radius 1 m, topped by a hemispherical dome of radius 1 m. Water is pumped out through a spout located 0.15 m above the dome.

8.3 Center of Mass

Overview

Discrete Systems

Definition (Center of Mass for a System of Particles). Consider n particles with masses $m_1, m_2, \dots, m_n$ located at points $x_1, x_2, \dots, x_n$ on a line. The *center of mass* $\bar{x}$ is defined by

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}.$$

For particles in the xy -plane, if the positions are (x_i, y_i) , then the center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}, \quad \bar{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}.$$

Theorem (Law of the Lever). If two masses m_1 and m_2 are placed on opposite sides of a fulcrum at distances d_1 and d_2 , respectively, then the rod will balance provided that

$$m_1 d_1 = m_2 d_2.$$

In the case where m_1 is located at x_1 , m_2 at x_2 , and the center of mass is at x , the balancing condition can be written as

$$m_1(x - x_1) = m_2(x_2 - x),$$

which implies

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Moments

Definition (Moment). The *moment* of a mass m about a point (or an axis) is the product of the mass and its distance from that point (or axis). For a system of particles, the moment about the origin is given by

$$M = \sum_{i=1}^n m_i x_i.$$

In the plane, we define:

$$M_y = \sum_{i=1}^n m_i x_i \quad (\text{moment about the } y\text{-axis}),$$

$$M_x = \sum_{i=1}^n m_i y_i \quad (\text{moment about the } x\text{-axis}).$$

Continuous Systems (Lamina)

For a flat plate (or lamina) with uniform density δ , the center of mass coincides with the *centroid* of the region.

Centroid of a Region with Uniform Density

Definition (Centroid). Let D be a plane region with area

$$A = \int_D dA.$$

The *centroid* $(\bar{x}, \bar{y})$ of D is defined by

$$\bar{x} = \frac{1}{A} \int_D x dA, \quad \bar{y} = \frac{1}{A} \int_D y dA.$$

Regions Bounded by Curves

Region Above the x -Axis: Suppose the region D is bounded above by a continuous function $f(x)$, below by the x -axis, and between $x = a$ and $x = b$. Then the area is

$$A = \int_a^b f(x) dx,$$

and the moments are

$$M_y = \int_a^b x f(x) dx, \quad M_x = \int_a^b \frac{[f(x)]^2}{2} dx.$$

Thus, the centroid is given by

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx, \quad \bar{y} = \frac{1}{A} \int_a^b \frac{[f(x)]^2}{2} dx.$$

Region Between Two Curves: If D is bounded by the curves $y = f(x)$ (upper curve) and $y = g(x)$ (lower curve) for $x \in [a, b]$, with $f(x) \geq g(x)$, then

$$A = \int_a^b [f(x) - g(x)] dx.$$

The moments are

$$M_y = \int_a^b x [f(x) - g(x)] dx, \quad M_x = \int_a^b \frac{[f(x)]^2 - [g(x)]^2}{2} dx,$$

so that the centroid is

$$\bar{x} = \frac{1}{A} \int_a^b x [f(x) - g(x)] dx, \quad \bar{y} = \frac{1}{A} \int_a^b \frac{[f(x)]^2 - [g(x)]^2}{2} dx.$$

Symmetry Principle

Theorem (Symmetry Principle). If a region D is symmetric with respect to a line ℓ , then the centroid of D lies on ℓ .

Center of Mass Problems

1. A system consists of three point masses:

- $m_1 = 2$ kg at $(1, 3)$
- $m_2 = 3$ kg at $(4, 5)$
- $m_3 = 4$ kg at $(6, 2)$

Compute the center of mass of the system.

2. A system consists of four point masses:

- $m_1 = 1$ kg at $(0, 0)$
- $m_2 = 2$ kg at $(2, 4)$
- $m_3 = 3$ kg at $(5, 1)$
- $m_4 = 4$ kg at $(3, 3)$

Compute the center of mass of the system.

3. Compute the center of mass of the region bounded by

$$y = x^2, \quad y = 0, \quad x = 1, \quad \text{and} \quad x = 2.$$

4. Compute the center of mass of the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = \pi,$$

5. Use symmetry to find the center of mass of a uniform semicircular lamina of radius R .

6. Use symmetry to determine the center of mass of a uniform triangular lamina with vertices at $(0, 0)$, $(a, 0)$, and (a, a) .

8.1, 10.1-10.2 Parametric Curves & Arc Length

Overview

Basic Definitions

Definition (Parametric Equations). A curve in the plane can be described by a pair of parametric equations

$$x = f(t), \quad y = g(t),$$

where t is a parameter (often representing time). The set of all points $(f(t), g(t))$ traced out as t varies over an interval $[a, b]$ is called a *parametric curve*.

Remark. Different parametric equations can represent the same geometric curve. However, the direction of traversal and the speed at which the curve is traced depend on how x and y change with respect to the parameter t .

Eliminating the Parameter

Sometimes we can solve one of the equations (e.g. $x = f(t)$) for t and substitute into the other ($y = g(t)$) to obtain a Cartesian equation $F(x, y) = 0$. This process is called *eliminating the parameter*. However, doing so can lose information about direction and speed.

Slopes and Tangents of Parametric Curves

Theorem (Slope of a Parametric Curve). Suppose $x = f(t)$ and $y = g(t)$ are differentiable functions and $\frac{dx}{dt} \neq 0$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Remark. $\frac{dy}{dx}$ is the slope of the tangent line to the parametric curve at a given t . From this, we can find conditions for horizontal or vertical tangents:

- **Horizontal tangent:** $\frac{dy}{dx} = 0 \iff \frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
- **Vertical tangent:** $\frac{dy}{dx}$ undefined $\iff \frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

Arc Length

Definition (Arc Length of a Curve $y = f(x)$). Let f be a continuous function on the interval $[a, b]$. Divide $[a, b]$ into n subintervals with endpoints $x_0, x_1, \dots, x_n$, and let $y_i = f(x_i)$. The polygonal approximation to the curve is

$$L_n = \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

The *arc length* L of the curve is defined as

$$L = \lim_{n \rightarrow \infty} L_n,$$

provided the limit exists.

Theorem (Arc Length Formula for $y = f(x)$). If f is differentiable on $[a, b]$ and f' is continuous, then the arc length L of the curve $y = f(x)$ is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Remark. This result is often written using Leibniz notation as

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

Definition (Arc Length for $x = g(y)$). If a curve is given by $x = g(y)$ for $y \in [c, d]$ with g differentiable and g' continuous, then the arc length L is

$$L = \int_c^d \sqrt{1 + (g'(y))^2} dy.$$

Definition (Arc Length for Parametric Equations). Let a curve C be defined by the parametric equations

$$x = f(t), \quad y = g(t), \quad t \in [\alpha, \beta],$$

where f and g are differentiable and their derivatives are continuous. Then the arc length L of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Remark. This formula is valid even if the curve cannot be written in the form $y = f(x)$. One must ensure the parametric curve is traversed exactly once over the chosen interval $[a, b]$.

Parametric Curves & Arc Length Problems

1. Given the parametric equations:

$$x = t^2 + 1, \quad y = 2t - 3,$$

find the slope of the curve at $t = 1$.

2. Consider the curve defined by:

$$x = \sin t, \quad y = \cos t, \quad 0 \leq t \leq 2\pi.$$

Find the points where the curve has a horizontal tangent.

3. Find the values of t where the parametric curve:

$$x = t^3 - 3t, \quad y = t^2 - 2t$$

has a vertical tangent.

4. Find the equation of the tangent line to the curve:

$$x = e^t, \quad y = e^{-t}$$

at $t = 0$.

5. Determine the slope of the tangent line for the parametric curve:

$$x = t - \sin t, \quad y = 1 - \cos t$$

at $t = \frac{\pi}{4}$.

6. Compute the equation of the tangent line to the curve:

$$x = \ln(t), \quad y = t^2$$

at $t = 1$.

7. Find all values of t where the parametric curve:

$$x = t^2 - 4t + 3, \quad y = t^3 - 3t^2$$

has a horizontal tangent.

8. Given the parametric equations:

$$x = 3t^2 + 2, \quad y = 4t^3 - 5,$$

find the slope of the tangent line at $t = -1$.

9. Find the equation of the tangent line to the parametric curve:

$$x = t^2 + 2t, \quad y = 3t - 1$$

at the point corresponding to $t = 2$.

10. For the parametric curve:

$$x = t^2 - 2t, \quad y = t^3 - 3t$$

determine the values of t where the curve has vertical tangents.

11. Find the arc length of the curve $f(x) = \ln(\cos x)$ over the interval $0 \leq x \leq \frac{\pi}{4}$.¹

12. Find the arc length of the curve $f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$ over the interval $0 \leq x \leq 2$.

13. Find the arc length of the curve $x = \frac{1}{3}y^{3/2} - y^{1/2}$ over the interval $1 \leq y \leq 4$.

14. Find the arc length of the curve $x = \frac{2}{3}y^{3/2}$ over the interval $0 \leq y \leq 4$.

15. Find the arc length of the curve defined by

$$x = t^2, \quad y = t^3, \quad 0 \leq t \leq 1.$$

16. Find the arc length of the cycloid given by

$$x = t - \sin t, \quad y = 1 - \cos t, \quad 0 \leq t \leq 2\pi.$$

¹These problems are hard to come up with. That is, how can we ensure that $\sqrt{1 + [f'(x)]^2}$ can be integrated? For a discussion of this, see <https://math.colorado.edu/~chda1090/arclengthprobs.pdf>

10.3-10.4 Polar Coordinates

Overview

Polar Coordinates and Conversion

Definition (Polar Coordinate System). A point in the plane is represented by *polar coordinates* (r, θ) where:

- r is the distance from the point to a fixed point O (the *pole*).
- θ is the angle (measured in radians) between the polar axis (usually the positive x -axis) and the ray from O to the point.

When $r < 0$, the point (r, θ) is equivalent to $(-r, \theta + \pi)$.

Theorem (Conversion Between Polar and Cartesian Coordinates). If a point has polar coordinates (r, θ) and Cartesian coordinates (x, y) , then:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Conversely, if (x, y) are given, then:

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.$$

Properties of Polar Coordinates

Theorem (Multiple Representations). A point can have infinitely many polar representations. In particular,

$$(r, \theta) = (r, \theta + 2\pi k) \quad \text{and} \quad (r, \theta) = (-r, \theta + (2k + 1)\pi)$$

for any integer k .

Theorem (Symmetry of Polar Graphs). Let $r = f(\theta)$ be a polar equation. Then:

1. If $f(-\theta) = f(\theta)$ for all θ , the graph is symmetric about the polar axis.
2. If the equation is unchanged when r is replaced by $-r$ (or equivalently when θ is replaced by $\theta + \pi$), then the graph is symmetric about the pole.
3. If $f(\pi - \theta) = f(\theta)$ for all θ , the graph is symmetric about the line $\theta = \frac{\pi}{2}$.

Area and Arc Length in Polar Coordinates

Theorem (Area of a Polar Region). Let $r = f(\theta)$ be a non-negative, continuous function on the interval $\theta \in [a, b]$, with $b - a \leq 2\pi$. Then the area A of the region bounded by the curve $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ is given by

$$A = \frac{1}{2} \int_a^b [f(\theta)]^2 d\theta.$$

Theorem (Arc Length of a Polar Curve). If a polar curve is given by $r = f(\theta)$ for $\theta \in [\alpha, \beta]$, where f is differentiable, then the length L of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + \left(\frac{df}{d\theta}\right)^2} d\theta.$$

Theorem. The area of a sector of a circle with radius r and central angle θ (in radians) is given by

$$\text{Area} = \frac{1}{2} r^2 \theta.$$

This result underlies the derivation of the polar area formula.

Tangents and Slopes in Polar Coordinates

Theorem (Slope of the Tangent Line for a Polar Curve). For a polar curve defined by $r = f(\theta)$, the Cartesian coordinates are

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta.$$

Differentiating with respect to θ gives

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta, \quad \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

Thus, the slope of the tangent line is

$$\frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$

Additional Notes

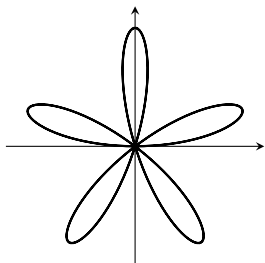
- A polar curve is defined by an equation of the form $r = f(\theta)$; its graph consists of all points (r, θ) satisfying the equation.
- Many common curves (such as circles, cardioids, limaçons, roses, and lemniscates) have elegant representations in polar coordinates.

Polar Coordinates Problems

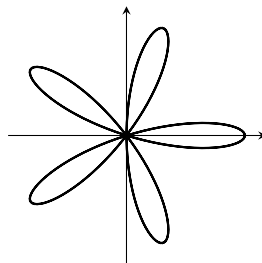
Multiple Choice Matching Problems

For each question, choose the letter (A, B, C, or D) that correctly matches the given description.

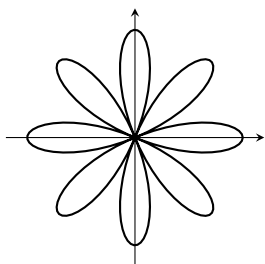
1. Which of the following graphs represents the polar function $r = \sin(5\theta)$?



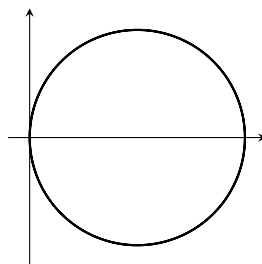
A.



B.

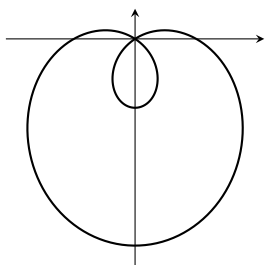


C.

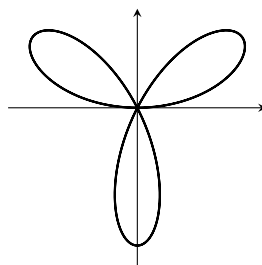


D.

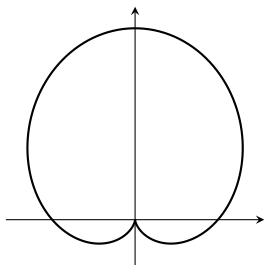
2. Which of the following graphs represents the polar function $r = 1 + \sin \theta$?



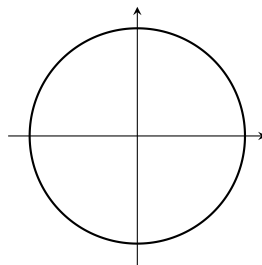
A.



B.

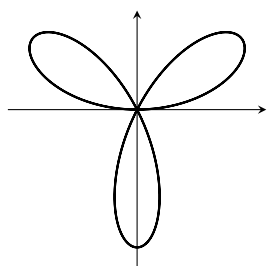


C.

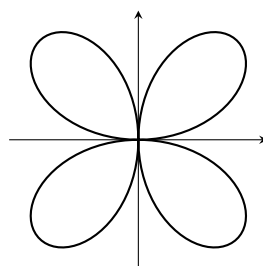


D.

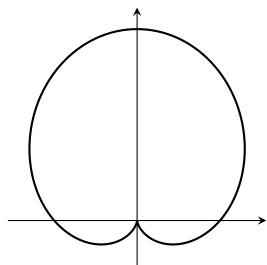
3. Which of the following graphs represents the polar function $r^2 = 2 \cos(2\theta)$?



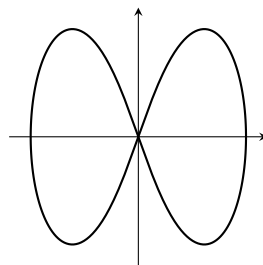
A.



B.

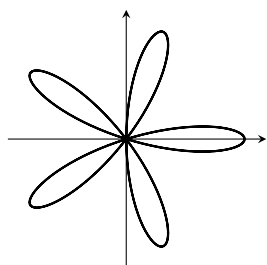


C.

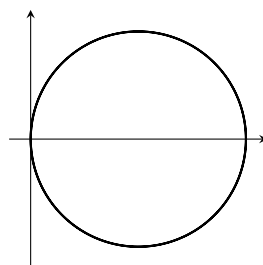


D.

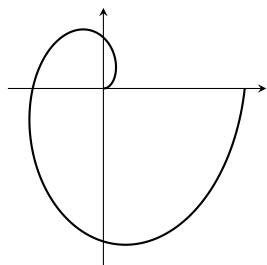
4. Which of the following graphs represents the polar function $r = 2 \cos \theta$?



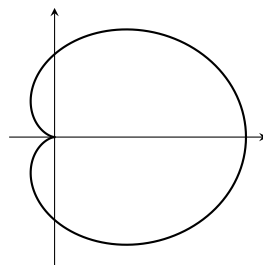
A.



B.

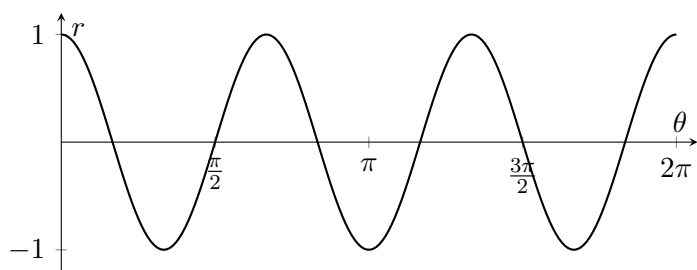


C.

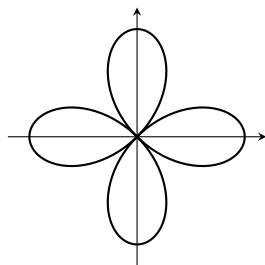


D.

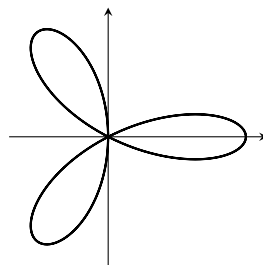
5. The following figure shows a graph of r as a function of θ .



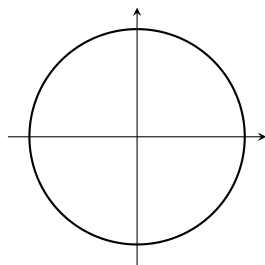
Which of the following corresponds to the curve on the Cartesian xy -plane, using polar coordinates?



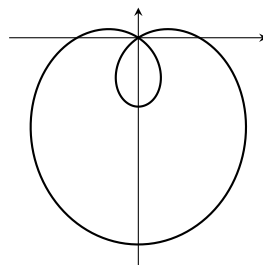
A.



B.

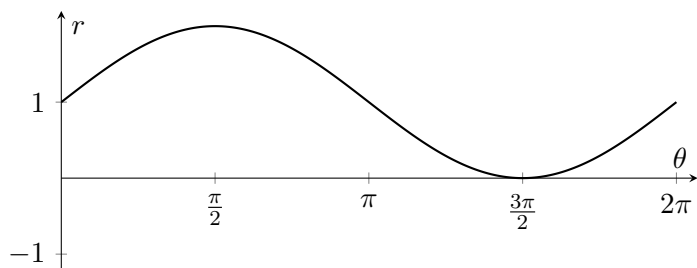


C.

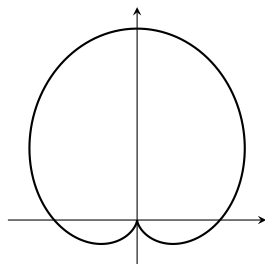


D.

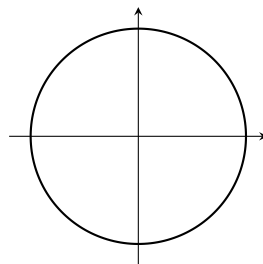
6. The following figure shows a graph of r as a function of θ .



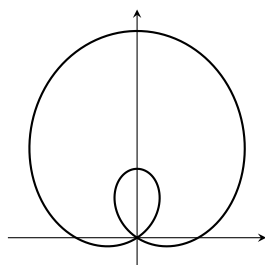
Which of the following corresponds to the curve on the Cartesian xy -plane, using polar coordinates?



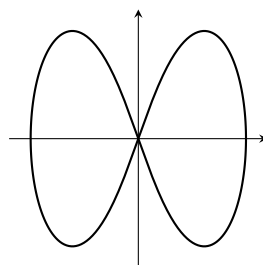
A.



B.

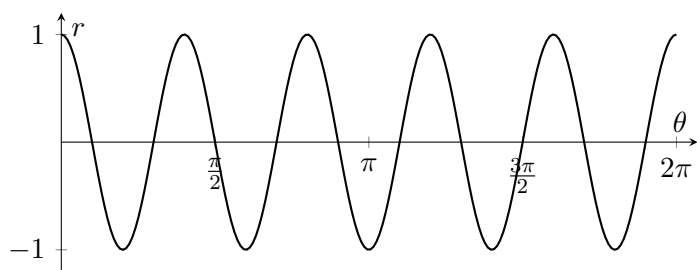


C.

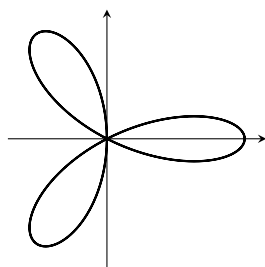


D.

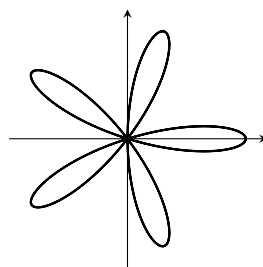
7. The following figure shows a graph of r as a function of θ .



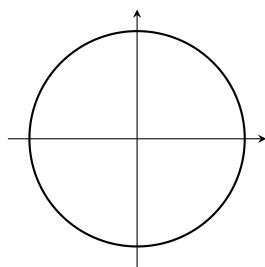
Which of the following corresponds to the curve on the Cartesian xy -plane, using polar coordinates?



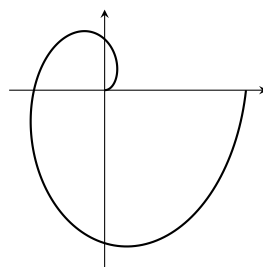
A.



B.

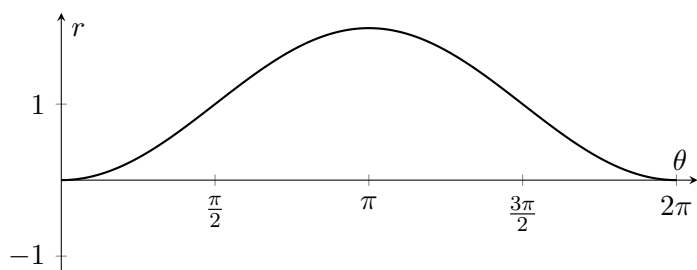


C.

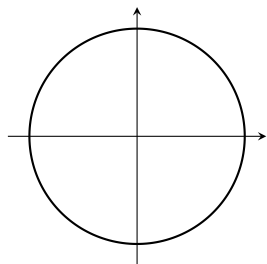


D.

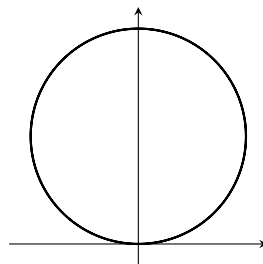
8. The following figure shows a graph of r as a function of θ .



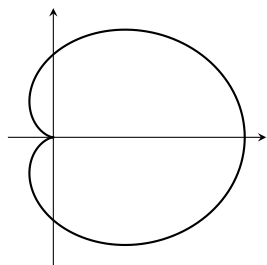
Which of the following corresponds to the curve on the Cartesian xy -plane, using polar coordinates?



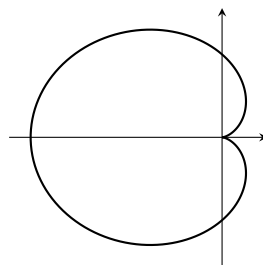
A.



B.



C.

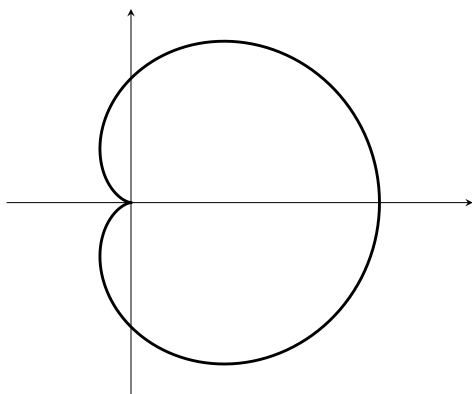


D.

Areas Between Polar Curves

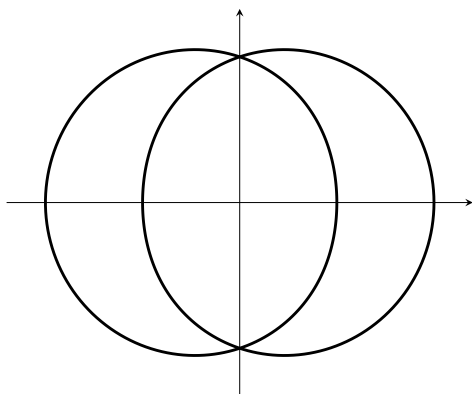
1. Set up an integral to find the area enclosed by the cardioid:

$$r = 2(1 + \cos \theta).$$



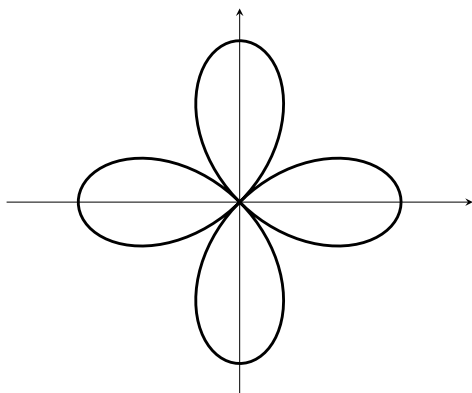
2. Find the area common to both polar curves:

$$r = 3 + \cos \theta, \quad r = 3 - \cos \theta.$$



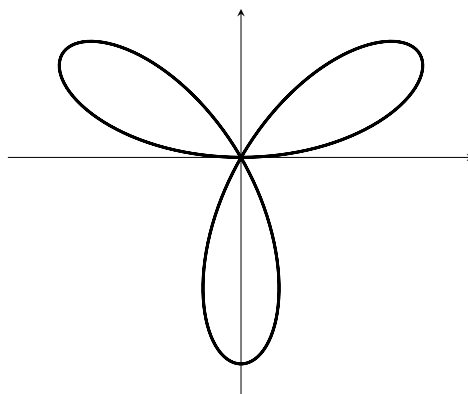
3. Find the area enclosed by the four-leaved rose:

$$r = 3 \cos(2\theta).$$

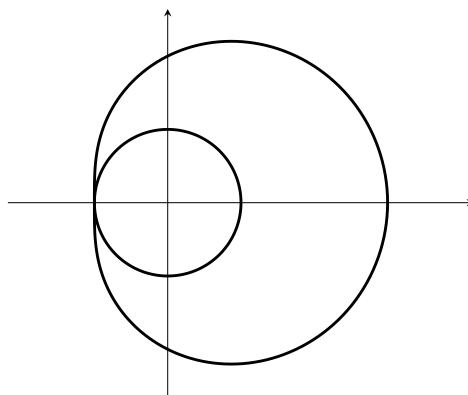


4. Compute the area inside one petal of the rose curve:

$$r = 2 \sin(3\theta).$$

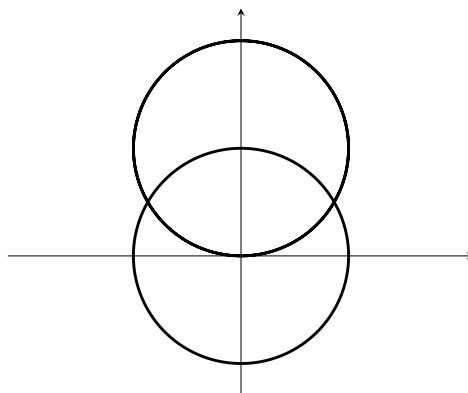


5. Find the area inside $r = 2 + \cos \theta$ and outside $r = 1$.



6. Find the area inside:

$$r = 6 \sin \theta, \quad r = 3.$$



Tangent Lines and Arc Length

1. Find all points of intersection of the curves:

$$r = 1 + \sin \theta, \quad r = 1 - \cos \theta.$$

2. Find all points of intersection of the curves:

$$r = 2 \cos 2\theta, \quad r = 1.$$

3. Find the exact length of the polar curve:

$$r = 3 \sin \theta, \quad 0 \leq \theta \leq \pi.$$

4. Find the exact length of the polar curve:

$$r = e^\theta, \quad 0 \leq \theta \leq \ln 2.$$

5. Find the slope of the tangent line to the curve:

$$r = 1 + 2 \sin \theta, \quad \theta = \frac{\pi}{6}.$$

6. Find points where the tangent line is horizontal or vertical for:

$$r = 2(1 - \cos \theta).$$

11.1 Sequences

Overview

Basic Definitions

Definition. An *infinite sequence* is a function $a : \mathbb{N} \rightarrow \mathbb{R}$, usually written as $a_1, a_2, a_3, \dots$ or equivalently as $\{a_n\}_{n=1}^{\infty}$.

Definition. A sequence $\{a_n\}$ converges to the limit L (written $\lim_{n \rightarrow \infty} a_n = L$) if for every $\varepsilon > 0$ there exists an integer N such that for all $n > N$ $|a_n - L| < \varepsilon$. If no such L exists, the sequence is said to *diverge*.

Limit Laws for Sequences

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences and let c be a constant. Then:

1. **Sum Law:**

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n.$$

2. **Difference Law:**

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n.$$

3. **Constant Multiple Law:**

$$\lim_{n \rightarrow \infty} (c a_n) = c \lim_{n \rightarrow \infty} a_n.$$

4. **Product Law:**

$$\lim_{n \rightarrow \infty} (a_n b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right).$$

5. **Quotient Law:** If $\lim_{n \rightarrow \infty} b_n \neq 0$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}.$$

6. **Power Law:** For any real number $p > 0$ (and with $a_n > 0$),

$$\lim_{n \rightarrow \infty} (a_n^p) = \left(\lim_{n \rightarrow \infty} a_n \right)^p.$$

Convergence Theorems

Theorem (Squeeze Theorem for Sequences). If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences satisfying $a_n \leq b_n \leq c_n$ for all $n \geq N$, and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

Theorem (Absolute Value Theorem). If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Theorem (Limits and Continuity). If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then $\lim_{n \rightarrow \infty} f(a_n) = f(L)$.

Theorem (Geometric Sequences). Consider the sequence $\{r^n\}$.

- If $-1 < r < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$.
- If $r = 1$, then $\lim_{n \rightarrow \infty} r^n = 1$.
- If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ diverges (in the case $r \leq -1$, the terms oscillate without approaching a single value).

Theorem (Monotonic Sequence Theorem). Every bounded, monotonic sequence converges.

A sequence is said to be:

- **Increasing** if $a_{n+1} \geq a_n$ for all n .
- **Decreasing** if $a_{n+1} \leq a_n$ for all n .
- **Monotonic** if it is either increasing or decreasing.

A sequence $\{a_n\}$ is *bounded above* if there exists a number M such that $a_n \leq M$ for all n , and *bounded below* if there exists m such that $a_n \geq m$ for all n . If both conditions hold, the sequence is called *bounded*.

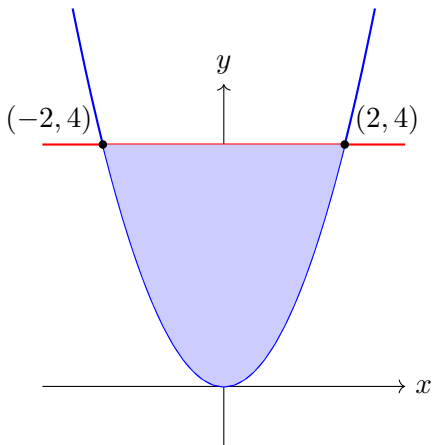
Sequences Problems

1. Determine whether the sequence $a_n = \frac{n}{n+1}$ converges or diverges. If it converges, find its limit.
2. Determine whether the sequence $b_n = (-1)^n$ converges or diverges. If it converges, find its limit.
3. Determine whether the sequence $c_n = \frac{1}{n^2}$ converges or diverges. If it converges, find its limit.
4. Determine whether the sequence $a_n = \frac{n^2}{n^2+1}$ converges or diverges. If it converges, find its limit.
5. Determine whether the sequence $b_n = \frac{\ln n}{n}$ converges or diverges. If it converges, find its limit.
6. Determine whether the sequence $c_n = \frac{n}{\sqrt{n^2+1}}$ converges or diverges. If it converges, find its limit.
7. Determine whether the sequence $d_n = \frac{(-1)^n}{n}$ converges or diverges. If it converges, find its limit.
8. Determine whether the sequence $a_n = \frac{\cos(3n+1)}{n^2}$ converges or diverges. If it converges, find its limit.
9. Determine whether the sequence $c_n = \frac{n^2-3n}{n^3+5}$ converges or diverges. If it converges, find its limit.
10. Determine whether the sequence $a_n = \frac{\sqrt{9n^5+4n^2}}{n^3}$ converges or diverges. If it converges, find its limit.
11. Determine whether the sequence $b_n = \frac{5n + \sin(n)}{n+10}$ converges or diverges. If it converges, find its limit.
12. Determine whether the sequence $c_n = \frac{n^3+2}{\sqrt{n^6+5n^2}}$ converge or diverge? If it converges, find its limit.
13. Determine whether the sequence $a_n = \frac{\sin(5n)}{n^3}$ converges or diverges. If it converges, find its limit.
14. Determine whether the sequence $b_n = \frac{3\sqrt{n} + n^3}{n^3 + \sqrt{n}}$ converges or diverges. If it converges, find its limit.
15. Determine whether the sequence $c_n = \frac{n^3-2n}{\sqrt{4n^6+7n}}$ converges or diverges. If it converges, find its limit.

Solutions

6.1 Areas Between Curves (Solutions)

1.



- **Step 1.** Find the points of intersection by setting

$$x^2 = 4 \implies x = -2 \text{ and } x = 2.$$

- **Step 2.** The area A between the curves is given by

$$A = \int_{-2}^2 [4 - x^2] dx.$$

- **Step 3.** Since the integrand is even, we can write

$$A = 2 \int_0^2 (4 - x^2) dx.$$

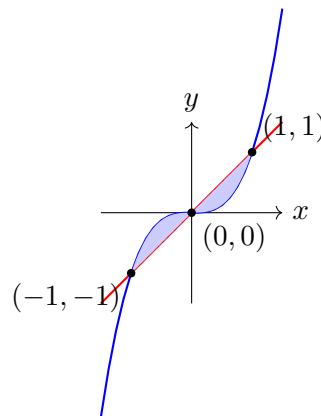
- **Step 4.** Compute the integral:

$$\begin{aligned} \int_0^2 (4 - x^2) dx &= \left[4x - \frac{x^3}{3} \right]_0^2 \\ &= \left(4(2) - \frac{2^3}{3} \right) - 0 \\ &= \left(8 - \frac{8}{3} \right) \\ &= \frac{24 - 8}{3} = \frac{16}{3}. \end{aligned}$$

- **Step 5.** Thus,

$$A = 2 \cdot \frac{16}{3} = \frac{32}{3}.$$

2.



- **Step 1.** Find the intersection points by solving

$$x^3 = x \implies x(x^2 - 1) = 0.$$

Thus, $x = 0$, $x = 1$, and $x = -1$.

- **Step 2.** For $x \in [-1, 0]$, note that $x^3 \geq x$ (e.g., at $x = -0.5$, $x^3 = -0.125$ and $x = -0.5$). For $x \in [0, 1]$, we have $x \geq x^3$.
- **Step 3.** Write the total area A as the sum of two integrals:

$$A = \int_{-1}^0 [x^3 - x] dx + \int_0^1 [x - x^3] dx.$$

- **Step 4.** Compute each integral.

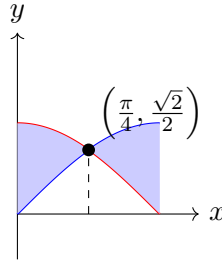
$$\int_0^1 [x - x^3] dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{4}.$$

$$\begin{aligned} \int_{-1}^0 [x^3 - x] dx &= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 \\ &= \left([0 - 0] - \left[\frac{1}{4} - \frac{1}{2} \right] \right) \\ &= 0 - \left(-\frac{1}{4} \right) \\ &= \frac{1}{4}. \end{aligned}$$

- **Step 5.** Thus, the total area is

$$A = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

3.



- **Step 1.** Find the intersection point in $[0, \frac{\pi}{2}]$ by setting

$$\sin x = \cos x \implies x = \frac{\pi}{4}.$$

- **Step 2.** For $0 \leq x \leq \frac{\pi}{4}$, $\cos x \geq \sin x$ and for $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$, $\sin x \geq \cos x$.
- **Step 3.** The total area is the sum of two integrals:

$$A = \int_0^{\pi/4} [\cos x - \sin x] dx + \int_{\pi/4}^{\pi/2} [\sin x - \cos x] dx.$$

- **Step 4.** Evaluate the first integral:

$$\begin{aligned} \int_0^{\pi/4} (\cos x - \sin x) dx &= \left[\sin x + \cos x \right]_0^{\pi/4} \\ &= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (0 + 1) \\ &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - 1 \\ &= \sqrt{2} - 1. \end{aligned}$$

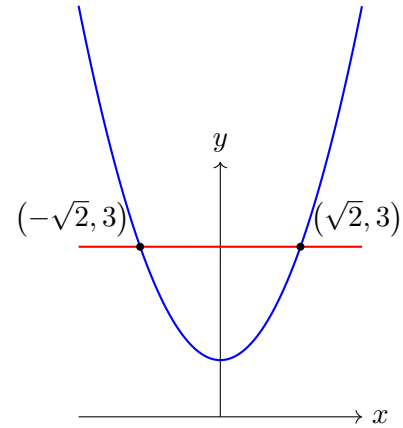
- **Step 5.** Evaluate the second integral:

$$\begin{aligned} \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx &= \left[-\cos x - \sin x \right]_{\pi/4}^{\pi/2} \\ &= \left[(0 - 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \right] \\ &= (-1) - (-\sqrt{2}) \\ &= \sqrt{2} - 1. \end{aligned}$$

- **Step 6.** Hence, the total area is

$$A = (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2(\sqrt{2} - 1).$$

4.



- **Step 1.** Find the points of intersection by setting

$$x^2 + 1 = 3 \implies x^2 = 2 \implies x = \pm\sqrt{2}.$$

- **Step 2.** The area is given by

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} [3 - (x^2 + 1)] dx = \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2) dx.$$

- **Step 3.** Since the integrand is even,

$$A = 2 \int_0^{\sqrt{2}} (2 - x^2) dx.$$

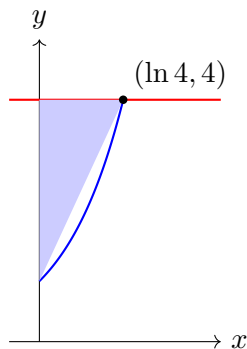
- **Step 4.** Evaluate the integral:

$$\begin{aligned} \int_0^{\sqrt{2}} (2 - x^2) dx &= \left[2x - \frac{x^3}{3} \right]_0^{\sqrt{2}} \\ &= 2\sqrt{2} - \frac{(\sqrt{2})^3}{3} \\ &= 2\sqrt{2} - \frac{2\sqrt{2}}{3} \\ &= \frac{6\sqrt{2} - 2\sqrt{2}}{3} = \frac{4\sqrt{2}}{3}. \end{aligned}$$

- **Step 5.** Therefore,

$$A = 2 \cdot \frac{4\sqrt{2}}{3} = \frac{8\sqrt{2}}{3}.$$

5.



- **Step 1.** Note that when $x = 0$, $e^0 = 1$ and when $x = \ln 4$, $e^{\ln 4} = 4$. Thus the region is bounded on top by $y = 4$ and below by $y = e^x$ over $x \in [0, \ln 4]$.
- **Step 2.** The area A is

$$A = \int_0^{\ln 4} [4 - e^x] dx.$$

- **Step 3.** Compute the integral:

$$\begin{aligned} \int_0^{\ln 4} (4 - e^x) dx &= [4x - e^x]_0^{\ln 4} \\ &= (4 \ln 4 - e^{\ln 4}) - (0 - e^0) \\ &= (4 \ln 4 - 4) + 1 \\ &= 4 \ln 4 - 3. \end{aligned}$$

6.2-6.3 Volumes (Solutions)

6.2-6.3 Volumes by Known Cross-Sections (Solutions)

1. Circular Base, Square Cross-Sections.

- **Base:** The base is the circle

$$x^2 + y^2 \leq 4.$$

- **Slicing:** A slice perpendicular to the x -axis at a given x meets the circle at

$$y = \pm\sqrt{4 - x^2},$$

so the width is

$$2\sqrt{4 - x^2}.$$

- **Cross-Section:** Each slice is a square with side length

$$s(x) = 2\sqrt{4 - x^2},$$

hence its area is

$$A(x) = (2\sqrt{4 - x^2})^2 = 16 - 4x^2.$$

- **Volume:** The solid extends from $x = -2$ to $x = 2$, so

$$V = \int_{-2}^2 (16 - 4x^2) dx.$$

- **Computation:**

$$\begin{aligned} V &= \int_{-2}^2 16 dx - 4 \int_{-2}^2 x^2 dx \\ &= 16 [x]_{-2}^2 - 4 \left[\frac{x^3}{3} \right]_{-2}^2 \\ &= 16(2 - (-2)) - 4 \left(\frac{8}{3} - \left(-\frac{8}{3} \right) \right) \\ &= 16 \cdot 4 - 4 \left(\frac{16}{3} \right) \\ &= 64 - \frac{64}{3} \\ &= \frac{192 - 64}{3} = \frac{128}{3}. \end{aligned}$$

- **Answer:** $V = \frac{128}{3}$ cubic units.

2. Triangular Base, Semicircular Cross-Sections.

- **Base:** The triangular base is bounded by

$$x = 0, \quad y = 0, \quad \text{and} \quad y = 4 - x.$$

- **Slicing:** For a fixed x (with $0 \leq x \leq 4$), the vertical slice extends from $y = 0$ to $y = 4 - x$. Thus, the diameter of the semicircular cross-section is

$$D(x) = 4 - x.$$

- **Cross-Section:**

- The radius is

$$r(x) = \frac{4 - x}{2}.$$

- The area of a semicircle is half that of a full circle:

$$A(x) = \frac{1}{2} \pi [r(x)]^2 = \frac{\pi(4 - x)^2}{8}.$$

- **Volume:** The volume is

$$V = \int_0^4 A(x) dx = \frac{\pi}{8} \int_0^4 (4 - x)^2 dx.$$

- **Substitution:** Let $u = 4 - x$ so that $du = -dx$. When $x = 0$, $u = 4$; when $x = 4$, $u = 0$. Then,

$$V = \frac{\pi}{8} \int_4^0 u^2 (-du) = \frac{\pi}{8} \int_0^4 u^2 du.$$

- **Compute the Integral:**

$$\int_0^4 u^2 du = \frac{u^3}{3} \Big|_0^4 = \frac{64}{3}.$$

- **Result:**

$$V = \frac{\pi}{8} \cdot \frac{64}{3} = \frac{8\pi}{3}.$$

- **Answer:** $V = \frac{8\pi}{3}$ cubic units.

3. Square Base, Equilateral Triangular Cross-Sections.

- **Slicing:** Slicing perpendicular to the x -axis, each slice has a base (in the y -direction) from $y = 0$ to $y = 3$ (length 3).
- **Cross-Section:** For an equilateral triangle of side s , the area is

$$A = \frac{\sqrt{3}}{4}s^2.$$

With $s = 3$, the area becomes

$$A(x) = \frac{\sqrt{3}}{4}(3^2) = \frac{9\sqrt{3}}{4}.$$

- **Volume:**

$$V = \int_0^3 A(x) dx = \frac{9\sqrt{3}}{4} \cdot 3 = \frac{27\sqrt{3}}{4}.$$

- **Answer:** $V = \frac{27\sqrt{3}}{4}$ cubic units.

4. Rectangular Base, Right Triangular Cross-Sections.

- **Slicing:** We slice perpendicular to the y -axis. At a fixed y , the slice has a length of 5 in the x -direction.
- **Cross-Section:** Each slice is a right triangle with base 5 and height y . Thus, the area is

$$A(y) = \frac{1}{2} \cdot 5 \cdot y = \frac{5y}{2}.$$

- **Volume:**

$$V = \int_0^2 \frac{5y}{2} dy.$$

- **Compute:**

$$\begin{aligned} V &= \frac{5}{2} \int_0^2 y dy \\ &= \frac{5}{2} \left[\frac{y^2}{2} \right]_0^2 \\ &= \frac{5}{2} \cdot \frac{4}{2} = 5. \end{aligned}$$

- **Answer:** $V = 5$ cubic units.

5. Circular Base, Equilateral Triangles.

- **Base:** The base is the circle

$$x^2 + y^2 \leq 1.$$

- **Slicing:** At a fixed y , the chord in the x -direction runs from

$$x = -\sqrt{1-y^2} \quad \text{to} \quad x = \sqrt{1-y^2},$$

with length

$$2\sqrt{1-y^2}.$$

- **Cross-Section:** This chord forms the base of an equilateral triangle, so its area is

$$A(y) = \frac{\sqrt{3}}{4} \left(2\sqrt{1-y^2} \right)^2 = \sqrt{3}(1-y^2).$$

- **Volume:**

$$V = \int_{-1}^1 \sqrt{3}(1-y^2) dy.$$

- **Compute the inner integral:**

$$\begin{aligned} \int_{-1}^1 (1-y^2) dy &= \left[y - \frac{y^3}{3} \right]_{-1}^1 \\ &= \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right] \\ &= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}. \end{aligned}$$

- **Result:**

$$V = \sqrt{3} \cdot \frac{4}{3} = \frac{4\sqrt{3}}{3}.$$

- **Answer:** $V = \frac{4\sqrt{3}}{3}$ cubic units.

6. Triangular Base, Rectangular Cross-Sections.

- **Base:** The triangular base is bounded by

$$x = 0, \quad y = 0, \quad x + y = 6.$$

For a fixed $x \in [0, 6]$, y runs from 0 to $6 - x$.
Define

$$L(x) = 6 - x.$$

- **Cross-Section:** The rectangle at x has:
 - **Base (in the xy -plane):** $L(x) = 6 - x$,
 - **Height (in the z -direction):** $2(6 - x)$.

Thus, its area is

$$A(x) = (6 - x) \cdot [2(6 - x)] = 2(6 - x)^2.$$

- **Volume:**

$$V = \int_0^6 2(6 - x)^2 dx.$$

- **Substitution:** Let $u = 6 - x$ so that $du = -dx$. Then,

$$\begin{aligned} V &= \int_{u=6}^0 2u^2 (-du) \\ &= \int_0^6 2u^2 du. \end{aligned}$$

- **Compute:**

$$\begin{aligned} \int_0^6 2u^2 du &= 2 \left[\frac{u^3}{3} \right]_0^6 \\ &= 2 \cdot \frac{216}{3} = 144. \end{aligned}$$

- **Answer:** $V = 144$ cubic units.

7. Elliptical Base, Square Cross-Sections.

- **Base:** The ellipse is given by

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

- **Slicing:** For a fixed y , solve for x^2 :

$$x^2 = 9 \left(1 - \frac{y^2}{4} \right) = 9 - \frac{9y^2}{4}.$$

Hence,

$$x = \pm \sqrt{9 - \frac{9y^2}{4}} = \pm \frac{3}{2} \sqrt{4 - y^2}.$$

The total width is

$$2 \cdot \frac{3}{2} \sqrt{4 - y^2} = 3\sqrt{4 - y^2}.$$

- **Cross-Section:** Since the cross-sections are squares,

$$A(y) = \left[3\sqrt{4 - y^2} \right]^2 = 9(4 - y^2) = 36 - 9y^2.$$

- **Volume:** Integrate with respect to y (from -2 to 2):

$$V = \int_{-2}^2 (36 - 9y^2) dy.$$

- **Computation:**

$$\int_{-2}^2 36 dy = 36 [y]_{-2}^2 = 36(4) = 144,$$

$$\int_{-2}^2 y^2 dy = \left[\frac{y^3}{3} \right]_{-2}^2 = \frac{8}{3} - \left(-\frac{8}{3} \right) = \frac{16}{3}.$$

Therefore,

$$V = 144 - 9 \left(\frac{16}{3} \right) = 144 - 48 = 96.$$

- **Answer:** $V = 96$ cubic units.

8. Isosceles Trapezoidal Cross-Sections over a Square Base.

- **Slicing:** For a fixed x , the cross-section extends in y from 0 to 4. Thus, the longer base is 4 and the shorter base is 2; the trapezoid's height (perpendicular to the xy -plane) is 1.

- **Cross-Section:** Its area is

$$A = \frac{4 + 2}{2} \times 1 = 3.$$

- **Volume:** Since the area is constant,

$$V = \int_0^4 3 dx = 12.$$

- **Answer:** $V = 12$ cubic units.

9. **Parabolic Region, Rectangular Cross-Sections (Height Proportional to y).**

- **Base:** For a given x , the region in y goes from 0 to $4 - x^2$; thus the width is

$$4 - x^2.$$

- **Cross-Section:** The rectangle has width $4 - x^2$ and (constant for the slice) height

$$2(4 - x^2).$$

Its area is

$$A(x) = (4 - x^2) \cdot 2(4 - x^2) = 2(4 - x^2)^2.$$

- **Volume:** The region in x is from 0 to 2, so

$$V = \int_0^2 2(4 - x^2)^2 dx.$$

- **Expand and Compute:** Note that

$$(4 - x^2)^2 = 16 - 8x^2 + x^4.$$

Then,

$$\begin{aligned} V &= 2 \int_0^2 (16 - 8x^2 + x^4) dx \\ &= 2 \left[\int_0^2 16 dx - 8 \int_0^2 x^2 dx + \int_0^2 x^4 dx \right]. \end{aligned}$$

- **Individual Integrals:**

$$\int_0^2 16 dx = 32, \quad \int_0^2 x^2 dx = \frac{8}{3}, \quad \int_0^2 x^4 dx = \frac{32}{5}.$$

- **Thus,**

$$\begin{aligned} V &= 2 \left(32 - 8 \cdot \frac{8}{3} + \frac{32}{5} \right) \\ &= 2 \left(32 - \frac{64}{3} + \frac{32}{5} \right) \\ &= 2 \left[\frac{32 \cdot 15 - 64 \cdot 5 + 32 \cdot 3}{15} \right] \\ &= 2 \left[\frac{480 - 320 + 96}{15} \right] \\ &= 2 \left[\frac{256}{15} \right] = \frac{512}{15}. \end{aligned}$$

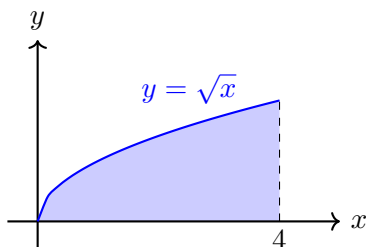
- **Answer:** $V = \frac{512}{15}$ cubic units.

6.2-6.3 Disk Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x -axis.

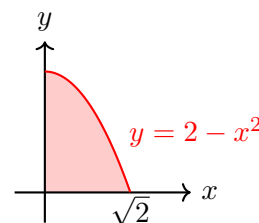


$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{x})^2 dx \\ &= \pi \int_0^4 x dx \\ &= \pi \left[\frac{x^2}{2} \right]_0^4 \\ &= \pi \left(\frac{16}{2} \right) \\ &= 8\pi. \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x^2, \quad y = 0, \quad x \in [0, \sqrt{2}]$$

about the x -axis.

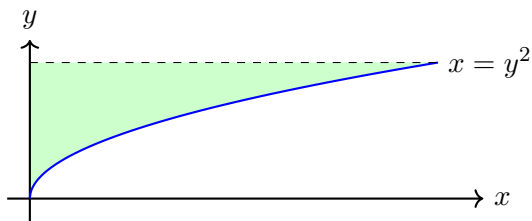


$$\begin{aligned} V &= \pi \int_0^{\sqrt{2}} (2 - x^2)^2 dx \\ &= \pi \int_0^{\sqrt{2}} (4 - 4x^2 + x^4) dx \\ &= \pi \left[4x - \frac{4x^3}{3} + \frac{x^5}{5} \right]_0^{\sqrt{2}} \\ &= \pi \left(4\sqrt{2} - \frac{4(\sqrt{2})^3}{3} + \frac{(\sqrt{2})^5}{5} \right) \\ &= \pi\sqrt{2} \left(4 - \frac{8}{3} + \frac{4}{5} \right) \\ &= \pi\sqrt{2} \left(\frac{60 - 40 + 12}{15} \right) \\ &= \pi\sqrt{2} \cdot \frac{32}{15} \\ &= \frac{32\sqrt{2}\pi}{15}. \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 0, \quad y = 3$$

about the y -axis.

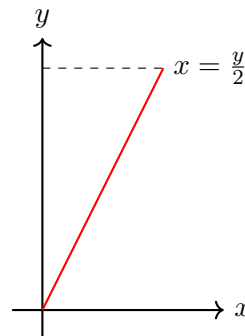


$$\begin{aligned} V &= \pi \int_0^3 (y^2)^2 dy \\ &= \pi \int_0^3 y^4 dy \\ &= \pi \left[\frac{y^5}{5} \right]_0^3 \\ &= \pi \cdot \frac{3^5}{5} \\ &= \frac{243\pi}{5}. \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$x = \frac{y}{2}, \quad x = 0, \quad y = 4$$

about the y -axis.

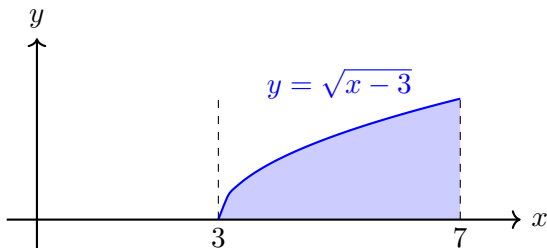


$$\begin{aligned} V &= \pi \int_0^4 \left(\frac{y}{2} \right)^2 dy \\ &= \pi \int_0^4 \frac{y^2}{4} dy \\ &= \frac{\pi}{4} \int_0^4 y^2 dy \\ &= \frac{\pi}{4} \left[\frac{y^3}{3} \right]_0^4 \\ &= \frac{\pi}{4} \cdot \frac{64}{3} \\ &= \frac{16\pi}{3}. \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x-3}, \quad y = 0, \quad x = 7$$

about the vertical line $x = 7$.



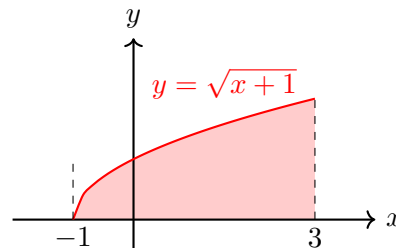
First, rewrite $y = \sqrt{x-3}$ as $x = y^2 + 3$. When $x = 7$, $y = \sqrt{7-3} = 2$, so using horizontal slices:

$$\begin{aligned} V &= \pi \int_0^2 \left[(7 - (y^2 + 3))^2 \right] dy \\ &= \pi \int_0^2 (4 - y^2)^2 dy \\ &= \pi \int_0^2 16 - 8y^2 + y^4 dy \\ &= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\ &= \pi \cdot \left[32 - \frac{64}{3} + \frac{32}{5} \right] \\ &= \frac{256\pi}{15}. \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x+1}, \quad y = 0, \quad x = 3$$

about the vertical line $x = 3$.



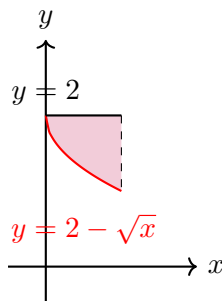
Rewrite $y = \sqrt{x+1}$ as $x = y^2 - 1$. When $x = 3$, $y = \sqrt{3+1} = 2$. Then,

$$\begin{aligned} V &= \pi \int_0^2 \left[(3 - (y^2 - 1))^2 \right] dy \\ &= \pi \int_0^2 (4 - y^2)^2 dy \\ &= \pi \int_0^2 16 - 8y^2 + y^4 dy \\ &= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\ &= \pi \cdot \left[32 - \frac{64}{3} + \frac{32}{5} \right] \\ &= \frac{256\pi}{15}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - \sqrt{x}, \quad y = 2, \quad x = 1$$

about the horizontal line $y = 2$.

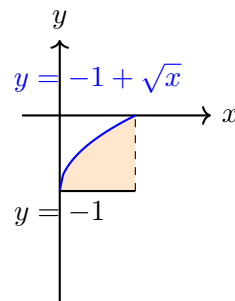


$$\begin{aligned} V &= \pi \int_0^1 \left[2 - (2 - \sqrt{x}) \right]^2 dx \\ &= \pi \int_0^1 (\sqrt{x})^2 dx \\ &= \pi \int_0^1 x dx \\ &= \pi \left[\frac{x^2}{2} \right]_0^1 \\ &= \frac{\pi}{2} \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = -1 + \sqrt{x}, \quad y = -1, \quad x = 1$$

about the horizontal line $y = -1$.



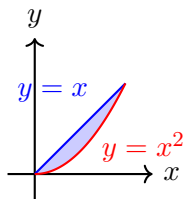
$$\begin{aligned} V &= \pi \int_0^1 \left[(-1 + \sqrt{x}) - (-1) \right]^2 dx \\ &= \pi \int_0^1 (\sqrt{x})^2 dx \\ &= \pi \int_0^1 x dx \\ &= \pi \left[\frac{x^2}{2} \right]_0^1 \\ &= \frac{\pi}{2} \end{aligned}$$

6.2-6.3 Washer Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = x^2$$

about the x -axis.

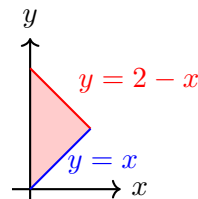


$$\begin{aligned} V &= \pi \int_0^1 \left[(R(x))^2 - (r(x))^2 \right] dx \\ &= \pi \int_0^1 \left[(x)^2 - (x^2)^2 \right] dx \\ &= \pi \int_0^1 \left[x^2 - x^4 \right] dx \\ &= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \pi \left(\frac{5-3}{15} \right) \\ &= \frac{2\pi}{15} \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = x, \quad x = 0$$

about the x -axis.

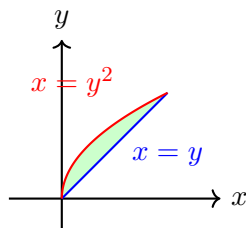


$$\begin{aligned} V &= \pi \int_0^1 \left[(R(x))^2 - (r(x))^2 \right] dx \\ &= \pi \int_0^1 \left[(2-x)^2 - (x)^2 \right] dx \\ &= \pi \int_0^1 \left[(4 - 4x + x^2) - x^2 \right] dx \\ &= \pi \int_0^1 (4 - 4x) dx \\ &= \pi \left[4x - 2x^2 \right]_0^1 \\ &= \pi(4 - 2) \\ &= 2\pi \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y, \quad x = y^2$$

about the y -axis.

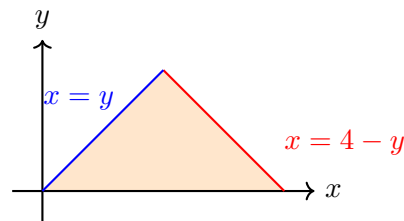


$$\begin{aligned} V &= \pi \int_0^1 \left[(R(y))^2 - (r(y))^2 \right] dy \\ &= \pi \int_0^1 \left[(y)^2 - (y^2)^2 \right] dy \\ &= \pi \int_0^1 \left[y^2 - y^4 \right] dy \\ &= \pi \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{5} \right) \\ &= \frac{2\pi}{15} \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = 4 - x, \quad \text{and} \quad y = 0$$

about the y -axis.

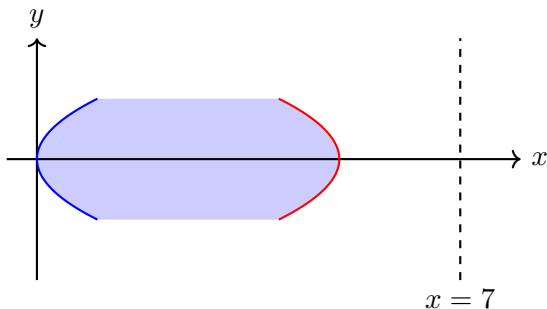


$$\begin{aligned} V &= \pi \int_0^2 \left[(R(y))^2 - (r(y))^2 \right] dy \\ &= \pi \int_0^2 \left[((4 - y))^2 - (y)^2 \right] dy \\ &= \pi \int_0^2 \left[(16 - 8y + y^2) - y^2 \right] dy \\ &= \pi \int_0^2 (16 - 8y) dy \\ &= \pi \left[16y - 4y^2 \right]_0^2 \\ &= \pi (32 - 16) \\ &= 16\pi \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 5 - y^2, \quad y = -1, \quad \text{and} \quad y = 1$$

about the vertical line $x = 7$.



For a given y , the distances from the vertical line $x = 7$ to the curves are:

$$\begin{aligned} r(y) &= 7 - (5 - y^2) = 2 + y^2 \\ R(y) &= 7 - y^2 \end{aligned}$$

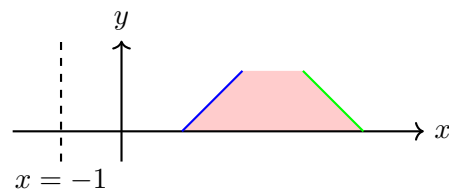
Thus, the volume is

$$\begin{aligned} V &= \pi \int_{-1}^1 \left[(7 - y^2)^2 - (2 + y^2)^2 \right] dy \\ &= \pi \int_{-1}^1 \left[(49 - 14y^2 + y^4) - (4 + 4y^2 + y^4) \right] dy \\ &= \pi \int_{-1}^1 (45 - 18y^2) dy \\ &= \pi \left[45y - 18 \left(\frac{y^3}{3} \right) \right]_{-1}^1 \\ &= \pi [45y - 6y^3]_{-1}^1 \\ &= \pi \left([45(1) - 6(1)^3] - [45(-1) - 6(-1)^3] \right) \\ &= \pi ((45 - 6) - (-45 + 6)) \\ &= \pi (39 + 39) \\ &= 78\pi \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x - 1, \quad y = 4 - x, \quad y = 0, \quad \text{and} \quad y = 1$$

about the vertical line $x = -1$.



For a fixed y with $0 \leq y \leq 1$, the region extends horizontally from

$$x = y + 1 \quad \text{to} \quad x = 4 - y.$$

Since the axis of rotation is $x = -1$, the distances from $x = -1$ to these curves are:

$$\begin{aligned} r(y) &= (y + 1) - (-1) = y + 2 \\ R(y) &= (4 - y) - (-1) = 5 - y. \end{aligned}$$

Thus, the cross-sectional area (washer) is:

$$A(y) = \pi \left[(5 - y)^2 - (y + 2)^2 \right].$$

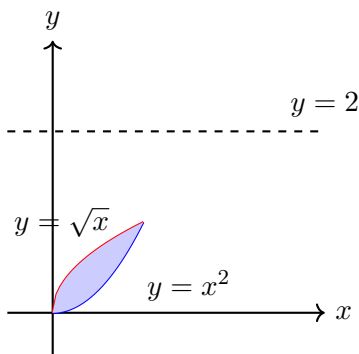
Hence, the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left[(5 - y)^2 - (y + 2)^2 \right] dy \\ &= \pi \int_0^1 \left[(25 - 10y + y^2) - (y^2 + 4y + 4) \right] dy \\ &= \pi \int_0^1 [25 - 10y + y^2 - y^2 - 4y - 4] dy \\ &= \pi \int_0^1 (21 - 14y) dy \\ &= \pi \left[21y - \frac{14y^2}{2} \right]_0^1 \\ &= \pi [21y - 7y^2]_0^1 \\ &= \pi ((21 - 7) - 0) \\ &= 14\pi \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the horizontal line $y = 2$.



Using the washer method with vertical slices, note that the region in x runs from $x = 0$ to $x = 1$. For a fixed x in this interval, the y -values range from $y = x^2$ (lower curve) to $y = \sqrt{x}$ (upper curve).

When this slice is rotated about the horizontal line $y = 2$, the distances from $y = 2$ to the curves are:

$$\begin{aligned} \text{Outer radius: } R(x) &= 2 - x^2 \\ \text{Inner radius: } r(x) &= 2 - \sqrt{x}. \end{aligned}$$

Thus, the cross-sectional area is:

$$A(x) = \pi \left[(2 - x^2)^2 - (2 - \sqrt{x})^2 \right].$$

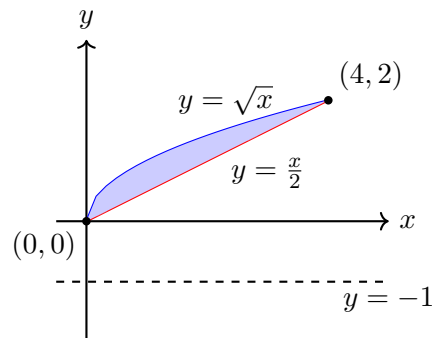
Hence, the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left[(2 - x^2)^2 - (2 - \sqrt{x})^2 \right] dx \\ &= \pi \int_0^1 \left[(4 - 4x^2 + x^4) - (4 - 4\sqrt{x} + x) \right] dx \\ &= \pi \int_0^1 \left(-4x^2 + x^4 + 4\sqrt{x} - x \right) dx \\ &= \pi \left[-\frac{4x^3}{3} + \frac{x^5}{5} + \frac{8x^{3/2}}{3} - \frac{x^2}{2} \right]_0^1 \\ &= \pi \left(-\frac{4}{3} + \frac{1}{5} + \frac{8}{3} - \frac{1}{2} \right) \\ &= \frac{31\pi}{30}. \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -1$.



For a fixed x in $[0, 4]$, the region extends vertically from

$$y = \frac{x}{2} \quad \text{to} \quad y = \sqrt{x}.$$

When this slice is rotated about $y = -1$, it forms a washer with

$$\begin{aligned} \text{Outer radius: } R(x) &= \sqrt{x} - (-1) = \sqrt{x} + 1 \\ \text{Inner radius: } r(x) &= \frac{x}{2} - (-1) = \frac{x}{2} + 1. \end{aligned}$$

Thus, the cross-sectional area is

$$A(x) = \pi \left[\left(\sqrt{x} + 1 \right)^2 - \left(\frac{x}{2} + 1 \right)^2 \right].$$

Hence, the volume is

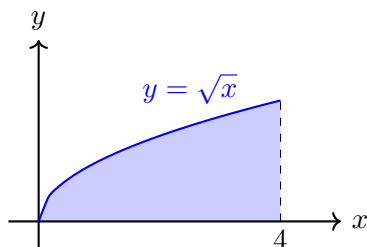
$$\begin{aligned} V &= \pi \int_0^4 \left[\left(\sqrt{x} + 1 \right)^2 - \left(\frac{x}{2} + 1 \right)^2 \right] dx \\ &= \pi \int_0^4 \left[(x + 2\sqrt{x} + 1) - \left(\frac{x^2}{4} + x + 1 \right) \right] dx \\ &= \pi \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx \\ &= \pi \left[\frac{4}{3} x^{3/2} - \frac{x^3}{12} \right]_0^4 \\ &= \pi \left[\frac{4}{3} (4^{3/2}) - \frac{4^3}{12} \right] \\ &= \pi \left[\frac{32}{3} - \frac{16}{3} \right] \\ &= \frac{16\pi}{3}. \end{aligned}$$

6.2-6.3 Cylindrical Shells Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x -axis.



Using horizontal shells (with y as the variable), note that:

- Express $y = \sqrt{x}$ as $x = y^2$.
- For a given horizontal slice at height y , the shell extends from $x = y^2$ to $x = 4$. Thus, the *height* is $h(y) = 4 - y^2$.
- The *radius* of the shell is the distance from y to the x -axis, i.e. $r(y) = y$.

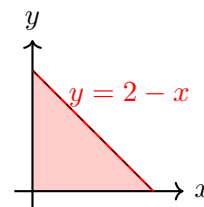
Then,

$$\begin{aligned} V &= 2\pi \int_0^2 (\text{radius}) \cdot (\text{height}) \, dy \\ &= 2\pi \int_0^2 y(4 - y^2) \, dy \\ &= 2\pi \left[\int_0^2 (4y - y^3) \, dy \right] \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi [8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = 0, \quad x \in [0, 2]$$

about the x -axis.



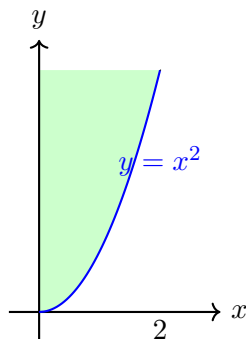
Rewriting $y = 2 - x$ as $x = 2 - y$, the region in y is from $y = 0$ to $y = 2$ with horizontal length $h(y) = 2 - y$ and radius $r(y) = y$. Then,

$$\begin{aligned} V &= 2\pi \int_0^2 y(2 - y) \, dy \\ &= 2\pi \int_0^2 (2y - y^2) \, dy \\ &= 2\pi \left[y^2 - \frac{y^3}{3} \right]_0^2 \\ &= 2\pi \left[4 - \frac{8}{3} \right] \\ &= 2\pi \left(\frac{12 - 8}{3} \right) \\ &= 2\pi \left(\frac{4}{3} \right) \\ &= \frac{8\pi}{3} \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 4, \quad x \in [0, 2]$$

about the y -axis.



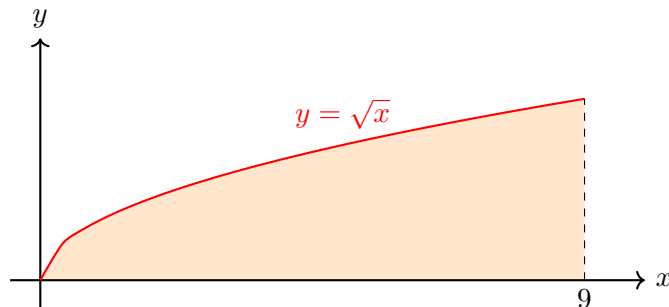
Using vertical shells, the radius is $r(x) = x$ and the height is $h(x) = 4 - x^2$. Thus,

$$\begin{aligned} V &= 2\pi \int_0^2 x(4 - x^2) dx \\ &= 2\pi \left[\int_0^2 (4x - x^3) dx \right] \\ &= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi [8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 9$$

about the y -axis.



For vertical shells, the radius is $r(x) = x$ and the height is $h(x) = \sqrt{x}$. Then,

$$\begin{aligned} V &= 2\pi \int_0^9 x \cdot \sqrt{x} dx \\ &= 2\pi \int_0^9 x^{3/2} dx \\ &= 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^9 \\ &= 2\pi \cdot \frac{2}{5} \cdot 9^{5/2} \\ &= \frac{4\pi}{5} \cdot (9^{5/2}). \end{aligned}$$

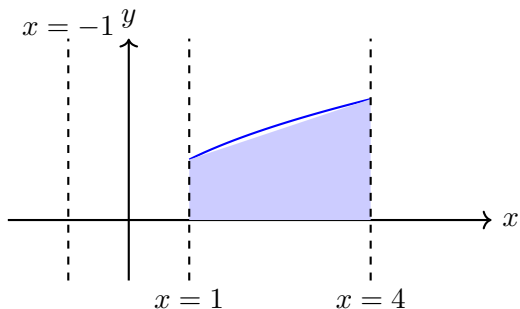
Since $9^{5/2} = (\sqrt{9})^5 = 3^5 = 243$, we have

$$V = \frac{4\pi}{5} \cdot 243 = \frac{972\pi}{5}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the vertical line $x = -1$.



Since the region is bounded vertically by $y = 0$ and $y = \sqrt{x}$ and horizontally by $x = 1$ and $x = 4$, we use the *shell method* with vertical slices. For a typical slice at position x , the height is

$$h(x) = \sqrt{x} - 0 = \sqrt{x},$$

and the radius (distance from x to the axis $x = -1$) is

$$r(x) = x - (-1) = x + 1.$$

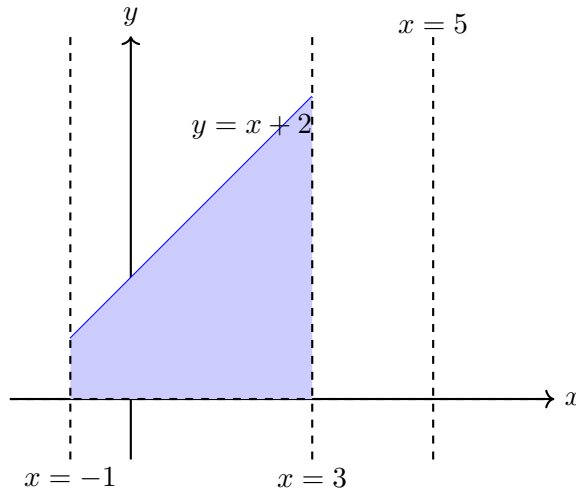
Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 r(x) h(x) dx \\ &= 2\pi \int_1^4 (x+1)\sqrt{x} dx \\ &= 2\pi \int_1^4 (x^{3/2} + x^{1/2}) dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} \right]_{x=1}^4 \\ &= 2\pi \left\{ \left[\frac{2}{5}(4^{5/2}) + \frac{2}{3}(4^{3/2}) \right] - \left[\frac{2}{5}(1^{5/2}) + \frac{2}{3}(1^{3/2}) \right] \right\} \\ &= \frac{512\pi}{15}. \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x + 2, \quad y = 0, \quad x = -1, \quad x = 3$$

about the vertical line $x = 5$.



Since the region is defined by $-1 \leq x \leq 3$ and $0 \leq y \leq x+2$, we use the *cylindrical shells* method. A typical vertical slice at x has:

$$\text{Height: } h(x) = x + 2,$$

and its distance from the axis $x = 5$ is

$$\text{Radius: } r(x) = 5 - x.$$

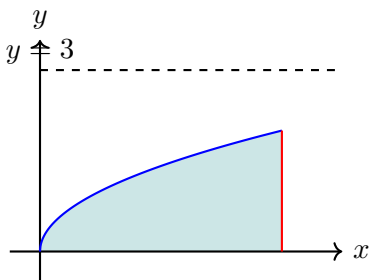
Thus, the volume is given by

$$\begin{aligned} V &= 2\pi \int_{-1}^3 r(x) h(x) dx \\ &= 2\pi \int_{-1}^3 (5-x)(x+2) dx \\ &= 2\pi \int_{-1}^3 [5(x+2) - x(x+2)] dx \\ &= 2\pi \int_{-1}^3 [5x + 10 - x^2 - 2x] dx \\ &= 2\pi \int_{-1}^3 [-x^2 + 3x + 10] dx \\ &= 2\pi \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 10x \right]_{x=-1}^3 \\ &= \frac{256\pi}{3}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4, \quad y = 0$$

about the horizontal line $y = 3$.



For a horizontal slice at y (with $0 \leq y \leq 2$):

- The *radius* is the distance from y to the line $y = 3$: $r(y) = 3 - y$.
- The *height* is the horizontal length: $h(y) = 4 - x$ where x runs from $x = y^2$ to $x = 4$; thus, $h(y) = 4 - y^2$.

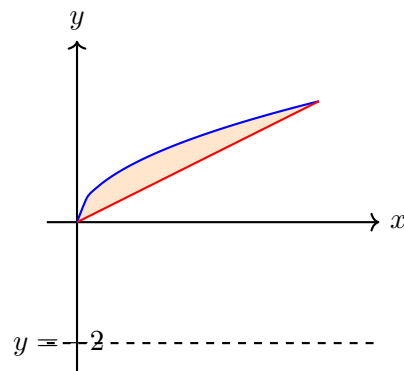
Then,

$$\begin{aligned} V &= 2\pi \int_{y=0}^2 (3 - y)(4 - y^2) dy \\ &= 2\pi \int_0^2 [12 - 3y^2 - 4y + y^3] dy \\ &= 2\pi \left[12y - \frac{3y^3}{3} - 2y^2 + \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left[12(2) - y^3|_2 - 2(2)^2 + \frac{(2)^4}{4} \right] \\ &= 2\pi [24 - 8 - 8 + 4] \\ &= 2\pi(12) \\ &= 24\pi \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -2$.



Using horizontal shells, for a slice at height y (with y from the intersection of the curves, $0 \leq y \leq 2$):

- The *radius* is the distance from y to $y = -2$: $r(y) = y - (-2) = y + 2$.
- The *height* is the horizontal length between the curves: $h(y) = 2y - y^2$.

Then,

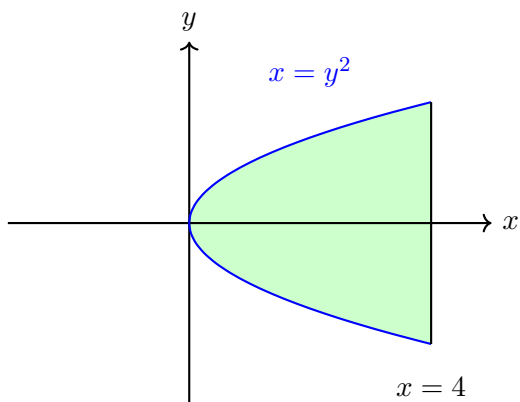
$$\begin{aligned} V &= 2\pi \int_{y=0}^2 (y + 2) [2y - y^2] dy \\ &= 2\pi \int_0^2 [(y + 2)(2y - y^2)] dy \\ &= 2\pi \int_0^2 [2y^2 - y^3 + 4y - 2y^2] dy \\ &= 2\pi \int_0^2 (4y - y^3) dy \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi [8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

6.2-6.3 Additional Problems (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4$$

about the y -axis.



Solution via Washer Method:

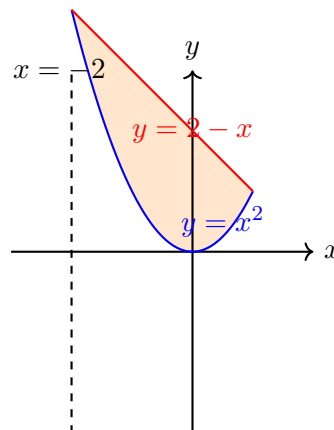
Using the Washer Method with y -limits $y = -2$ to 2 , outer radius $R(y) = 4$ and inner radius $r(y) = y^2$:

$$\begin{aligned} V &= \pi \int_{-2}^2 [4^2 - (y^2)^2] dy = \pi \int_{-2}^2 (16 - y^4) dy \\ &= 2\pi \int_0^2 (16 - y^4) dy = 2\pi \left[16y - \frac{y^5}{5} \right]_0^2 \\ &= 2\pi \left(32 - \frac{32}{5} \right) = 2\pi \left(\frac{160 - 32}{5} \right) = \frac{256\pi}{5}. \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x$$

about the vertical line $x = -2$.



Solution via the Shell Method:

Using vertical slices, for x from -2 to 1 the height of a typical slice is

$$h(x) = (2 - x) - x^2,$$

and its distance from the axis $x = -2$ (the shell radius) is

$$r(x) = x - (-2) = x + 2.$$

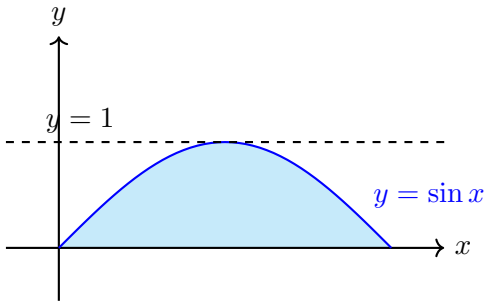
Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_{-2}^1 [(x + 2)((2 - x) - x^2)] dx \\ &= 2\pi \int_{-2}^1 (x + 2)(2 - x - x^2) dx \\ &= 2\pi \int_{-2}^1 [4 - 3x^2 - x^3] dx \\ &= 2\pi \left[4x - x^3 - \frac{x^4}{4} \right]_{x=-2}^1 \\ &= 2\pi \left[\frac{11}{4} - (-4) \right] = 2\pi \left(\frac{11}{4} + 4 \right) = 2\pi \left(\frac{11 + 16}{4} \right) \\ &= 2\pi \left(\frac{27}{4} \right) = \frac{27\pi}{2} \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

about the line $y = 1$.



Solution via the Washer Method:

Since the region lies entirely below $y = 1$, a vertical slice at a fixed x (with $0 \leq x \leq \pi$) extends from $y = 0$ to $y = \sin x$. When rotated about $y = 1$, this slice generates a washer with

$$\text{Outer radius: } R = 1 - 0 = 1$$

$$\text{Inner radius: } r = 1 - \sin x$$

Thus, the area of the washer is

$$A(x) = \pi [R^2 - r^2] = \pi [1^2 - (1 - \sin x)^2].$$

Expanding the inner square,

$$(1 - \sin x)^2 = 1 - 2 \sin x + \sin^2 x,$$

so that

$$A(x) = \pi [1 - (1 - 2 \sin x + \sin^2 x)] = \pi (2 \sin x - \sin^2 x).$$

Hence, the volume is given by

$$\begin{aligned} V &= \int_{x=0}^{\pi} A(x) dx \\ &= \pi \int_0^{\pi} (2 \sin x - \sin^2 x) dx. \end{aligned}$$

We evaluate the integrals separately:

$$\int_0^{\pi} 2 \sin x dx = 2 [-\cos x]_0^{\pi} = 2 [-(-1) + 1] = 4,$$

and

$$\int_0^{\pi} \sin^2 x dx = \frac{\pi}{2}.$$

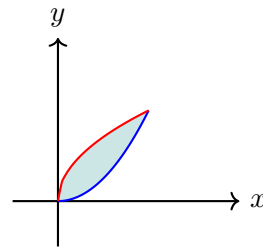
Thus,

$$V = \pi \left(4 - \frac{\pi}{2} \right) = \frac{\pi}{2} (8 - \pi).$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the y -axis.



Solution via the Washer Method:

$$\begin{aligned} V_{\text{washers}} &= \int_{y=0}^1 A(y) dy \\ &= \pi \int_0^1 (y - y^4) dy \\ &= \pi \left[\frac{y^2}{2} - \frac{y^5}{5} \right]_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{5} \right) \\ &= \pi \left(\frac{5-2}{10} \right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

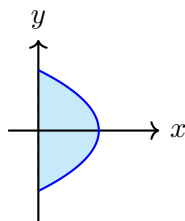
Solution via the Shell Method:

$$\begin{aligned} V_{\text{shells}} &= 2\pi \int_0^1 (x^{3/2} - x^3) dx \\ &= 2\pi \left[\frac{2}{5} x^{5/2} - \frac{x^4}{4} \right]_0^1 \\ &= 2\pi \left(\frac{2}{5} - \frac{1}{4} \right) \\ &= 2\pi \left(\frac{8-5}{20} \right) \\ &= 2\pi \left(\frac{3}{20} \right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = 1 - y^2, \quad x = 0, \quad y \in [-1, 1]$$

about the y -axis.



Solution Using the Disk Method:

Since rotation is about the y -axis, the radius of a typical disk at height y is

$$R(y) = 1 - y^2.$$

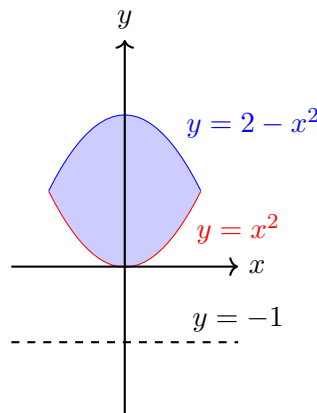
Thus, the volume is

$$\begin{aligned} V &= \pi \int_{y=-1}^1 (1 - y^2)^2 dy \\ &= \pi \int_{-1}^1 (1 - 2y^2 + y^4) dy \\ &= \pi \left[\int_{-1}^1 1 dy - 2 \int_{-1}^1 y^2 dy + \int_{-1}^1 y^4 dy \right] \\ &= \pi \left[2 - 2 \left(\frac{2}{3} \right) + \frac{2}{5} \right] \\ &= \pi \left[2 - \frac{4}{3} + \frac{2}{5} \right] \\ &= \pi \left[\frac{30}{15} - \frac{20}{15} + \frac{6}{15} \right] \\ &= \pi \left(\frac{16}{15} \right). \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x^2$$

about the line $y = -1$.



Solution Using the Washer Method:

For a fixed x with $x \in [-1, 1]$, the region extends vertically from the lower curve $y = x^2$ to the upper curve $y = 2 - x^2$. When rotated about $y = -1$, the distances from $y = -1$ are:

$$\text{Inner radius: } r(x) = x^2 - (-1) = x^2 + 1$$

$$\text{Outer radius: } R(x) = (2 - x^2) - (-1) = 3 - x^2.$$

Thus, the washer cross-sectional area is

$$A(x) = \pi \left[(3 - x^2)^2 - (x^2 + 1)^2 \right].$$

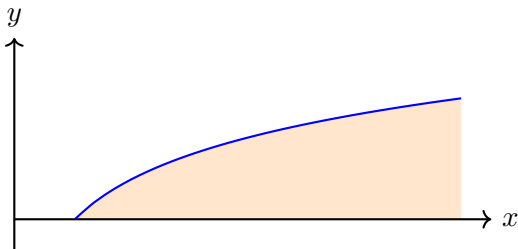
The volume is

$$\begin{aligned} V &= \pi \int_{x=-1}^1 \left[(3 - x^2)^2 - (x^2 + 1)^2 \right] dx \\ &= \pi \int_{-1}^1 (8 - 8x^2) dx \\ &= 8\pi \int_{-1}^1 (1 - x^2) dx \\ &= 8\pi \left[\int_{-1}^1 1 dx - \int_{-1}^1 x^2 dx \right] \\ &= 8\pi \left[(1 - (-1)) - \left(\frac{1^3 - (-1)^3}{3} \right) \right] \\ &= \frac{32\pi}{3}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln x, \quad y = 0, \quad x = 1, \quad x = e^2$$

about the y -axis.



Solution Using the Washer Method:

Since we are rotating about the y -axis, it is convenient to express x in terms of y . From

$$y = \ln x \implies x = e^y,$$

and note that when $x = e^2$ we have $y = \ln(e^2) = 2$. Thus, for a fixed y between 0 and 2, the region extends in x from the curve $x = e^y$ to the vertical line $x = e^2$. When rotated about the y -axis, the horizontal slice produces an annular cross section with

Inner radius: $R_{\text{in}}(y) = e^y$, Outer radius: $R_{\text{out}}(y) = e^2$.

The area of the washer is then

$$A(y) = \pi \left[(e^2)^2 - (e^y)^2 \right] = \pi (e^4 - e^{2y}).$$

Thus, the volume by washers is

$$\begin{aligned} V_{\text{washers}} &= \pi \int_{y=0}^2 (e^4 - e^{2y}) dy \\ &= \pi \left[e^4 y - \frac{e^{2y}}{2} \right]_{y=0}^2 \\ &= \pi \left[e^4(2) - \frac{e^4}{2} - \left(0 - \frac{1}{2} \right) \right] \\ &= \pi \left(2e^4 - \frac{e^4}{2} + \frac{1}{2} \right) \\ &= \pi \left(\frac{4e^4 - e^4 + 1}{2} \right) \\ &= \frac{\pi(3e^4 + 1)}{2}. \end{aligned}$$

Solution Using the Shell Method:

Using vertical slices, the region is described by x from 1 to e^2 and, for each x , y runs from $y = 0$ to $y = \ln x$. A typical vertical slice at x has height

$$h(x) = \ln x,$$

and its distance (radius) from the y -axis is

$$r(x) = x.$$

Thus, the volume by shells is given by

$$V_{\text{shells}} = 2\pi \int_{x=1}^{e^2} x (\ln x) dx.$$

We compute the integral using integration by parts. Let

$$\begin{aligned} u = \ln x &\implies du = \frac{1}{x} dx \\ dv = x dx &\implies v = \frac{x^2}{2} \end{aligned}$$

Then,

$$\begin{aligned} \int x \ln x dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4}. \end{aligned}$$

Evaluating from $x = 1$ to $x = e^2$:

$$\begin{aligned} &\left. \frac{x^2}{2} \ln x - \frac{x^2}{4} \right|_{x=1}^{e^2} \\ &= \left[\frac{(e^2)^2}{2} \ln(e^2) - \frac{(e^2)^2}{4} \right] - \left[\frac{1}{2} \ln 1 - \frac{1}{4} \right] \\ &= \left[\frac{e^4}{2} \cdot 2 - \frac{e^4}{4} \right] - \left[0 - \frac{1}{4} \right] \\ &= \left[e^4 - \frac{e^4}{4} \right] + \frac{1}{4} \\ &= \frac{4e^4 - e^4 + 1}{4} \\ &= \frac{3e^4 + 1}{4}. \end{aligned}$$

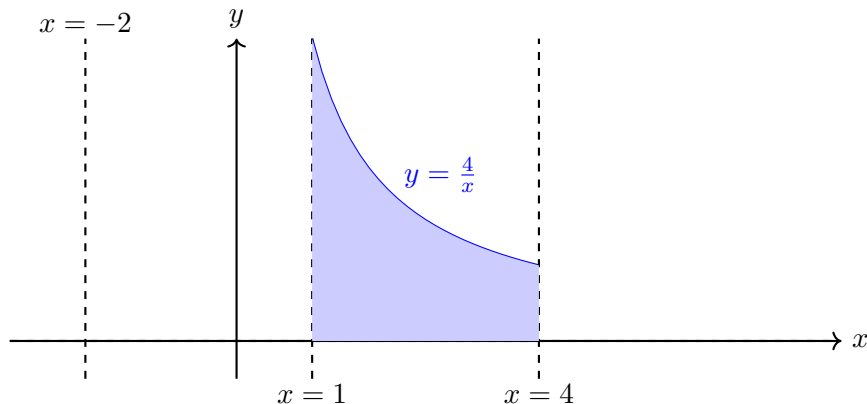
Thus,

$$\begin{aligned} V_{\text{shells}} &= 2\pi \cdot \frac{3e^4 + 1}{4} \\ &= \frac{\pi(3e^4 + 1)}{2}. \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \frac{4}{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the line $x = -2$.



Solution Using the Shell Method:

A vertical slice at position x (with $1 \leq x \leq 4$) has height

$$h(x) = \frac{4}{x},$$

and its distance (radius) from the axis $x = -2$ is

$$r(x) = x - (-2) = x + 2.$$

Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 r(x) h(x) dx \\ &= 2\pi \int_1^4 (x + 2) \left(\frac{4}{x} \right) dx \\ &= 2\pi \int_1^4 \frac{4(x + 2)}{x} dx \\ &= 8\pi \int_1^4 \left(1 + \frac{2}{x} \right) dx \\ &= 8\pi \left[\int_1^4 1 dx + 2 \int_1^4 \frac{1}{x} dx \right] \\ &= 8\pi \left[(4 - 1) + 2 \ln x \Big|_1^4 \right] \\ &= 8\pi [3 + 2 \ln(4)] \end{aligned}$$

6.4 Work (Solutions)

Spring Problems (Solutions)

1. A spring has a natural length of 20 m. A force of 12 N is required to stretch the spring to 25 m. Determine the work required to stretch the spring from 20 m to 30 m.

- By Hooke's Law, the force required to stretch a spring is:

$$F = kx,$$

where k is the spring constant and x is the displacement from the natural length.

- Given that a force of 12 N is required to stretch the spring to 25 m:

$$12 = k(5).$$

Solving for k :

$$k = \frac{12}{5} = 2.4.$$

- The work done to stretch the spring from $x = a$ to $x = b$ is given by:

$$W = \int_a^b kx \, dx.$$

- Here, we compute the work to stretch from 20 m to 30 m, which corresponds to $x = 0$ to $x = 10$ m:

$$W = \int_0^{10} 2.4x \, dx.$$

- Computing the integral:

$$\begin{aligned} W &= 2.4 \int_0^{10} x \, dx \\ &= 2.4 \left[\frac{x^2}{2} \right]_0^{10} \\ &= 2.4 \left(\frac{100}{2} - \frac{0}{2} \right) \\ &= 2.4 \times 50 \\ &= 120 \text{ J.} \end{aligned}$$

2. A spring has a natural length of 15 m. A force of 10 N is required to stretch the spring to 18 m. Determine the work required to stretch the spring from 16 m to 22 m.

- Hooke's Law states that $F = kx$, where k is the spring constant and x is the displacement from the natural length.
- Given that a force of 10 N stretches the spring to 18 m, we find k :

$$10 = k(18 - 15)$$

$$10 = k(3)$$

$$k = \frac{10}{3} \text{ N/m}$$

- The work done to stretch the spring from $x_1 = 16$ m to $x_2 = 22$ m is given by:

$$\begin{aligned} W &= \int_{x_1}^{x_2} kx \, dx \\ &= \int_1^7 \frac{10}{3}x \, dx \\ &= \frac{10}{3} \left[\frac{x^2}{2} \right]_1^7 \\ &= \frac{10}{3} \left(\frac{49}{2} - \frac{1}{2} \right) \\ &= \frac{10}{3} \times \frac{48}{2} \\ &= \frac{10}{3} \times 24 \\ &= 80 \text{ J} \end{aligned}$$

3. A spring has a natural length of 30 m. A force of 8 N is required to stretch the spring to 35 m. Determine the work required to compress the spring from 30 m to 20 m.

- Using Hooke's Law, we first determine the spring constant k :

$$8 = k(35 - 30)$$

$$8 = k(5)$$

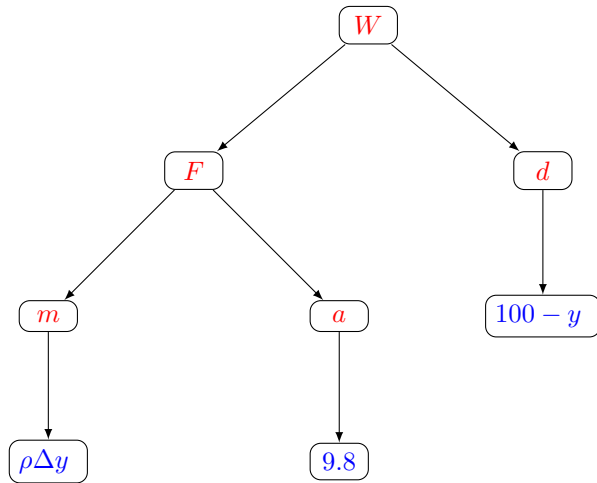
$$k = \frac{8}{5} \text{ N/m}$$

- The work required to compress the spring from $x_1 = 30$ m to $x_2 = 20$ m is:

$$\begin{aligned} W &= \int_{x_1}^{x_2} kx \, dx \\ &= \int_0^{-10} \frac{8}{5}x \, dx \\ &= \frac{8}{5} \left[\frac{x^2}{2} \right]_0^{-10} \\ &= \frac{8}{5} \left(\frac{100}{2} - 0 \right) \\ &= \frac{8}{5} \times 50 \\ &= 80 \text{ J} \end{aligned}$$

Cable Problems (Solutions)

1. A **100-meter-long** cable with a **linear density of 5 kg/m** is hanging from a winch at the top of a well. The cable is initially fully extended into the well and is lifted **to the top**. Compute the work required to lift the entire cable.



Step 1: Define Variables and Divide the Cable into Slices

- Let y be the height above the **bottom of the well**, with $y = 0$ at the bottom and $y = 100$ at the top.
- Partition the interval $[0, 100]$ into n slices of equal height Δy .
- The mass of a slice of cable at height y_i^* is:

$$m_i = \rho \Delta y = 5 \Delta y.$$

- The force due to gravity acting on the slice is:

$$F_i = g \cdot m_i = 9.8 \cdot (5 \Delta y).$$

Step 2: Compute the Work on Each Slice

- Each slice at height y_i^* must be lifted from its original position y_i^* to the top ($y = 100$).
- The lifting distance for the slice is:

$$d_i = 100 - y_i^*.$$

- The work done to lift the i th slice is:

$$W_i = F_i \cdot d_i = (9.8 \cdot 5 \Delta y) \cdot (100 - y_i^*).$$

Step 3: Express Work as an Integral

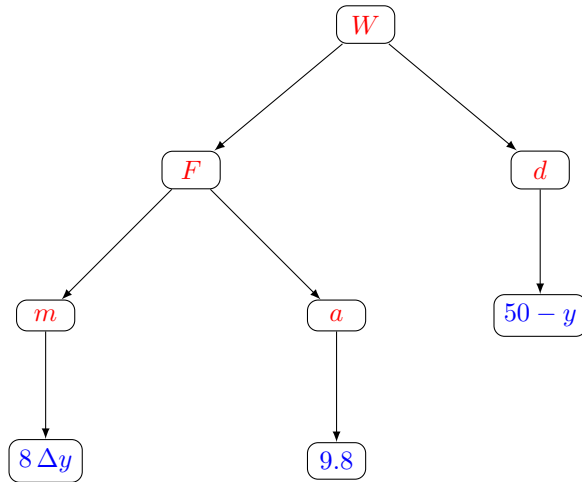
- Summing over all slices and taking the limit as $n \rightarrow \infty$, the total work is:

$$W = \int_0^{100} 9.8 \cdot 5 \cdot (100 - y) dy.$$

- Evaluating the integral:

$$\begin{aligned}
 W &= 49 \int_0^{100} (100 - y) dy \\
 &= 49 \left[100y - \frac{y^2}{2} \right]_0^{100} \\
 &= 49 \left(100(100) - \frac{100^2}{2} \right) \\
 &= 49 (10000 - 5000) \\
 &= 49 \times 5000 \\
 &= 245000 \text{ J.}
 \end{aligned}$$

2. A **50-meter-long** chain with a **linear density of 8 kg/m** is hanging from a pulley at the top of a mine shaft. The chain is initially fully extended into the shaft and is lifted **to the top**. Compute the work required to lift the entire chain.



Step 1: Define Variables and Divide the Chain into Slices

- Let y denote the height above the **bottom of the shaft** (with $y = 0$ at the bottom and $y = 50$ at the top).
- Divide the interval $[0, 50]$ into n slices of equal thickness Δy .
- The mass of a slice at height y_i^* is:

$$m_i = \rho \Delta y = 8 \Delta y.$$

- The weight (force due to gravity) on the slice is:

$$F_i = m_i g = 8(9.8) \Delta y = 78.4 \Delta y.$$

Step 2: Compute the Work on Each Slice

- A slice at height y_i^* is lifted to the top, a distance of:

$$d_i = 50 - y_i^*.$$

- The work done on this slice is approximately:

$$W_i = F_i \cdot d_i = 78.4 (50 - y_i^*) \Delta y.$$

Step 3: Express Work as an Integral

- Summing over all slices and taking the limit as $n \rightarrow \infty$ gives:

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 78.4 (50 - y_i^*) \Delta y \\ &= 78.4 \int_0^{50} (50 - y) dy. \end{aligned}$$

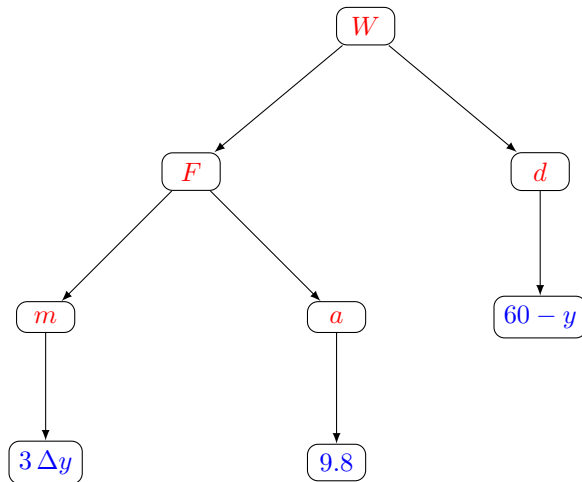
- Evaluate the integral:

$$\begin{aligned} \int_0^{50} (50 - y) dy &= \left[50y - \frac{y^2}{2} \right]_0^{50} \\ &= 50(50) - \frac{50^2}{2} \\ &= 2500 - 1250 \\ &= 1250. \end{aligned}$$

- Thus, the total work is:

$$W = 78.4 \times 1250 = 98\,000 \text{ J.}$$

3. A **60-meter-long** rope with a **linear density of 3 kg/m** is hanging over the edge of a cliff, with one end secured at the top and the other end dangling freely. The rope is slowly lifted until it is fully coiled at the top of the cliff. Compute the work required to lift the rope.



Step 1: Define Variables and Divide the Rope into Slices

- Let y denote the height above the **bottom of the rope** (with $y = 0$ at the bottom and $y = 60$ at the top).
- Divide the interval $[0, 60]$ into n slices of equal length Δy .
- The mass of a slice is:

$$m_i = \rho \Delta y = 3 \Delta y.$$

- The weight on the slice is:

$$F_i = m_i g = 3(9.8) \Delta y = 29.4 \Delta y.$$

Step 2: Compute the Work on Each Slice

- Each slice at height y_i^* is lifted a distance:

$$d_i = 60 - y_i^*.$$

- The work done on the slice is:

$$W_i = F_i \cdot d_i = 29.4 (60 - y_i^*) \Delta y.$$

Step 3: Express Work as an Integral

- The total work is:

$$W = 29.4 \int_0^{60} (60 - y) dy.$$

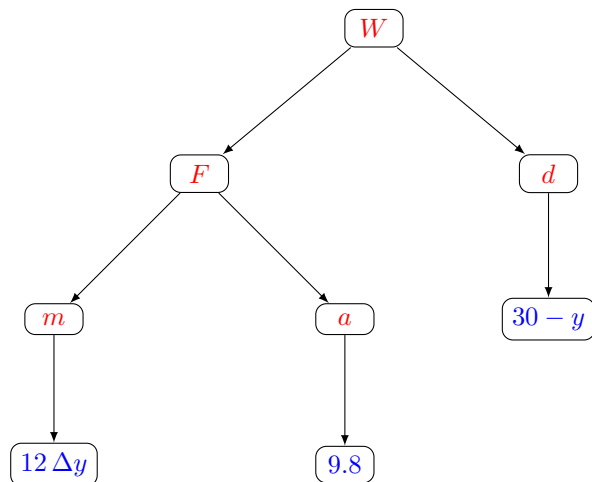
- Evaluate the integral:

$$\begin{aligned} \int_0^{60} (60 - y) dy &= \left[60y - \frac{y^2}{2} \right]_0^{60} \\ &= 3600 - 1800 \\ &= 1800. \end{aligned}$$

- Hence, the work is:

$$W = 29.4 \times 1800 = 52\,920 \text{ J}.$$

4. A **30-meter-long** anchor chain with a **linear density of 12 kg/m** is hanging from the side of a ship, with one end attached to the ship and the other submerged in the water. The chain is hoisted onto the deck of the ship. Compute the work required to lift the entire chain onto the ship.



Step 1: Define Variables and Divide the Chain into Slices

- Let y denote the height above the **bottom of the chain** (with $y = 0$ at the submerged end and $y = 30$ at the deck).
- Divide the interval $[0, 30]$ into n slices of thickness Δy .
- The mass of a slice is:

$$m_i = \rho \Delta y = 12 \Delta y.$$

- The weight on the slice is:

$$F_i = m_i g = 12(9.8) \Delta y = 117.6 \Delta y.$$

Step 2: Compute the Work on Each Slice

- A slice at height y_i^* is lifted a distance:

$$d_i = 30 - y_i^*.$$

- The work done on the slice is:

$$W_i = F_i \cdot d_i = 117.6 (30 - y_i^*) \Delta y.$$

Step 3: Express Work as an Integral

- The total work is:

$$W = 117.6 \int_0^{30} (30 - y) dy.$$

- Evaluate the integral:

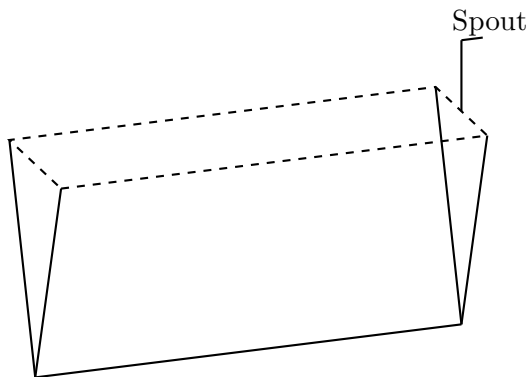
$$\begin{aligned} \int_0^{30} (30 - y) dy &= \left[30y - \frac{y^2}{2} \right]_0^{30} \\ &= 900 - 450 \\ &= 450. \end{aligned}$$

- Thus, the work is:

$$W = 117.6 \times 450 = 52\,920 \text{ J.}$$

Tank Problems (Solutions)

1. **Rectangular Tank with Triangular Ends.** A tank is 6 m long (into the page) and its end view is an isosceles triangle with a base of 2 m and a height of 3 m. The tank is filled with water, and the water is pumped out through a spout located 0.5 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the bottom (vertex) and $y = 3$ at the top.

Partition the interval $[0, 3]$ into n subintervals of equal width Δy . For the i th subinterval, choose a representative point y_i^* .

By similar triangles, the width of the tank at y_i^* is

$$w_i = \frac{2}{3} y_i^*.$$

Since the tank is 6 m long, the cross-sectional area of a slice is

$$A_i = 6 \cdot w_i = 6 \left(\frac{2}{3} y_i^* \right) = 4 y_i^*.$$

Thus, the volume of the i th slice is given by

$$V_i = A_i \Delta y = 4 y_i^* \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the i th slice is obtained by multiplying the mass by gravitational acceleration:

$$F_i = \rho g V_i = 1000 \cdot 9.8 \cdot (4 y_i^* \Delta y).$$

Each slice must be lifted to the spout, which is at a height of $3 + 0.5 = 3.5$ m. Hence, the lifting distance for the i th slice is

$$d_i = 3.5 - y_i^*.$$

Thus, the work done on the i th slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot 4 y_i^* (3.5 - y_i^*) \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

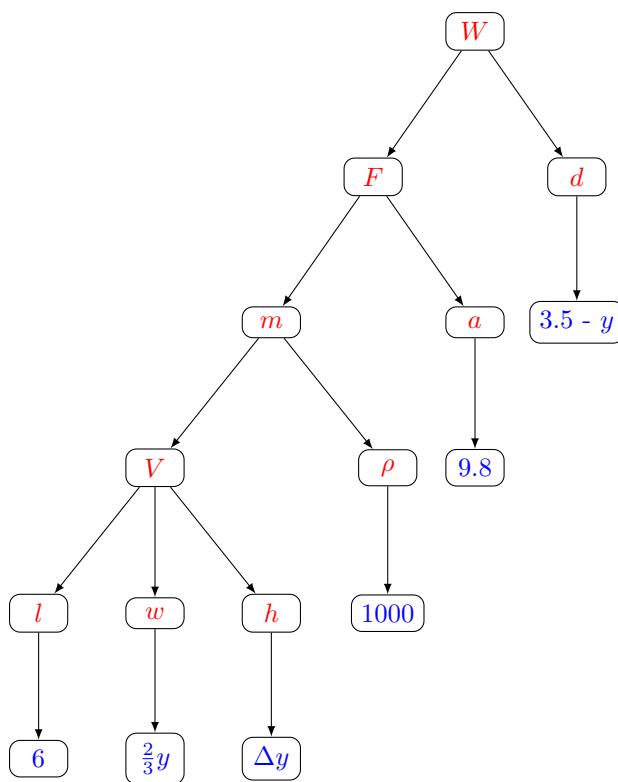
The total work required to pump the water is given by the Riemann sum

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n [1000 \cdot 9.8 \cdot 4 y_i^* (3.5 - y_i^*) \Delta y].$$

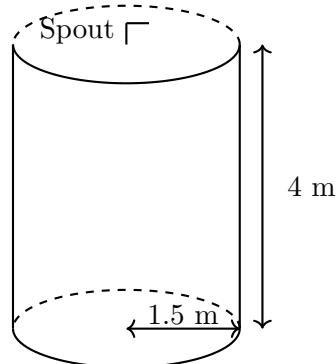
This is equivalent to the integral

$$W = \int_0^3 1000 \cdot 9.8 \cdot 4 y (3.5 - y) dy.$$

Tree Diagram Representation:



2. **Cylindrical Tank with a Spout.** A vertical cylindrical tank is 4 m high with a radius of 1.5 m. The tank is completely full of water, and the water is pumped out through a spout located 0.3 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Let $y = 0$ at the bottom of the cylinder and $y = 4$ at the top. Partition $[0, 4]$ into n subintervals of equal width Δy . In the i th subinterval, pick a representative point y_i^* .

The cross-section at height y_i^* is a circle of radius 1.5. Hence, its area is

$$A = \pi \times (1.5)^2 = 2.25\pi.$$

The volume of the i th slice is

$$V_i = A \Delta y = 2.25\pi \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the i th slice is

$$F_i = \rho g V_i = 1000 \cdot 9.8 \cdot (2.25\pi \Delta y).$$

Each slice must be lifted to the spout, which is at $y = 4 + 0.3 = 4.3$. Hence, the lifting distance is

$$d_i = 4.3 - y_i^*.$$

Thus, the work for the i th slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot 2.25\pi (4.3 - y_i^*) \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

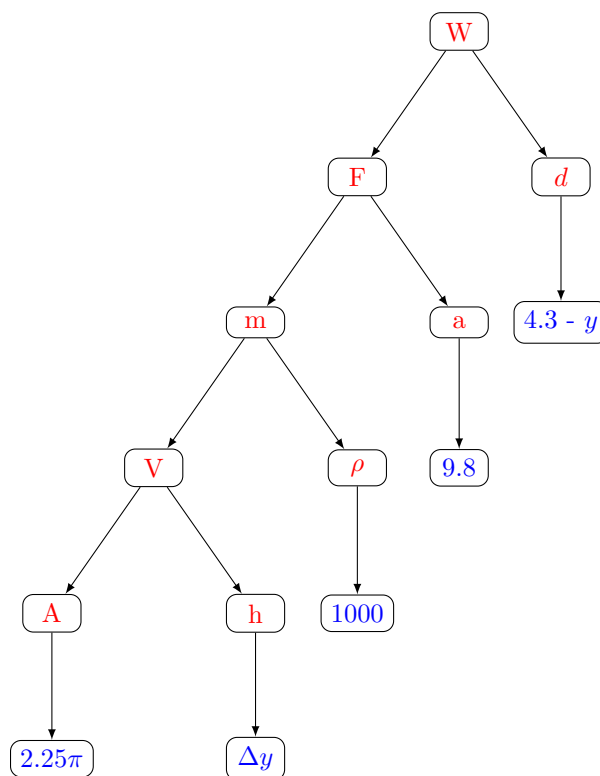
Summing over all slices,

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n [1000 \cdot 9.8 \cdot 2.25\pi (4.3 - y_i^*) \Delta y].$$

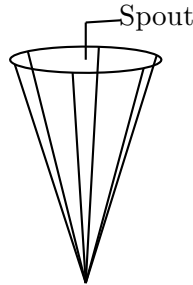
Equivalently, in integral form,

$$W = \int_0^4 1000 \cdot 9.8 \cdot 2.25\pi (4.3 - y) dy.$$

Tree Diagram Representation:



3. **Inverted Conical Tank.** An inverted conical tank has a height of 3 m and a top (open) radius of 1 m. The tank is completely full of water, and the water is pumped out through a spout located 0.2 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the tip (bottom) and $y = 3$ at the top. Partition the interval $[0, 3]$ into n equal subintervals of width Δy ; in the i th subinterval choose a representative point y_i^* .

By similar triangles, the radius at height y_i^* is

$$r_i = \frac{1}{3} y_i^*,$$

so the cross-sectional area is

$$A_i = \pi(r_i)^2 = \pi\left(\frac{y_i^*}{3}\right)^2 = \frac{\pi(y_i^*)^2}{9}.$$

Thus, the volume of the i th slice is

$$V_i = A_i \Delta y = \frac{\pi(y_i^*)^2}{9} \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the i th slice is given by

$$F_i = \rho g V_i = 1000 \cdot 9.8 \cdot \frac{\pi(y_i^*)^2}{9} \Delta y.$$

The water must be pumped up to the spout, which is located 0.2 m above the top. Since the top is at $y = 3$, the spout is at $y = 3.2$. Therefore, the lifting distance is

$$d_i = 3.2 - y_i^*.$$

Thus, the work done on the i th slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot \frac{\pi(y_i^*)^2}{9} (3.2 - y_i^*) \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

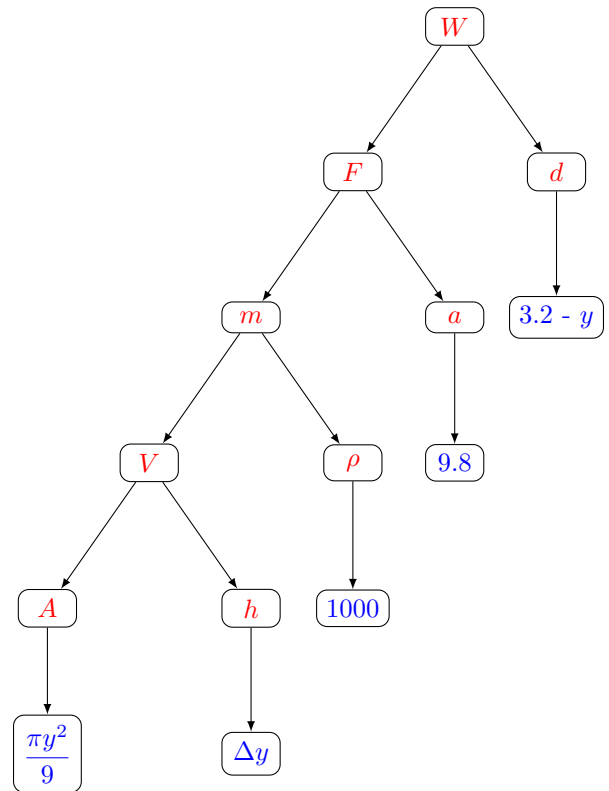
Summing over all slices, the total work is

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1000 \cdot 9.8 \cdot \frac{\pi(y_i^*)^2}{9} (3.2 - y_i^*) \Delta y \right],$$

which is equivalent to the integral

$$W = \int_0^3 1000 \cdot 9.8 \cdot \frac{\pi y^2}{9} (3.2 - y) dy.$$

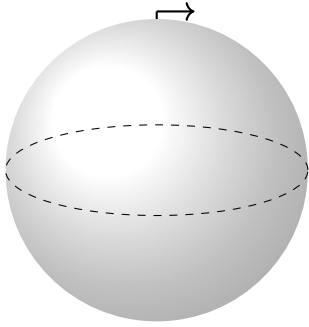
Tree Diagram Representation:



Thus, the work required to pump the water is

$$W = \int_0^3 1000 \cdot 9.8 \cdot \frac{\pi y^2}{9} (3.2 - y) dy.$$

4. **Spherical Tank.** A spherical tank of radius 2 m is completely filled with water. A spout is located 0.1 m above the top of the sphere.



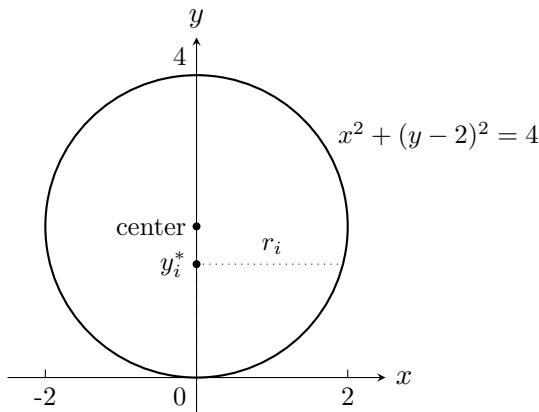
Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the bottom of the sphere, and $y = 4$ at the top. Partition the interval $[0, 4]$ into n subintervals of equal width Δy ; in the i th subinterval, choose a representative point y_i^* .

At height y_i^* , the horizontal cross section of the sphere is a circle whose radius is given by

$$r_i = \sqrt{2^2 - (y_i^* - 2)^2} = \sqrt{4 - (y_i^* - 2)^2}.$$

To see this, consider the side view of the spherical tank:



Then at height y_i^* , the radius of the cross section is the corresponding x -coordinate on the graph of $x^2 + (y - 2)^2 = 4$. Plugging in y_i^* for y , we solve

$$r_i = x = \sqrt{4 - (y_i^* - 2)^2}$$

Thus, the area of the slice is

$$A_i = \pi (r_i)^2 = \pi [4 - (y_i^* - 2)^2].$$

The volume of the i th slice is then

$$V_i = A_i \Delta y = \pi [4 - (y_i^* - 2)^2] \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the i th slice is

$$F_i = \rho g V_i = 1000 \cdot 9.8 \cdot \pi [4 - (y_i^* - 2)^2] \Delta y.$$

The spout is 0.1 m above the top of the sphere (top is at $y = 4$), so the spout is at $y = 4.1$. Hence, the lifting distance for the i th slice is

$$d_i = 4.1 - y_i^*.$$

Thus, the work done on the i th slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot \pi [4 - (y_i^* - 2)^2] (4.1 - y_i^*) \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

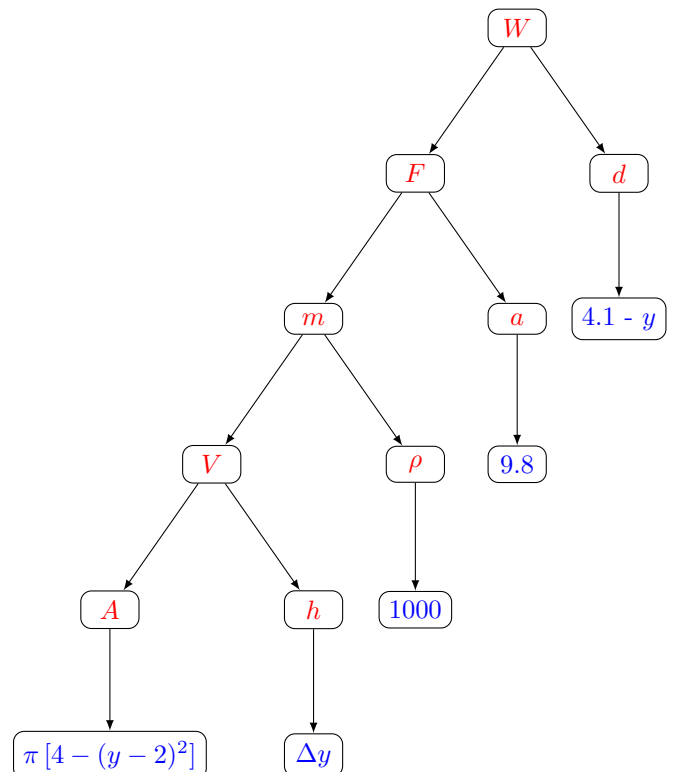
The total work required to pump the water is given by the Riemann sum

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n [1000 \cdot 9.8 \cdot \pi [4 - (y_i^* - 2)^2] (4.1 - y_i^*) \Delta y],$$

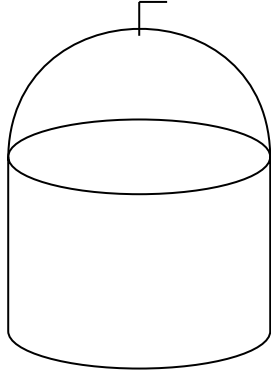
which is equivalent to the integral

$$W = \int_0^4 1000 \cdot 9.8 \cdot \pi [4 - (y - 2)^2] (4.1 - y) dy.$$

Tree Diagram Representation:



5. **Composite Tank – Cylinder with Hemispherical Top.** A tank consists of a cylindrical section 3 m high with a circular cross section of radius 1 m, topped by a hemispherical dome of radius 1 m. Water is pumped out through a spout located 0.15 m above the dome.



Step 1: Divide the Tank into Slices.

(a) Cylindrical Section (for $0 \leq y \leq 3$):

- Partition $[0, 3]$ into n subintervals of width Δy ; choose a representative point y_i^* in each.
- The cross-sectional area is constant:

$$A_{\text{cyl}} = \pi(1)^2 = \pi.$$

- The volume of the i th slice is

$$V_i^{\text{cyl}} = A_{\text{cyl}} \Delta y = \pi \Delta y.$$

(b) Hemispherical Dome (for $3 \leq y \leq 4$):

- Partition $[3, 4]$ into subintervals of width Δy ; choose a representative point y_i^* in each.
- The hemisphere is the upper half of a sphere of radius 1.
- At height y_i^* , the horizontal cross-sectional radius is

$$r_i = \sqrt{1 - (y_i^* - 3)^2},$$

(see the previous problem for an explanation), so the area is

$$A_i^{\text{hemi}} = \pi (r_i)^2 = \pi [1 - (y_i^* - 3)^2].$$

- The volume of the i th slice is then

$$V_i^{\text{hemi}} = A_i^{\text{hemi}} \Delta y.$$

Step 2: Compute the Work on Each Slice.

Let the density $\rho = 1000$ and gravitational acceleration $g = 9.8$.

(a) Cylindrical Section:

$$F_i^{\text{cyl}} = \rho g V_i^{\text{cyl}} = 1000 \cdot 9.8 \cdot \pi \Delta y.$$

Each slice must be lifted to the spout at $y = 4.15$; therefore, the lifting distance is

$$d_i^{\text{cyl}} = 4.15 - y_i^*.$$

Thus, the work on the i th cylindrical slice is

$$W_i^{\text{cyl}} = F_i^{\text{cyl}} \cdot d_i^{\text{cyl}} = 1000 \cdot 9.8 \cdot \pi (4.15 - y_i^*) \Delta y.$$

(b) Hemispherical Dome:

$$F_i^{\text{hemi}} = \rho g V_i^{\text{hemi}} = 1000 \cdot 9.8 \cdot \pi [1 - (y_i^* - 3)^2] \Delta y.$$

The lifting distance is the same:

$$d_i^{\text{hemi}} = 4.15 - y_i^*.$$

Thus, the work on the i th hemispherical slice is

$$\begin{aligned} W_i^{\text{hemi}} &= F_i^{\text{hemi}} \cdot d_i^{\text{hemi}} \\ &= 1000 \cdot 9.8 \cdot \pi [1 - (y_i^* - 3)^2] (4.15 - y_i^*) \Delta y. \end{aligned}$$

Step 3: Write the Total Work as a Riemann Sum.

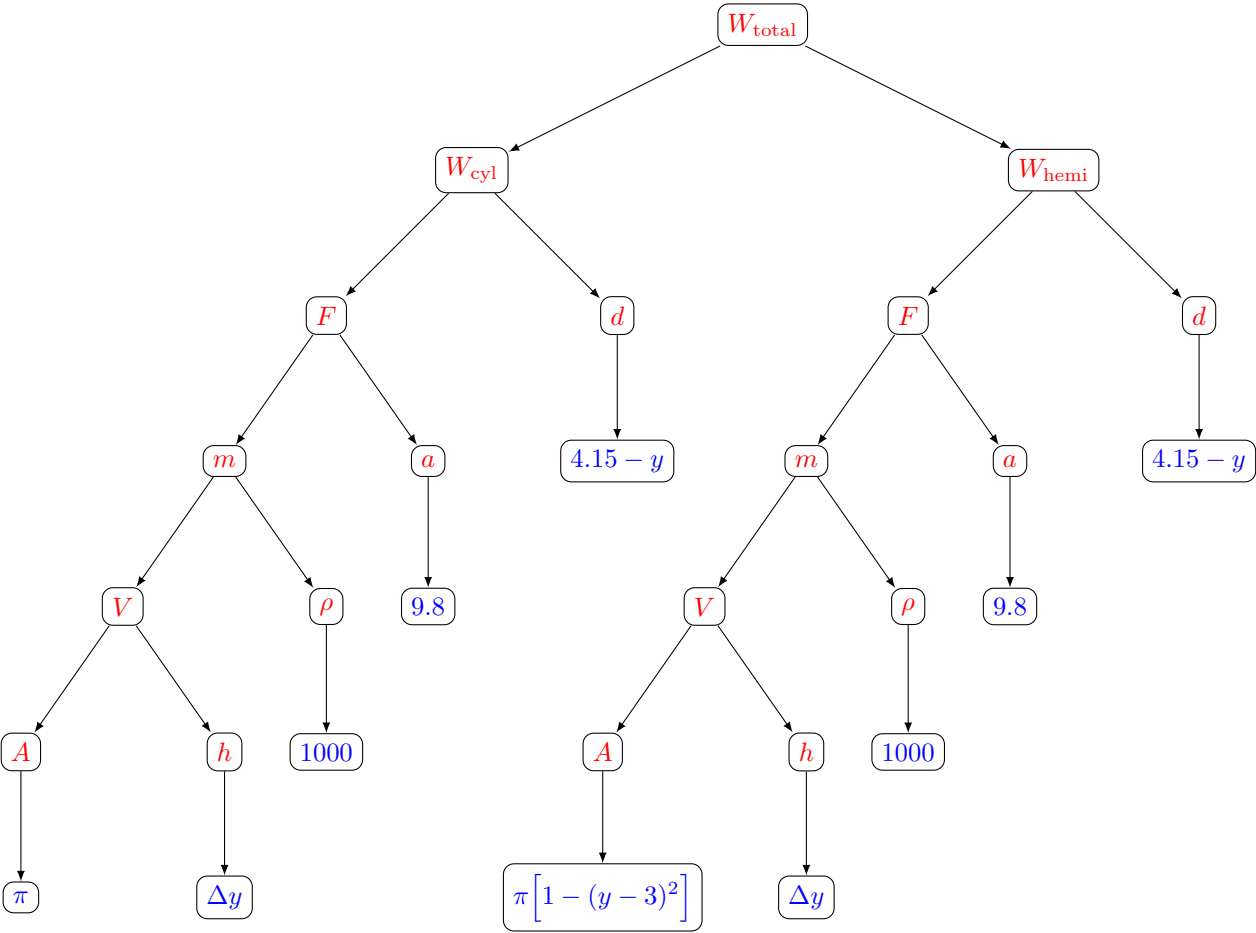
The total work is the sum of the work on the cylindrical section and the hemispherical dome:

$$W = \lim_{n \rightarrow \infty} \left[\sum_{i=1}^{n_{\text{cyl}}} W_i^{\text{cyl}} + \sum_{i=1}^{n_{\text{hemi}}} W_i^{\text{hemi}} \right],$$

which is equivalent to

$$\begin{aligned} W &= \int_0^3 1000 \cdot 9.8 \cdot \pi (4.15 - y) dy \\ &\quad + \int_3^4 1000 \cdot 9.8 \cdot \pi [1 - (y - 3)^2] (4.15 - y) dy. \end{aligned}$$

Tree Diagram Representation:



6.5 Average Value of a Function (Solutions)

1. $f(x) = x^2 + 1, \quad x \in [0, 3].$

Average value:

$$\text{Average} = \frac{1}{3-0} \int_0^3 (x^2 + 1) dx.$$

Compute the integral:

$$\begin{aligned} \int_0^3 (x^2 + 1) dx &= \left[\frac{x^3}{3} + x \right]_0^3 \\ &= \left(\frac{3^3}{3} + 3 \right) - (0 + 0) \\ &= (9 + 3) = 12. \end{aligned}$$

Hence,

$$\text{Average} = \frac{12}{3} = 4.$$

Finding c : We want $f(c) = 4$, i.e.

$$\begin{aligned} c^2 + 1 = 4 &\implies c^2 = 3 \\ &\implies c = \sqrt{3} \quad (\text{since } c \geq 0 \text{ in } [0, 3]). \end{aligned}$$

Thus, $c = \sqrt{3}$.

2. $f(x) = (x - 1)^2, \quad x \in [0, 2].$

Average value:

$$\text{Average} = \frac{1}{2-0} \int_0^2 (x - 1)^2 dx.$$

First expand or integrate directly:

$$\begin{aligned} \int_0^2 (x - 1)^2 dx &= \int_0^2 (x^2 - 2x + 1) dx \\ &= \left[\frac{x^3}{3} - x^2 + x \right]_0^2 \\ &= \frac{2}{3} \end{aligned}$$

Divide by the interval length $(2 - 0) = 2$:

$$\text{Average} = \frac{\frac{2}{3}}{2} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

Finding c : We want $(c - 1)^2 = \frac{1}{3}$. That is

$$c - 1 = \pm \frac{1}{\sqrt{3}}.$$

Within $[0, 2]$, $c = 1 - \frac{1}{\sqrt{3}}$ is less than 1 and we must check if it remains ≥ 0 . Numerically, $1 - \frac{1}{\sqrt{3}} \approx 1 - 0.577 \approx 0.423 > 0$, so it is indeed in $[0, 2]$. Also,

$$c = 1 + \frac{1}{\sqrt{3}}$$

is about 1.577, which is also in $[0, 2]$. Hence there are two solutions:

$$c_1 = 1 - \frac{1}{\sqrt{3}}, \quad c_2 = 1 + \frac{1}{\sqrt{3}}.$$

3. $f(x) = 2 \cos x + 1, \quad x \in [0, \pi].$

Average value:

$$\text{Average} = \frac{1}{\pi - 0} \int_0^\pi (2 \cos x + 1) dx.$$

Compute the integral:

$$\begin{aligned} \int_0^\pi (2 \cos x + 1) dx &= [2 \sin x + x]_0^\pi \\ &= (2 \sin \pi + \pi) - (2 \sin 0 + 0) \\ &= \pi \end{aligned}$$

Thus,

$$\text{Average} = \frac{\pi}{\pi} = 1.$$

Finding c : We want $2 \cos c + 1 = 1$, so $2 \cos c = 0 \Rightarrow \cos c = 0$. Within $[0, \pi]$, this happens at $c = \frac{\pi}{2}$.

4. $f(x) = e^x, \quad x \in [0, 1].$

Average value:

$$\text{Average} = \frac{1}{1 - 0} \int_0^1 e^x dx = \int_0^1 e^x dx.$$

$$\int_0^1 e^x dx = e^x \Big|_0^1 = e - 1.$$

Hence the average value is

$$\text{Average} = e - 1.$$

Finding c : We want $e^c = e - 1$. Taking the natural logarithm,

$$c = \ln(e - 1).$$

Since $e - 1 > 1$, $\ln(e - 1)$ is positive and is indeed in the interval $[0, 1]$ (numerically about 0.54).

5. $f(x) = x, \quad x \in [-2, 2].$

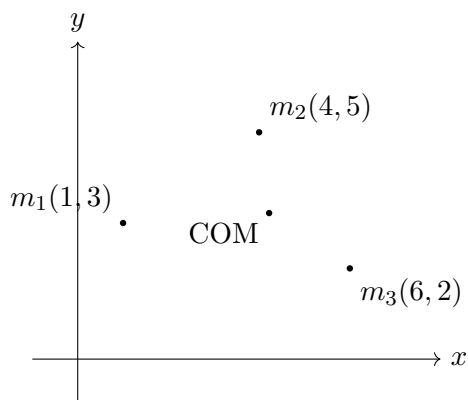
Average value:

$$\begin{aligned}\text{Average} &= \frac{1}{2 - (-2)} \int_{-2}^2 x \, dx \\ &= \frac{1}{4} \left[\frac{x^2}{2} \right]_{-2}^2 \\ &= \frac{1}{4} \left(\frac{4}{2} - \frac{4}{2} \right) \\ &= \frac{1}{4} (2 - 2) \\ &= 0\end{aligned}$$

Finding c : We need $x = 0$. Thus, $c = 0$ in the interval $[-2, 2]$ satisfies $f(c) = 0$.

8.3 Center of Mass (Solutions)

1. Three Point Masses.



- **Total mass:**

$$M = 2 + 3 + 4 = 9 \text{ kg.}$$

- **Compute the moments:**

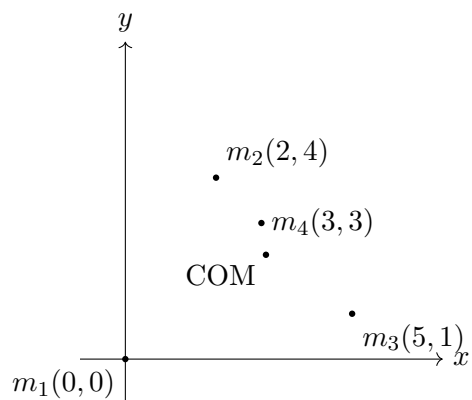
$$\begin{aligned} M_x &= 2(1) + 3(4) + 4(6) \\ &= 2 + 12 + 24 = 38 \end{aligned}$$

$$\begin{aligned} M_y &= 2(3) + 3(5) + 4(2) \\ &= 6 + 15 + 8 = 29. \end{aligned}$$

- **Center of Mass:**

$$\begin{aligned} \bar{x} &= \frac{38}{9}, \\ \bar{y} &= \frac{29}{9}. \end{aligned}$$

2. Four Point Masses.



- **Total mass:**

$$M = 1 + 2 + 3 + 4 = 10 \text{ kg.}$$

- **Compute the moments:**

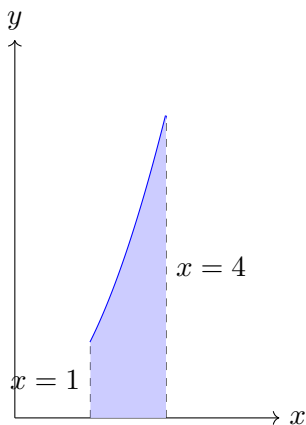
$$\begin{aligned} M_x &= 1(0) + 2(2) + 3(5) + 4(3) \\ &= 0 + 4 + 15 + 12 = 31 \end{aligned}$$

$$\begin{aligned} M_y &= 1(0) + 2(4) + 3(1) + 4(3) \\ &= 0 + 8 + 3 + 12 = 23. \end{aligned}$$

- **Center of Mass:**

$$\begin{aligned} \bar{x} &= \frac{31}{10} = 3.1, \\ \bar{y} &= \frac{23}{10} = 2.3. \end{aligned}$$

3. **Region Bounded by** $y = x^2$, $y = 0$, $x = 1$, and $x = 2$.



• **Area:**

$$\begin{aligned} A &= \int_1^2 (x^2 - 0) dx = \int_1^2 x^2 dx \\ &= \left[\frac{x^3}{3} \right]_1^2 = \frac{2^3 - 1^3}{3} = \frac{8 - 1}{3} = \frac{7}{3}. \end{aligned}$$

• **x -coordinate of the centroid:**

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_1^2 x(x^2) dx = \frac{1}{A} \int_1^2 x^3 dx \\ &= \frac{1}{7/3} \left[\frac{x^4}{4} \right]_1^2 = \frac{3}{7} \left(\frac{16 - 1}{4} \right) \\ &= \frac{3}{7} \cdot \frac{15}{4} = \frac{45}{28}. \end{aligned}$$

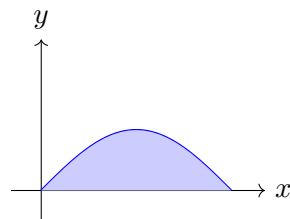
• **y -coordinate of the centroid:**

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int_1^2 \frac{(x^2)^2}{2} dx = \frac{1}{A} \int_1^2 \frac{x^4}{2} dx \\ &= \frac{1}{7/3} \cdot \frac{1}{2} \left[\frac{x^5}{5} \right]_1^2 = \frac{3}{7} \cdot \frac{1}{2} \left(\frac{32 - 1}{5} \right) \\ &= \frac{3}{7} \cdot \frac{31}{10} = \frac{93}{70}. \end{aligned}$$

Thus, the center of mass is

$$\left(\frac{45}{28}, \frac{93}{70} \right).$$

4. **Region Bounded by** $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.



• **Area:**

$$\begin{aligned} A &= \int_0^\pi \sin x dx = [-\cos x]_0^\pi \\ &= [-\cos \pi] - [-\cos 0] = (1) - (-1) = 2. \end{aligned}$$

• **x -coordinate of the centroid:**

$$\bar{x} = \frac{1}{A} \int_0^\pi x \sin x dx.$$

Using integration by parts (with $u = x$, $dv = \sin x dx$):

$$\begin{aligned} \int_0^\pi x \sin x dx &= [-x \cos x]_0^\pi + \int_0^\pi \cos x dx \\ &= [-\pi \cos \pi + 0] + [\sin x]_0^\pi \\ &= -\pi(-1) + (0 - 0) = \pi. \end{aligned}$$

Thus,

$$\bar{x} = \frac{\pi}{2}.$$

• **y -coordinate of the centroid:**

$$\bar{y} = \frac{1}{A} \int_0^\pi \frac{[\sin x]^2}{2} dx = \frac{1}{2A} \int_0^\pi \sin^2 x dx.$$

Using the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$:

$$\begin{aligned} \int_0^\pi \sin^2 x dx &= \int_0^\pi \frac{1 - \cos 2x}{2} dx \\ &= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^\pi \\ &= \frac{1}{2} (\pi - 0) = \frac{\pi}{2}. \end{aligned}$$

Therefore,

$$\bar{y} = \frac{1}{2 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{8}.$$

Thus, the center of mass is

$$\left(\frac{\pi}{2}, \frac{\pi}{8} \right).$$

5. Uniform Semicircular Lamina.

A uniform semicircular lamina of radius R (with the flat side along the x -axis) is symmetric about the y -axis. Therefore,

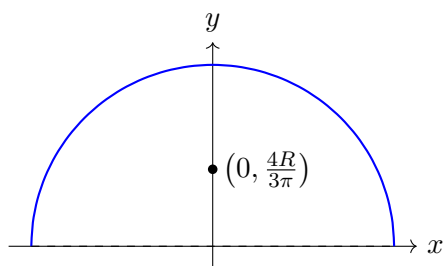
$$\bar{x} = 0.$$

It is a well-known result that the y -coordinate of the centroid is given by:

$$\bar{y} = \frac{4R}{3\pi}.$$

Thus, the center of mass is

$$\left(0, \frac{4R}{3\pi}\right).$$



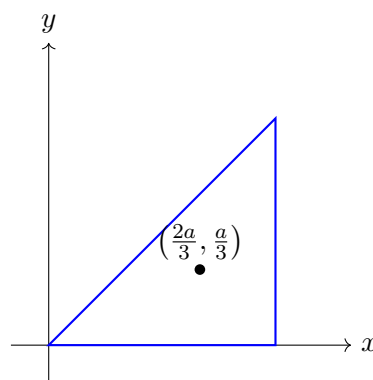
6. Uniform Triangular Lamina.

Consider the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, a) . The centroid (center of mass) of a triangle is the average of its vertices:

$$\begin{aligned}\bar{x} &= \frac{0 + a + a}{3} = \frac{2a}{3}, \\ \bar{y} &= \frac{0 + 0 + a}{3} = \frac{a}{3}.\end{aligned}$$

Thus, the center of mass is

$$\left(\frac{2a}{3}, \frac{a}{3}\right).$$



8.1, 10.1-10.2 Parametric Curves & Arc Length (Solutions)

1. Given the parametric equations:

$$x = t^2 + 1, \quad y = 2t - 3,$$

the slope of the curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Computing derivatives:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2.$$

Evaluating at $t = 1$:

$$\frac{dy}{dx} = \frac{2}{2(1)} = 1.$$

2. For the curve:

$$x = \sin t, \quad y = \cos t,$$

horizontal tangents occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$:

$$\frac{dy}{dt} = -\sin t = 0.$$

Solving $\sin t = 0$:

$$t = 0, \pi, 2\pi.$$

Checking dx/dt :

$$\frac{dx}{dt} = \cos t.$$

At $t = 0, 2\pi$, $\cos t = 1$ and at $t = \pi$, $\cos t = -1$, so $dx/dt \neq 0$. Thus, the horizontal tangent points are:

$$(0, 1), \quad (0, -1), \quad (0, 1).$$

3. For the curve:

$$x = t^3 - 3t, \quad y = t^2 - 2t,$$

vertical tangents occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$:

$$\frac{dx}{dt} = 3t^2 - 3, \quad \frac{dy}{dt} = 2t - 2.$$

Solving $3t^2 - 3 = 0$:

$$t^2 = 1 \quad \Rightarrow \quad t = \pm 1.$$

Checking dy/dt :

$$\frac{dy}{dt} = 2(1) - 2 = 0, \quad \frac{dy}{dt} = 2(-1) - 2 = -4 \neq 0.$$

Since $dy/dt = 0$ at $t = 1$, we use L'Hôpital's Rule:

$$\lim_{t \rightarrow 1} \frac{dy/dt}{dx/dt} = \lim_{t \rightarrow 1} \frac{2}{6t} = \frac{2}{6(1)} = \frac{1}{3}.$$

Since the limit is finite, $t = 1$ does not produce a vertical tangent. Thus, the only valid vertical tangent occurs at $t = -1$.

4. For the curve:

$$x = e^t, \quad y = e^{-t},$$

the slope is:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t}}{e^t} = -e^{-2t}.$$

Evaluating at $t = 0$:

$$\frac{dy}{dx} = -1.$$

Equation of the tangent line at $(1, 1)$:

$$y - 1 = -1(x - 1) \quad \Rightarrow \quad y = -x + 2.$$

5. For the curve:

$$x = t - \sin t, \quad y = 1 - \cos t,$$

the slope is:

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}.$$

Evaluating at $t = \frac{\pi}{4}$:

$$\frac{dy}{dx} = \frac{\sin \frac{\pi}{4}}{1 - \cos \frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1 - \frac{\sqrt{2}}{2}} = \sqrt{2} + 1.$$

6. For the curve:

$$x = \ln t, \quad y = t^2,$$

the slope is:

$$\frac{dy}{dx} = \frac{2t}{\frac{1}{t}} = 2t^2.$$

Evaluating at $t = 1$:

$$\frac{dy}{dx} = 2.$$

Equation of the tangent line at $(0, 1)$:

$$y - 1 = 2(x - 0) \Rightarrow y = 2x + 1.$$

7. For the curve:

$$x = t^2 - 4t + 3, \quad y = t^3 - 3t^2,$$

horizontal tangents occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$:

$$\frac{dy}{dt} = 3t^2 - 6t, \quad \frac{dx}{dt} = 2t - 4.$$

Solving $3t^2 - 6t = 0$:

$$3t(t - 2) = 0 \Rightarrow t = 0, 2.$$

Checking dx/dt :

$$\frac{dx}{dt} = 2t - 4.$$

At $t = 0$, $dx/dt = -4 \neq 0$, and at $t = 2$, $dx/dt = 0$, requiring L'Hôpital's Rule. Differentiating again:

$$\frac{d^2x}{dt^2} = 2, \quad \frac{d^2y}{dt^2} = 6t - 6.$$

Evaluating at $t = 2$:

$$\frac{d^2y}{dt^2} = 6(2) - 6 = 6, \quad \frac{d^2x}{dt^2} = 2.$$

Thus, using L'Hôpital's Rule:

$$\lim_{t \rightarrow 2} \frac{dy/dt}{dx/dt} = \lim_{t \rightarrow 2} \frac{6t - 6}{2} = \frac{6(2) - 6}{2} = \frac{6}{2} = 3.$$

Since the limit is finite, there is no horizontal tangent at $t = 2$, leaving only $t = 0$ as a valid solution.

8. For the curve:

$$x = 3t^2 + 2, \quad y = 4t^3 - 5,$$

the slope is:

$$\frac{dy}{dx} = \frac{12t^2}{6t} = 2t.$$

Evaluating at $t = -1$:

$$\frac{dy}{dx} = -2.$$

9. For the curve:

$$x = t^2 + 2t, \quad y = 3t - 1,$$

the slope is:

$$\frac{dy}{dx} = \frac{3}{2t + 2}.$$

Evaluating at $t = 2$:

$$\frac{dy}{dx} = \frac{3}{4 + 2} = \frac{1}{2}.$$

Equation of the tangent line at $(8, 5)$:

$$y - 5 = \frac{1}{2}(x - 8) \Rightarrow y = \frac{1}{2}x + 1.$$

10. For the curve:

$$x = t^2 - 2t, \quad y = t^3 - 3t,$$

vertical tangents occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$:

$$\frac{dx}{dt} = 2t - 2, \quad \frac{dy}{dt} = 3t^2 - 3.$$

Solving $2t - 2 = 0$:

$$t = 1.$$

Checking dy/dt at $t = 1$:

$$\frac{dy}{dt} = 3(1)^2 - 3 = 0.$$

Since both derivatives vanish, we use L'Hôpital's Rule:

$$\lim_{t \rightarrow 1} \frac{dy/dt}{dx/dt} = \lim_{t \rightarrow 1} \frac{6t}{2} = \frac{6(1)}{2} = 3.$$

Since the limit is finite, no vertical tangent occurs at $t = 1$, and thus, there are no vertical tangents.

11. The arc length formula is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

For $f(x) = \ln(\cos x)$:

- Compute the derivative:

$$f'(x) = \frac{d}{dx} \ln(\cos x) = -\tan x.$$

- Compute $1 + (f'(x))^2$:

$$1 + \tan^2 x = \sec^2 x.$$

- Compute the arc length integral:

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{\sec^2 x} dx \\ &= \int_0^{\pi/4} \sec x dx. \end{aligned}$$

- Use the integral result:

$$\int \sec x dx = \ln |\sec x + \tan x|.$$

- Evaluate from 0 to $\frac{\pi}{4}$:

$$\begin{aligned} L &= \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln |\sec 0 + \tan 0| \\ &= \ln(\sqrt{2} + 1) - \ln 1 \\ &= \ln(\sqrt{2} + 1). \end{aligned}$$

12. The arc length formula is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

For $f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$:

- Compute the derivative:

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{e^x}{2} + \frac{e^{-x}}{2} \right) \\ &= \frac{e^x}{2} - \frac{e^{-x}}{2}. \end{aligned}$$

- Compute $1 + (f'(x))^2$:

$$\begin{aligned} 1 + \left(\frac{e^x}{2} - \frac{e^{-x}}{2} \right)^2 &= 1 + \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4 + e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4}. \end{aligned}$$

- Recognize the identity:

$$e^{2x} + 2 + e^{-2x} = (e^x + e^{-x})^2.$$

- Substitute and simplify:

$$\begin{aligned} 1 + (f'(x))^2 &= \frac{(e^x + e^{-x})^2}{4} \\ &= \left(\frac{e^x + e^{-x}}{2} \right)^2. \end{aligned}$$

- Compute the arc length integral:

$$L = \int_0^2 \frac{e^x + e^{-x}}{2} dx.$$

- Split the integral:

$$L = \frac{1}{2} \int_0^2 e^x dx + \frac{1}{2} \int_0^2 e^{-x} dx.$$

- Compute each integral:

$$\int e^x dx = e^x, \quad \int e^{-x} dx = -e^{-x}.$$

- Evaluate from 0 to 2:

$$\begin{aligned} L &= \frac{1}{2} [e^x]_0^2 + \frac{1}{2} [-e^{-x}]_0^2 \\ &= \frac{1}{2} (e^2 - 1) + \frac{1}{2} (-e^{-2} + 1) \\ &= \frac{e^2 - 1 - e^{-2} + 1}{2} \\ &= \frac{e^2 - e^{-2}}{2}. \end{aligned}$$

13. The arc length formula for a function $x = g(y)$ over $[a, b]$ is:

$$L = \int_a^b \sqrt{1 + (g'(y))^2} dy.$$

For $x = g(y) = \frac{1}{3}y^{3/2} - y^{1/2}$:

- Compute the derivative:

$$\begin{aligned} g'(y) &= \frac{d}{dy} \left(\frac{1}{3}y^{3/2} - y^{1/2} \right) \\ &= \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}. \end{aligned}$$

- Compute $1 + (g'(y))^2$:

$$\begin{aligned} 1 + \left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \right)^2 &= 1 + \frac{y - 2 + y^{-1}}{4} \\ &= \frac{y + 2 + y^{-1}}{4}. \end{aligned}$$

- Recognize the identity:

$$y + 2 + y^{-1} = \left(y^{1/2} + y^{-1/2} \right)^2.$$

- Substitute and simplify:

$$\begin{aligned} 1 + (g'(y))^2 &= \frac{(y^{1/2} + y^{-1/2})^2}{4} \\ &= \left(\frac{y^{1/2} + y^{-1/2}}{2} \right)^2. \end{aligned}$$

- Compute the arc length integral:

$$L = \int_1^4 \frac{y^{1/2} + y^{-1/2}}{2} dy.$$

- Split the integral:

$$L = \frac{1}{2} \int_1^4 y^{1/2} dy + \frac{1}{2} \int_1^4 y^{-1/2} dy.$$

- Compute each integral:

$$\int y^{1/2} dy = \frac{2}{3}y^{3/2}, \quad \int y^{-1/2} dy = 2y^{1/2}.$$

- Evaluate from 1 to 4:

$$\begin{aligned} L &= \frac{1}{2} \left[\frac{2}{3}y^{3/2} \right]_1^4 + \frac{1}{2} \left[2y^{1/2} \right]_1^4 \\ &= \frac{1}{2} \left(\frac{2}{3}(4^{3/2}) - \frac{2}{3}(1^{3/2}) \right) + \frac{1}{2} \left(2(4^{1/2}) - 2(1^{1/2}) \right) \\ &= \frac{10}{3}. \end{aligned}$$

14. The arc length formula for a function $x = g(y)$ over $[a, b]$ is:

$$L = \int_a^b \sqrt{1 + (g'(y))^2} dy.$$

For $x = g(y) = \frac{2}{3}y^{3/2}$:

- Compute the derivative:

$$\begin{aligned} g'(y) &= \frac{d}{dy} \left(\frac{2}{3}y^{3/2} \right) \\ &= \frac{2}{3} \cdot \frac{3}{2}y^{1/2} \\ &= y^{1/2}. \end{aligned}$$

- Compute $1 + (g'(y))^2$:

$$1 + (y^{1/2})^2 = 1 + y.$$

- Compute the arc length integral:

$$L = \int_0^4 \sqrt{1 + y} dy.$$

- Use the substitution $u = 1 + y$, so $du = dy$:

$$L = \int_1^5 \sqrt{u} du.$$

- Compute the integral:

$$\int \sqrt{u} du = \frac{2}{3}u^{3/2}.$$

- Evaluate from 1 to 5:

$$\begin{aligned} L &= \frac{2}{3} \left[u^{3/2} \right]_1^5 \\ &= \frac{2}{3} \left(5^{3/2} - 1^{3/2} \right) \\ &= \frac{2}{3} \left(5\sqrt{5} - 1 \right) \\ &= \frac{10\sqrt{5} - 2}{3}. \end{aligned}$$

15. Parametric Curve $x = t^2$, $y = t^3$, $t \in [0, 1]$. Here,

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2.$$

The arc length is

$$L = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2} dt = \int_0^1 \sqrt{4t^2 + 9t^4} dt.$$

Factor t^2 under the square root:

$$\sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}.$$

Thus,

$$L = \int_0^1 t\sqrt{4 + 9t^2} dt.$$

Let $u = 4 + 9t^2$ so that $du = 18t dt$ or $t dt = \frac{du}{18}$.
When $t = 0$, $u = 4$; when $t = 1$, $u = 13$. Then,

$$L = \frac{1}{18} \int_4^{13} \sqrt{u} du = \frac{1}{18} \cdot \frac{2}{3} u^{3/2} \Big|_4^{13} = \frac{1}{27} \left(13^{3/2} - 4^{3/2} \right).$$

16. Cycloid $x = t - \sin t$, $y = 1 - \cos t$, $t \in [0, 2\pi]$.

Differentiate:

$$\frac{dx}{dt} = 1 - \cos t, \quad \frac{dy}{dt} = \sin t.$$

Then,

$$L = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt.$$

Simplify the integrand:

$$\begin{aligned}(1 - \cos t)^2 + \sin^2 t &= 1 - 2\cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2\cos t.\end{aligned}$$

Thus,

$$L = \int_0^{2\pi} \sqrt{2 - 2\cos t} dt.$$

Using the identity $2 - 2\cos t = 4\sin^2 \frac{t}{2}$, we have

$$L = \int_0^{2\pi} 2 \left| \sin \frac{t}{2} \right| dt.$$

Since $\sin \frac{t}{2} \geq 0$ for $t \in [0, 2\pi]$, we can drop the absolute value:

$$L = 2 \int_0^{2\pi} \sin \frac{t}{2} dt.$$

Let $u = \frac{t}{2}$ so that $dt = 2du$ and the limits become $u = 0$ to $u = \pi$. Then,

$$L = 2 \int_0^{\pi} \sin u (2du) = 4 \int_0^{\pi} \sin u du.$$

We have

$$\begin{aligned}\int_0^{\pi} \sin u du &= [-\cos u]_0^{\pi} \\ &= (-\cos \pi) - (-\cos 0) \\ &= 2\end{aligned}$$

Thus,

$$L = 4 \cdot 2 = 8.$$

10.3-10.4 Polar (Solutions)

Multiple Choice Matching (Solutions)

1. A
2. C
3. D
4. B
5. B
6. A
7. B
8. D

Areas Between Polar Curves (Solutions)

1. Set up an integral to find the area enclosed by the cardioid:

$$r = 2(1 + \cos \theta).$$

Solution:

- The formula for the area enclosed by a polar curve is:

$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

- Substituting $r = 2(1 + \cos \theta)$:

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} [2(1 + \cos \theta)]^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} [4(1 + 2\cos \theta + \cos^2 \theta)] d\theta. \end{aligned}$$

- Expanding the integral:

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} 4 + 8\cos \theta + 4\cos^2 \theta d\theta \\ &= \frac{1}{2} \left[\int_0^{2\pi} 4 d\theta + \int_0^{2\pi} 8\cos \theta d\theta + \int_0^{2\pi} 4\cos^2 \theta d\theta \right]. \end{aligned}$$

- Evaluating each integral:

$$\begin{aligned} - \int_0^{2\pi} 4 d\theta &= 4(2\pi) = 8\pi. \\ - \int_0^{2\pi} 8\cos \theta d\theta &= 0. \\ - \text{Using the identity } \cos^2 \theta &= \frac{1+\cos 2\theta}{2}: \end{aligned}$$

$$\int_0^{2\pi} 4\cos^2 \theta d\theta = 4 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta.$$

- Splitting the integral:

$$\begin{aligned} \int_0^{2\pi} 4\cos^2 \theta d\theta &= 4 \left[\frac{1}{2} \int_0^{2\pi} 1 d\theta + \frac{1}{2} \int_0^{2\pi} \cos 2\theta d\theta \right] \\ &= 4 \left[\frac{1}{2}(2\pi) + \frac{1}{2}(0) \right] = 4[\pi] = 4\pi. \end{aligned}$$

- Summing the results:

$$A = \frac{1}{2} [8\pi + 0 + 4\pi] = \frac{1}{2}(12\pi) = 6\pi.$$

- Final Answer:**

$$\boxed{6\pi}$$

2. Find the area common to both polar curves:

$$r = 3 + \cos \theta, \quad r = 3 - \cos \theta.$$

Solution:

- To find the intersection points, set $r_1 = r_2$:

$$3 + \cos \theta = 3 - \cos \theta.$$

- Solving for θ :

$$\cos \theta = -\cos \theta \quad \Rightarrow \quad \cos \theta = 0.$$

- This occurs at:

$$\theta = \frac{\pi}{2}, \quad \theta = \frac{3\pi}{2}.$$

3. Find the area enclosed by the four-leaved rose:

$$r = 3 \cos(2\theta).$$

Solution:

- **Determine Limits for One Petal:**

$$3 \cos(2\theta) = 0 \Rightarrow \cos(2\theta) = 0.$$

This yields

$$2\theta = \pm \frac{\pi}{2} \Rightarrow \theta = \pm \frac{\pi}{4},$$

so one petal is traced when

$$\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right].$$

- **Area of One Petal:**

$$\begin{aligned} A_{\text{petal}} &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (3 \cos(2\theta))^2 d\theta \\ &= \frac{9}{2} \int_{-\pi/4}^{\pi/4} \cos^2(2\theta) d\theta. \end{aligned}$$

- **Substitute $u = 2\theta$:**

$$d\theta = \frac{du}{2}, \quad \theta = -\frac{\pi}{4} \Rightarrow u = -\frac{\pi}{2}, \quad \theta = \frac{\pi}{4} \Rightarrow u = \frac{\pi}{2}.$$

Hence,

$$\begin{aligned} A_{\text{petal}} &= \frac{9}{2} \int_{-\pi/4}^{\pi/4} \cos^2(2\theta) d\theta \\ &= \frac{9}{2} \int_{-\pi/2}^{\pi/2} \cos^2(u) \frac{du}{2} \\ &= \frac{9}{4} \int_{-\pi/2}^{\pi/2} \cos^2(u) du \\ &= \frac{9}{4} \int_{-\pi/2}^{\pi/2} \frac{1 + \cos(2u)}{2} du \\ &= \frac{9}{8} \int_{-\pi/2}^{\pi/2} (1 + \cos(2u)) du \\ &= \frac{9}{8} \left[\left(u + \frac{\sin(2u)}{2} \right) \right]_{-\pi/2}^{\pi/2} \\ &= \frac{9}{8} \cdot \pi \end{aligned}$$

- **Total Area:**

$$A_{\text{total}} = 4 \cdot A_{\text{petal}} = 4 \cdot \frac{9\pi}{8} = \frac{9\pi}{2}.$$

4. Compute the area inside one petal of the rose curve:

$$r = 2 \sin(3\theta).$$

Solution:

- The formula for the area enclosed by a polar curve is:

$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

- A three-petal rose $r = 2 \sin(3\theta)$ has three identical petals.
- One petal is traced when θ runs from 0 to $\frac{\pi}{3}$.
- The area of one petal is:

$$A = \frac{1}{2} \int_0^{\pi/3} (2 \sin(3\theta))^2 d\theta.$$

- Expanding the square:

$$A = \frac{1}{2} \int_0^{\pi/3} 4 \sin^2(3\theta) d\theta.$$

- Using the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$:

$$A = \frac{1}{2} \int_0^{\pi/3} 4 \times \frac{1 - \cos 6\theta}{2} d\theta.$$

- Splitting the integral:

$$A = \frac{1}{2} \left[\int_0^{\pi/3} 2 d\theta - \int_0^{\pi/3} 2 \cos 6\theta d\theta \right].$$

- Evaluating:

$$- \int_0^{\pi/3} 2 d\theta = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}.$$

$$- \int_0^{\pi/3} 2 \cos 6\theta d\theta = 2 \times \frac{\sin 6\theta}{6} \Big|_0^{\pi/3} = 0.$$

- So the final area is:

$$A = \frac{1}{2} \times \frac{2\pi}{3} = \frac{\pi}{3}.$$

- **Final Answer:**

$$\boxed{\frac{\pi}{3}}$$

5. Find the area inside $r = 2 + \cos \theta$ and outside $r = 1$.

Solution:

The area enclosed between two polar curves is given by:

$$A = \frac{1}{2} \int_a^b (r_{\text{outer}}^2 - r_{\text{inner}}^2) d\theta.$$

Here, $r_{\text{outer}} = 2 + \cos \theta$ and $r_{\text{inner}} = 1$.

Step 1: Find the Limits of Integration

The curves intersect when:

$$2 + \cos \theta = 1.$$

Solving for θ : $\cos \theta = -1 \Rightarrow \theta = \pi$. Since we are computing the total enclosed area, we integrate from $\theta = 0$ to $\theta = 2\pi$.

Step 2: Integrate

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} ((2 + \cos \theta)^2 - 1^2) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (4 + 4 \cos \theta + \cos^2 \theta - 1) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (3 + 4 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} \left[\int_0^{2\pi} 3 d\theta + \int_0^{2\pi} 4 \cos \theta d\theta + \int_0^{2\pi} \cos^2 \theta d\theta \right]. \end{aligned}$$

Computing each integral:

$$\begin{aligned} \int_0^{2\pi} 3 d\theta &= 3(2\pi) = 6\pi, \\ \int_0^{2\pi} 4 \cos \theta d\theta &= 0. \end{aligned}$$

Using $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$, we rewrite:

$$\int_0^{2\pi} \cos^2 \theta d\theta = \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta.$$

Splitting:

$$\int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{2} \int_0^{2\pi} 1 d\theta + \frac{1}{2} \int_0^{2\pi} \cos 2\theta d\theta.$$

Since $\int_0^{2\pi} \cos 2\theta d\theta = 0$, we obtain:

$$\frac{1}{2} \cdot 2\pi = \pi.$$

Step 3: Compute the Final Result

$$\begin{aligned} A &= \frac{1}{2} (6\pi + 0 + \pi) \\ &= \frac{7\pi}{2}. \end{aligned}$$

6. Find the area inside $r = 6 \sin \theta$ and outside $r = 3$.

Solution:

The area enclosed between two polar curves is given by:

$$A = \frac{1}{2} \int_a^b (r_{\text{outer}}^2 - r_{\text{inner}}^2) d\theta.$$

Here, $r_{\text{outer}} = 6 \sin \theta$ and $r_{\text{inner}} = 3$.

Step 1: Find the Limits of Integration

The curves intersect when:

$$6 \sin \theta = 3 \Rightarrow \sin \theta = \frac{1}{2}, \quad \theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}.$$

We integrate from $\theta = \frac{\pi}{6}$ to $\theta = \frac{5\pi}{6}$.

Step 2: Set Up the Integral

$$\begin{aligned} A &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} ((6 \sin \theta)^2 - 3^2) d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (36 \sin^2 \theta - 9) d\theta. \end{aligned}$$

Using $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$, we substitute:

$$\begin{aligned} A &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} \left(36 \cdot \frac{1 - \cos 2\theta}{2} - 9 \right) d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (9 - 18 \cos 2\theta) d\theta. \end{aligned}$$

Step 3: Evaluate the Integral

Computing each term:

$$\begin{aligned} \int_{\pi/6}^{5\pi/6} 9 d\theta &= 9 \left(\frac{5\pi}{6} - \frac{\pi}{6} \right) = 9 \cdot \frac{4\pi}{6} = 6\pi, \\ \int_{\pi/6}^{5\pi/6} 18 \cos 2\theta d\theta &= 18 \cdot \frac{\sin 2\theta}{2} \Big|_{\pi/6}^{5\pi/6} \\ &= 9 \left(\sin \frac{5\pi}{3} - \sin \frac{\pi}{3} \right) \\ &= -9\sqrt{3} \end{aligned}$$

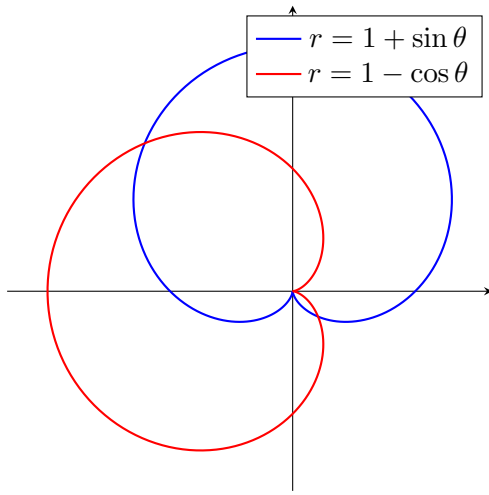
Step 4: Compute the Final Result

$$\begin{aligned} A &= \frac{1}{2} (6\pi + 9\sqrt{3}) \\ &= 3\pi + \frac{9\sqrt{3}}{2}. \end{aligned}$$

Tangent Lines and Arc Length

1. Find all points of intersection of the curves:

$$r = 1 + \sin \theta, \quad r = 1 - \cos \theta.$$



Solution:

Step 1: Justifying the Range $[0, 2\pi]$

While the equation $r = 1 + \sin \theta$ completes one full trace over $0 \leq \theta \leq \pi$, and $r = 1 - \cos \theta$ also completes its trace over $[0, \pi]$, we must consider θ in the full range $[0, 2\pi]$.

The reason is that in polar coordinates, a curve may intersect itself or intersect another curve at different angles, sometimes occurring at θ values outside $[0, \pi]$. Additionally, the same point can have multiple representations in polar form, meaning that intersections might not be obvious just by looking at $[0, \pi]$.

Step 2: Finding Intersection Points

To find intersections, we set the equations equal to each other:

$$1 + \sin \theta = 1 - \cos \theta.$$

Simplifying:

$$\sin \theta + \cos \theta = 0.$$

Dividing by $\cos \theta$ (where valid):

$$\tan \theta = -1.$$

The general solutions to $\tan \theta = -1$ are:

$$\theta = \frac{3\pi}{4} + k\pi, \quad k \in \mathbb{Z}.$$

Restricting to $[0, 2\pi]$, the valid solutions are:

$$\theta = \frac{3\pi}{4}, \quad \theta = \frac{7\pi}{4}.$$

Step 3: Compute r at These Intersections

Substituting into $r = 1 + \sin \theta$:

$$r = 1 + \sin \frac{3\pi}{4} = 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}.$$

$$r = 1 + \sin \frac{7\pi}{4} = 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}.$$

Thus, two intersection points are:

$$\left(\frac{2 + \sqrt{2}}{2}, \frac{3\pi}{4} \right), \quad \left(\frac{2 - \sqrt{2}}{2}, \frac{7\pi}{4} \right).$$

Step 4: Checking for the Origin as an Intersection

A point is at the origin if $r = 0$ for some θ . Setting $r = 0$ in both equations:

- From $r = 1 + \sin \theta$, setting $1 + \sin \theta = 0$ gives:

$$\sin \theta = -1 \Rightarrow \theta = \frac{3\pi}{2}.$$

- From $r = 1 - \cos \theta$, setting $1 - \cos \theta = 0$ gives:

$$\cos \theta = 1 \Rightarrow \theta = 0, 2\pi.$$

Since both equations independently achieve $r = 0$ at different values of θ , they both pass through the origin, meaning $(0, 0)$ must be included as an intersection.

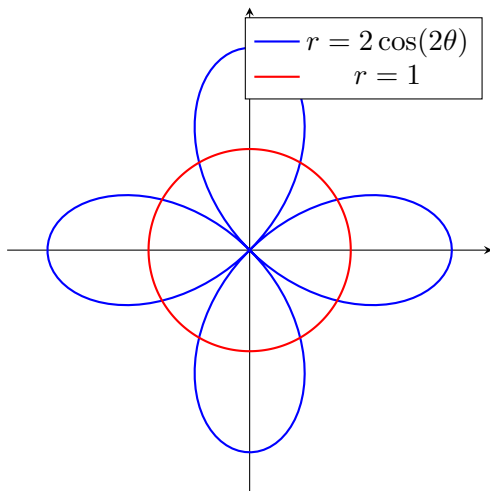
Step 5: Final Answer Including the Origin

The full set of intersection points is:

$$\left(\frac{2 + \sqrt{2}}{2}, \frac{3\pi}{4} \right), \quad \left(\frac{2 - \sqrt{2}}{2}, \frac{7\pi}{4} \right), \quad (0, 0).$$

2. Find all points of intersection of the curves:

$$r = 2 \cos 2\theta, \quad r = 1.$$



Solution:

Step 1: Solve for Intersections with $r = 1$

To find where the curves intersect, we first solve:

$$2 \cos 2\theta = 1.$$

Solving for θ :

$$\cos 2\theta = \frac{1}{2}.$$

The general solutions for $\cos x = \frac{1}{2}$ are:

$$2\theta = \pm \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}.$$

Dividing by 2:

$$\theta = \pm \frac{\pi}{6} + k\pi.$$

Restricting to $0 \leq \theta < 2\pi$, the valid solutions are:

$$\theta = \frac{\pi}{6}, \quad \theta = \frac{5\pi}{6}, \quad \theta = \frac{7\pi}{6}, \quad \theta = \frac{11\pi}{6}.$$

Since we set $r = 1$, the intersection points are:

$$\left(1, \frac{\pi}{6}\right), \quad \left(1, \frac{5\pi}{6}\right), \quad \left(1, \frac{7\pi}{6}\right), \quad \left(1, \frac{11\pi}{6}\right).$$

Step 2: Solve for Intersections with $r = -1$

A point (r, θ) is equivalent to $(-r, \theta + \pi)$, so we solve:

$$2 \cos 2\theta = -1.$$

Dividing by 2:

$$\cos 2\theta = -\frac{1}{2}.$$

The general solutions for $\cos x = -\frac{1}{2}$ are:

$$2\theta = \pm \frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}.$$

Dividing by 2:

$$\theta = \pm \frac{\pi}{3} + k\pi.$$

Restricting to $0 \leq \theta < 2\pi$, the valid solutions are:

$$\theta = \frac{\pi}{3}, \quad \theta = \frac{2\pi}{3}, \quad \theta = \frac{4\pi}{3}, \quad \theta = \frac{5\pi}{3}.$$

Since these correspond to $r = -1$, we convert to equivalent positive r representations:

$$\left(-1, \frac{\pi}{3}\right) = \left(1, \frac{4\pi}{3}\right),$$

$$\left(-1, \frac{2\pi}{3}\right) = \left(1, \frac{5\pi}{3}\right),$$

$$\left(-1, \frac{4\pi}{3}\right) = \left(1, \frac{\pi}{3}\right),$$

$$\left(-1, \frac{5\pi}{3}\right) = \left(1, \frac{2\pi}{3}\right).$$

Final Answer: The full set of intersection points is:

$$\left(1, \frac{\pi}{6}\right), \quad \left(1, \frac{5\pi}{6}\right), \quad \left(1, \frac{7\pi}{6}\right), \quad \left(1, \frac{11\pi}{6}\right),$$

$$\left(1, \frac{\pi}{3}\right), \quad \left(1, \frac{2\pi}{3}\right), \quad \left(1, \frac{4\pi}{3}\right), \quad \left(1, \frac{5\pi}{3}\right).$$

3. Find the exact length of the polar curve:

$$r = 3 \sin \theta, \quad 0 \leq \theta \leq \pi.$$

Solution: The arc length of a polar curve $r = f(\theta)$ over the interval $a \leq \theta \leq b$ is given by:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Step 1: Compute $\frac{dr}{d\theta}$ Differentiating $r = 3 \sin \theta$ with respect to θ :

$$\frac{dr}{d\theta} = 3 \cos \theta.$$

Step 2: Compute the integral Substituting into the arc length formula:

$$L = \int_0^\pi \sqrt{(3 \sin \theta)^2 + (3 \cos \theta)^2} d\theta.$$

Expanding:

$$L = \int_0^\pi \sqrt{9 \sin^2 \theta + 9 \cos^2 \theta} d\theta.$$

Factoring out 9:

$$L = \int_0^\pi \sqrt{9(\sin^2 \theta + \cos^2 \theta)} d\theta.$$

Using $\sin^2 \theta + \cos^2 \theta = 1$:

$$L = \int_0^\pi \sqrt{9} d\theta = \int_0^\pi 3 d\theta.$$

Step 3: Evaluate the integral

$$L = 3 \int_0^\pi d\theta = 3[\theta]_0^\pi.$$

$$L = 3(\pi - 0) = 3\pi.$$

Final Answer:

$$L = 3\pi.$$

4. Find the exact length of the polar curve:

$$r = e^\theta, \quad 0 \leq \theta \leq \ln 2.$$

Solution: The arc length of a polar curve $r = f(\theta)$ over the interval $a \leq \theta \leq b$ is given by:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Step 1: Compute $\frac{dr}{d\theta}$ Differentiating $r = e^\theta$ with respect to θ :

$$\frac{dr}{d\theta} = e^\theta.$$

Step 2: Compute the integral Substituting into the arc length formula:

$$L = \int_0^{\ln 2} \sqrt{(e^\theta)^2 + (e^\theta)^2} d\theta.$$

$$L = \int_0^{\ln 2} \sqrt{e^{2\theta} + e^{2\theta}} d\theta.$$

Factoring:

$$L = \int_0^{\ln 2} \sqrt{2e^{2\theta}} d\theta.$$

Since $\sqrt{2e^{2\theta}} = \sqrt{2}e^\theta$, we simplify:

$$L = \int_0^{\ln 2} \sqrt{2}e^\theta d\theta.$$

Step 3: Evaluate the integral

$$L = \sqrt{2} \int_0^{\ln 2} e^\theta d\theta.$$

The integral of e^θ is itself:

$$L = \sqrt{2} \left[e^\theta \right]_0^{\ln 2}.$$

Evaluating:

$$L = \sqrt{2}(e^{\ln 2} - e^0).$$

Since $e^{\ln 2} = 2$ and $e^0 = 1$, we obtain:

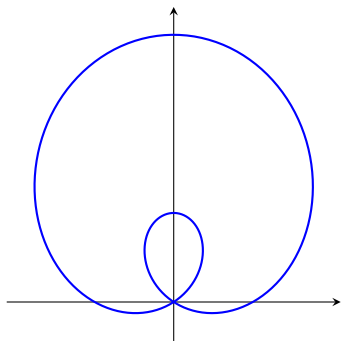
$$L = \sqrt{2}(2 - 1) = \sqrt{2} \cdot 1 = \sqrt{2}.$$

Final Answer:

$$L = \sqrt{2}.$$

5. Find the slope of the tangent line to the curve:

$$r = 1 + 2 \sin \theta, \quad \theta = \frac{\pi}{6}.$$



Solution: The slope of the tangent line to a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Step 1: Compute Cartesian Coordinates

The Cartesian coordinates of a polar curve are:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Substituting $r = 1 + 2 \sin \theta$:

$$x = (1 + 2 \sin \theta) \cos \theta, \quad y = (1 + 2 \sin \theta) \sin \theta.$$

Step 2: Compute Derivatives Differentiating x with respect to θ :

$$\frac{dx}{d\theta} = (2 \cos \theta) \cos \theta - (1 + 2 \sin \theta) \sin \theta.$$

Differentiating y with respect to θ :

$$\frac{dy}{d\theta} = (2 \cos \theta) \sin \theta + (1 + 2 \sin \theta) \cos \theta.$$

Step 3: Evaluate at $\theta = \frac{\pi}{6}$ First, compute necessary trigonometric values:

$$\sin \frac{\pi}{6} = \frac{1}{2}, \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Compute r at $\theta = \frac{\pi}{6}$:

$$r = 1 + 2 \sin \frac{\pi}{6} = 1 + 2 \cdot \frac{1}{2} = 2.$$

Compute $\frac{dx}{d\theta}$:

$$\begin{aligned} \frac{dx}{d\theta} &= (2 \cos \frac{\pi}{6}) \cos \frac{\pi}{6} - (1 + 2 \sin \frac{\pi}{6}) \sin \frac{\pi}{6} \\ &= \left(2 \cdot \frac{\sqrt{3}}{2}\right) \frac{\sqrt{3}}{2} - (2) \frac{1}{2} \\ &= (\sqrt{3}) \frac{\sqrt{3}}{2} - 1 \\ &= \frac{3}{2} - 1 = \frac{1}{2}. \end{aligned}$$

Compute $\frac{dy}{d\theta}$:

$$\begin{aligned} \frac{dy}{d\theta} &= (2 \cos \frac{\pi}{6}) \sin \frac{\pi}{6} + (1 + 2 \sin \frac{\pi}{6}) \cos \frac{\pi}{6} \\ &= \left(2 \cdot \frac{\sqrt{3}}{2}\right) \frac{1}{2} + (2) \frac{\sqrt{3}}{2} \\ &= (\sqrt{3}) \frac{1}{2} + \sqrt{3} \\ &= \frac{\sqrt{3}}{2} + \sqrt{3} = \frac{3\sqrt{3}}{2}. \end{aligned}$$

Step 4: Compute the Slope

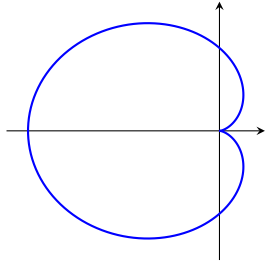
$$\frac{dy}{dx} = \frac{\frac{3\sqrt{3}}{2}}{\frac{1}{2}} = 3\sqrt{3}.$$

Final Answer:

$$\frac{dy}{dx} = 3\sqrt{3}.$$

6. Find points where the tangent line is horizontal or vertical for:

$$r = 2(1 - \cos \theta).$$



Solution:

The slope of the tangent line to a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Horizontal tangents occur where $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$. Vertical tangents occur where $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

Step 1: Compute First Derivatives

The Cartesian coordinates of the curve are:

$$x = r \cos \theta = 2(1 - \cos \theta) \cos \theta,$$

$$y = r \sin \theta = 2(1 - \cos \theta) \sin \theta.$$

Differentiating x with respect to θ :

$$\begin{aligned} \frac{dx}{d\theta} &= 2((1 - \cos \theta)(-\sin \theta) + \cos \theta \sin \theta) \\ &= 2(-\sin \theta + 2 \cos \theta \sin \theta) \\ &= 2 \sin \theta (2 \cos \theta - 1). \end{aligned}$$

Differentiating y with respect to θ :

$$\begin{aligned} \frac{dy}{d\theta} &= 2((1 - \cos \theta) \cos \theta + \sin \theta \sin \theta) \\ &= 2(\cos \theta - \cos^2 \theta + \sin^2 \theta) \\ &= 2(\cos \theta - \cos 2\theta). \end{aligned}$$

Step 2: Solve $\frac{dy}{d\theta} = 0$

Setting $\frac{dy}{d\theta} = 0$:

$$\cos \theta - \cos 2\theta = 0.$$

Rearranging:

$$\cos \theta = \cos 2\theta.$$

The general solutions to $\cos A = \cos B$ are:

$$A = B + 2k\pi \quad \text{or} \quad A = -B + 2k\pi.$$

Applying $A = \theta$ and $B = 2\theta$, we get:

$$\theta = \frac{2k\pi}{3}.$$

Restricting to $0 \leq \theta \leq 2\pi$, the valid solutions are:

$$\theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 2\pi.$$

Step 3: Solve $\frac{dx}{d\theta} = 0$

Setting $\frac{dx}{d\theta} = 0$:

$$2 \sin \theta (2 \cos \theta - 1) = 0.$$

Solving each factor separately:

- $\sin \theta = 0$ gives:

$$\theta = 0, \pi, 2\pi.$$

- $2 \cos \theta - 1 = 0$ gives:

$$\cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \quad \frac{5\pi}{3}.$$

Step 4: Apply L'Hôpital's Rule at $\theta = 0$ and $\theta = 2\pi$

At $\theta = 0$ and $\theta = 2\pi$, both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$, so we differentiate again:

$$\frac{d^2x}{d\theta^2} = 2(2 \cos^2 \theta - \cos \theta - 1 + \cos 2\theta).$$

$$\frac{d^2y}{d\theta^2} = 2(-\sin \theta + 2 \sin 2\theta).$$

Evaluating at $\theta = 0$:

$$\frac{d^2x}{d\theta^2} = 2, \quad \frac{d^2y}{d\theta^2} = 0.$$

Since $\frac{d^2x}{d\theta^2} \neq 0$ and $\frac{d^2y}{d\theta^2} = 0$, $\theta = 0$ gives a horizontal tangent.

Evaluating at $\theta = 2\pi$:

$$\frac{d^2x}{d\theta^2} = 2, \quad \frac{d^2y}{d\theta^2} = 0.$$

Since $\frac{d^2x}{d\theta^2} \neq 0$ and $\frac{d^2y}{d\theta^2} = 0$, $\theta = 2\pi$ gives a horizontal tangent.

Final Answer:

The curve has horizontal tangents at:

$$\theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 2\pi.$$

The curve has vertical tangents at:

$$\theta = \pi, \quad \frac{\pi}{3}, \quad \frac{5\pi}{3}.$$

11.1 Sequences (Solutions)

1. We rewrite:

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}}.$$

Since $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$, we get:

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1.$$

Thus, the sequence converges to 1.

2. The sequence oscillates between 1 and -1 and does not settle to a single value. Since it does not approach a single limit, the sequence diverges.

3. Since $\frac{1}{n^2} > 0$ for all n and $\frac{1}{n^2} \rightarrow 0$ as $n \rightarrow \infty$, the sequence converges to 0.

4. Dividing the numerator and denominator by n^2 , we get:

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}}.$$

Since $\frac{1}{n^2} \rightarrow 0$, the limit is 1. Thus, the sequence converges to 1.

5. Applying L'Hôpital's Rule to the related function $f(x) = \frac{\ln x}{x}$, we differentiate:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Thus, $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$, meaning the sequence converges to 0.

6. Dividing by n :

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n^2}}}.$$

Since $\frac{1}{n^2} \rightarrow 0$, we get:

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = \frac{1}{\sqrt{1}} = 1.$$

Thus, the sequence converges to 1.

7. Since $|d_n| = \frac{1}{n} \rightarrow 0$, d_n converges to 0 as well.

8. Since $-1 \leq \cos(3n+1) \leq 1$, we have

$$-\frac{1}{n^2} \leq \frac{\cos(3n+1)}{n^2} \leq \frac{1}{n^2}.$$

Since $\frac{1}{n^2} \rightarrow 0$ as $n \rightarrow \infty$, by the Squeeze Theorem, the sequence converges to 0.

9. Dividing numerator and denominator by n^3 :

$$c_n = \frac{\frac{n^2}{n^3} - \frac{3n}{n^3}}{\frac{n^3}{n^3} + \frac{5}{n^3}} = \frac{\frac{1}{n} - \frac{3}{n^2}}{1 + \frac{5}{n^3}}.$$

Taking the limit as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} c_n = \frac{0 - 0}{1 + 0} = 0.$$

Thus, the sequence converges to 0.

10. Factor out n^4 inside the square root:

$$\sqrt{9n^4 + 4n^2} = \sqrt{n^4 \left(9 + \frac{4}{n^2}\right)} = n^2 \sqrt{9 + \frac{4}{n^2}}.$$

Thus,

$$a_n = \frac{n^2 \sqrt{9 + \frac{4}{n^2}}}{n^3} = \frac{\sqrt{9 + \frac{4}{n^2}}}{n}.$$

As $n \rightarrow \infty$, the term $\sqrt{9 + \frac{4}{n^2}} \rightarrow 3$. Hence,

$$a_n \rightarrow \frac{3}{n} \rightarrow 0.$$

Therefore, the sequence $\{a_n\}$ converges and its limit is 0.

11. Dividing numerator and denominator by n :

$$b_n = \frac{5 + \frac{\sin(n)}{n}}{1 + \frac{10}{n}}.$$

Since $\frac{\sin(n)}{n} \rightarrow 0$ and $\frac{10}{n} \rightarrow 0$, we get:

$$\lim_{n \rightarrow \infty} b_n = \frac{5 + 0}{1 + 0} = 5.$$

Thus, the sequence converges to 5.

12. Dividing by n^3 in the numerator and n^3 inside the square root in the denominator:

$$c_n = \frac{1 + \frac{2}{n^3}}{\sqrt{1 + \frac{5}{n^4}}}.$$

Taking the limit:

$$\lim_{n \rightarrow \infty} c_n = \frac{1 + 0}{\sqrt{1 + 0}} = 1.$$

Thus, the sequence converges to 1.

13. Since $|\sin(5n)| \leq 1$, we have:

$$-\frac{1}{n^3} \leq \frac{\sin(5n)}{n^3} \leq \frac{1}{n^3}.$$

Since $\frac{1}{n^3} \rightarrow 0$, by the Squeeze Theorem, $\lim_{n \rightarrow \infty} a_n = 0$. The sequence converges to 0.

14. Dividing numerator and denominator by n^3 :

$$b_n = \frac{\frac{3\sqrt{n}}{n^3} + 1}{1 + \frac{\sqrt{n}}{n^3}}.$$

Taking the limit as $n \rightarrow \infty$, the terms with $\frac{\sqrt{n}}{n^3} \rightarrow 0$, so:

$$\lim_{n \rightarrow \infty} b_n = \frac{0 + 1}{1 + 0} = 1.$$

Thus, the sequence converges to 1.

15. Dividing by n^3 :

$$c_n = \frac{1 - \frac{2}{n^2}}{\sqrt{4 + \frac{7}{n^5}}}.$$

Taking the limit:

$$\lim_{n \rightarrow \infty} c_n = \frac{1 - 0}{\sqrt{4 + 0}} = \frac{1}{2}.$$

Thus, the sequence converges to $\frac{1}{2}$.