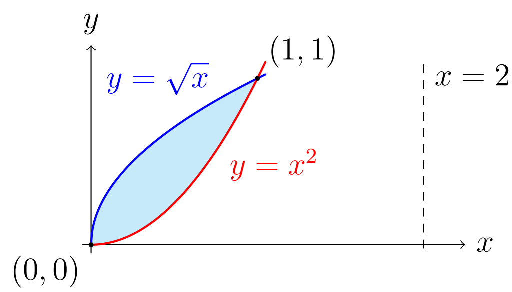
MATH 2300 – Exam 2 Review Problems (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \qquad y = \sqrt{x}$$

about the line $x = 2$.

Solution.



The curves intersect where

$$x^2 = \sqrt{x}.$$

Thus,

$$x^4 = x \quad \Rightarrow \quad x(x^3 - 1) = 0,$$

so the intersection points are $x = 0$ and $x = 1$.
Using cylindrical shells about the vertical line $x = 2$,

$$\text{radius} = 2 - x, \qquad \text{height} = \sqrt{x} - x^2.$$

Therefore,

$$V = 2\pi \int_0^1 (2 - x)(\sqrt{x} - x^2)\, dx.$$

Expand:

$$(2 - x)(\sqrt{x} - x^2) = 2x^{1/2} - x^{3/2} - 2x^2 + x^3.$$

So

$$V = 2\pi \int_0^1 \left(2x^{1/2} - x^{3/2} - 2x^2 + x^3\right)\, dx.$$

Now integrate:

$$V = 2\pi \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2} - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^1$$

$$= 2\pi \left(\frac{4}{3} - \frac{2}{5} - \frac{2}{3} + \frac{1}{4}\right) = 2\pi \left(\frac{31}{60}\right) = \frac{31\pi}{30}.$$

$$\boxed{V = \frac{31\pi}{30}}$$

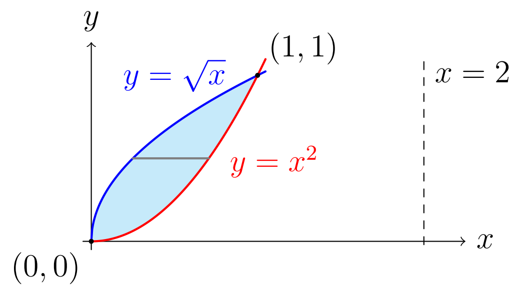
□

1. **(Alternate Solution)** Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \qquad y = \sqrt{x}$$

about the line $x = 2$.

Solution.



Since we are using washers about the vertical line $x = 2$, we write the curves as functions of y:

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}, \qquad y = \sqrt{x} \quad \Rightarrow \quad x = y^2.$$

The curves intersect at

$$(0,0) \quad \text{and} \quad (1,1),$$

so

$$0 \le y \le 1.$$

For a horizontal slice, the left boundary is $x = y^2$ and the right boundary is $x = \sqrt{y}$. Thus,

$$R(y) = 2 - y^2, \qquad r(y) = 2 - \sqrt{y}.$$

So the volume is

$$V = \pi \int_0^1 \left(R(y)^2 - r(y)^2\right) dy = \pi \int_0^1 \left[(2-y^2)^2 - (2-\sqrt{y})^2\right] dy.$$

Expand:

$$V = \pi \int_0^1 \left(4 - 4y^2 + y^4 - (4 - 4\sqrt{y} + y)\right) dy$$
$$= \pi \int_0^1 \left(y^4 - 4y^2 - y + 4y^{1/2}\right) dy.$$

Now integrate:

$$V = \pi \left[\frac{y^5}{5} - \frac{4y^3}{3} - \frac{y^2}{2} + \frac{8y^{3/2}}{3}\right]_0^1$$
$$= \pi \left(\frac{1}{5} - \frac{4}{3} - \frac{1}{2} + \frac{8}{3}\right) = \pi \left(\frac{1}{5} + \frac{4}{3} - \frac{1}{2}\right) = \pi \left(\frac{31}{30}\right).$$

$$\boxed{V = \frac{31\pi}{30}}$$

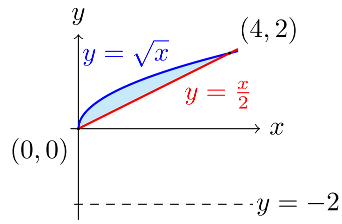
□

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \qquad y = \frac{x}{2}$$

about the horizontal line $y = -2$.

Solution.



First find the intersection points:

$$\sqrt{x} = \frac{x}{2}.$$

Then

$$2\sqrt{x} = x.$$

Let $u = \sqrt{x}$. Then $x = u^2$, so

$$2u = u^2 \quad \Rightarrow \quad u(u-2) = 0.$$

Thus $u = 0$ or $u = 2$, which gives

$$x = 0,\ 4.$$

Using washers with respect to x, the outer radius is

$$R(x) = \sqrt{x} + 2,$$

and the inner radius is

$$r(x) = \frac{x}{2} + 2.$$

So

$$V = \pi \int_0^4 \left(R(x)^2 - r(x)^2\right) dx = \pi \int_0^4 \left[\left(\sqrt{x}+2\right)^2 - \left(\frac{x}{2}+2\right)^2\right] dx.$$

Simplify:

$$V = \pi \int_0^4 \left(x + 4\sqrt{x} + 4 - \left(\frac{x^2}{4} + 2x + 4\right)\right) dx$$
$$= \pi \int_0^4 \left(4\sqrt{x} - x - \frac{x^2}{4}\right) dx.$$

Now integrate:

$$V = \pi \left[\frac{8}{3}x^{3/2} - \frac{x^2}{2} - \frac{x^3}{12}\right]_0^4$$
$$= \pi \left(\frac{8}{3}\cdot 8 - \frac{16}{2} - \frac{64}{12}\right) = \pi \left(\frac{64}{3} - 8 - \frac{16}{3}\right) = \pi(16 - 8).$$

$$\boxed{V = 8\pi}$$

□

3. Determine the work required to stretch a spring with natural length 15 cm from 16 cm to 22 cm, given that a force of 10 N is required to stretch the spring to 18 cm.

Solution.

By Hooke's Law,

$$F(x) = kx,$$

where x is the displacement from the natural length.

Since a force of 10 N stretches the spring from 15 cm to 18 cm, the displacement is

$$x = 3 \text{ cm} = 0.03 \text{ m}.$$

Thus,

$$10 = k(0.03),$$

so

$$k = \frac{10}{0.03} = \frac{1000}{3}.$$

To stretch the spring from 16 cm to 22 cm, the displacement changes from

$$x = 1 \text{ cm} = 0.01 \text{ m} \qquad \text{to} \qquad x = 7 \text{ cm} = 0.07 \text{ m}.$$

Therefore, the work is

$$W = \int_{0.01}^{0.07} kx\,dx = \int_{0.01}^{0.07} \frac{1000}{3} x\,dx.$$

Now integrate:

$$W = \frac{1000}{3}\left[\frac{x^2}{2}\right]_{0.01}^{0.07} = \frac{1000}{3} \cdot \frac{1}{2}\left(0.07^2 - 0.01^2\right).$$

So

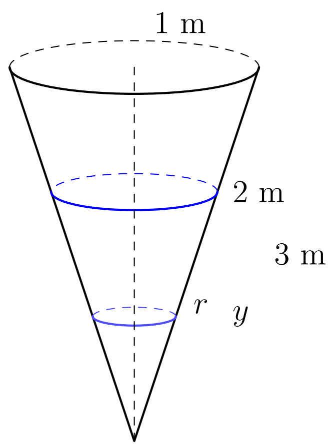
$$W = \frac{500}{3}(0.0049 - 0.0001) = \frac{500}{3}(0.0048) = 0.8.$$

$$\boxed{W = 0.8 \text{ J}}$$

□

4. Set up an integral for the work required to pump all of the water out over the top rim of an inverted conical tank of height 3 m and top radius 1 m, where the water is 2 m deep.

Solution.



Let y be the height of a thin slice of water measured from the tip at the bottom, so

$$0 \le y \le 2.$$

By similar triangles,

$$\frac{r}{y} = \frac{1}{3},$$

so

$$r = \frac{y}{3}.$$

Thus the volume of a thin slice is

$$dV = \pi r^2\, dy = \pi \left(\frac{y}{3}\right)^2 dy.$$

Its weight is

$$\rho g\, dV = 1000(9.8)\pi \left(\frac{y}{3}\right)^2 dy.$$

Since the water must be pumped to the top rim at height 3, the slice must be lifted

$$3 - y$$

meters.

Therefore, the work is

$$W = \int_0^2 1000(9.8)\pi \left(\frac{y}{3}\right)^2 (3 - y)\, dy.$$

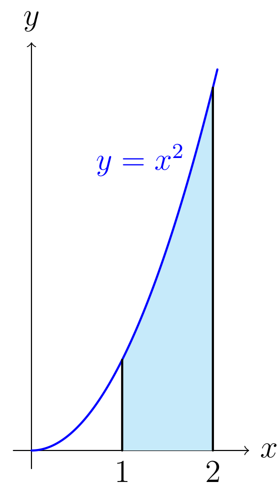
$$\boxed{W = \int_0^2 1000(9.8)\pi \left(\frac{y}{3}\right)^2 (3 - y)\, dy}$$

□

5. Compute the center of mass of the region bounded by

$$y = x^2, \qquad y = 0, \qquad x = 1, \qquad x = 2.$$

Solution.



Assuming uniform density, the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{A}, \frac{M_x}{A} \right).$$

First find the area:

$$A = \int_1^2 x^2\,dx = \left[\frac{x^3}{3} \right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

Now compute M_y:

$$M_y = \int_1^2 x(x^2)\,dx = \int_1^2 x^3\,dx = \left[\frac{x^4}{4} \right]_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$$

Now compute M_x:

$$M_x = \frac{1}{2} \int_1^2 (x^2)^2\,dx = \frac{1}{2} \int_1^2 x^4\,dx = \frac{1}{2} \left[\frac{x^5}{5} \right]_1^2 = \frac{1}{2} \left(\frac{32}{5} - \frac{1}{5} \right) = \frac{31}{10}.$$

Therefore,

$$\bar{x} = \frac{M_y}{A} = \frac{15/4}{7/3} = \frac{45}{28},$$
$$\bar{y} = \frac{M_x}{A} = \frac{31/10}{7/3} = \frac{93}{70}.$$

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{45}{28}, \frac{93}{70} \right)}$$

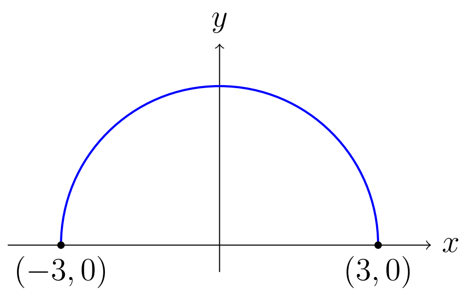
□

6. Give a parametrization of the upper semicircle

$$x^2 + y^2 = 9, \qquad y \geq 0,$$

traced from $(-3, 0)$ to $(3, 0)$ from left to right.

Solution.



A convenient parametrization is

$$x = 3\sin t, \qquad y = 3\cos t, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Indeed,

$$x^2 + y^2 = 9\sin^2 t + 9\cos^2 t = 9,$$

and on the interval

$$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2},$$

we have

$$\cos t \geq 0,$$

so

$$y = 3\cos t \geq 0.$$

Also,

$$t = -\frac{\pi}{2} \Rightarrow (-3, 0), \qquad t = \frac{\pi}{2} \Rightarrow (3, 0),$$

so the curve is traced from left to right.

$$\boxed{x = 3\sin t, \quad y = 3\cos t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}}$$

□

7. Find the equation of the tangent line to the parametric curve

$$x = e^t, \qquad y = e^{-t}$$

at the point $(1, 1)$.

Solution.

First find the value of t at the point $(1, 1)$:

$$x = e^t = 1 \quad \Rightarrow \quad t = 0.$$

Now compute the slope:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Since

$$\frac{dx}{dt} = e^t \qquad \text{and} \qquad \frac{dy}{dt} = -e^{-t},$$

we get

$$\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -e^{-2t}.$$

At $t = 0$,

$$\frac{dy}{dx} = -e^0 = -1.$$

So the tangent line at $(1, 1)$ is

$$y - 1 = -1(x - 1).$$

Thus,

$$\boxed{y = -x + 2.}$$

□

8. Set up an integral that computes the arc length of the curve

$$x = t^2, \qquad y = t^3, \qquad 0 \le t \le 1.$$

Solution.

For a parametric curve, the arc length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Here,

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 3t^2.$$

So the arc length is

$$L = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt.$$

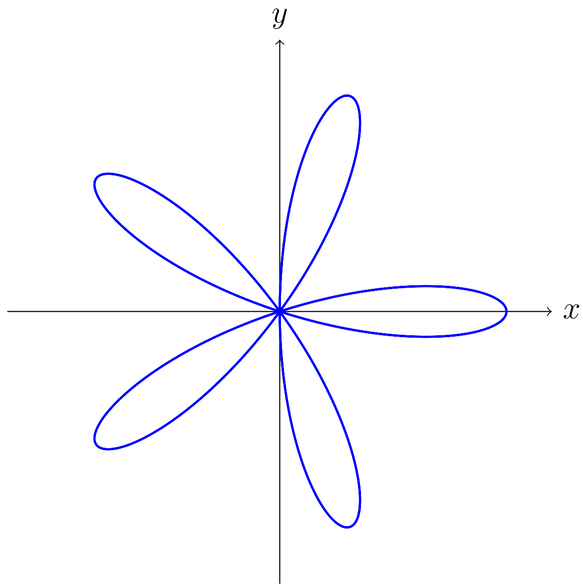
$$\boxed{L = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt}$$

$\square$

9. Draw the graph of the polar curve

$$r = \cos(5\theta).$$

Solution.



This is a 5-petaled rose, with one petal centered on the positive x-axis. □

10. Find the area inside $r = 6\sin\theta$ and outside $r = 3$.

Solution.

We use the polar area formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta.$$

First find the points of intersection:

$$\begin{aligned} 6\sin\theta &= 3 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}. \end{aligned}$$

On the interval

$$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6},$$

the curve $r = 6\sin\theta$ lies outside $r = 3$. Thus

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((6\sin\theta)^2 - 3^2\right) d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36\sin^2\theta - 9\right) d\theta.$$

Use

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}:$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36 \cdot \frac{1-\cos(2\theta)}{2} - 9\right) d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (18 - 18\cos(2\theta) - 9)\ d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (9 - 18\cos(2\theta))\ d\theta. \end{aligned}$$

Now evaluate:

$$A = \frac{1}{2}\left[9\theta - 9\sin(2\theta)\right]_{\pi/6}^{5\pi/6}.$$

So

$$A = \frac{1}{2}\left(9 \cdot \frac{4\pi}{6} - 9\sin\frac{5\pi}{3} + 9\sin\frac{\pi}{3}\right).$$

Since

$$\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}, \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

we get

$$A = \frac{1}{2}\left(6\pi + \frac{9\sqrt{3}}{2} + \frac{9\sqrt{3}}{2}\right) = \frac{1}{2}\left(6\pi + 9\sqrt{3}\right).$$

Therefore,

$$\boxed{A = 3\pi + \frac{9\sqrt{3}}{2}}.$$

□

11. Determine whether the sequence

$$a_n = \frac{5n + \sin(n)}{n + 10}$$

converges or diverges. If it converges, find its limit.

Solution.

We examine

$$\lim_{n\to\infty} \frac{5n + \sin(n)}{n + 10}.$$

Divide numerator and denominator by n:

$$a_n = \frac{5 + \frac{\sin(n)}{n}}{1 + \frac{10}{n}}.$$

Now as $n \to \infty$,

$$\frac{\sin(n)}{n} \to 0 \qquad \text{and} \qquad \frac{10}{n} \to 0.$$

Therefore,

$$\lim_{n\to\infty} a_n = \frac{5+0}{1+0} = 5.$$

So the sequence **converges**, and its limit is

$$\boxed{5}.$$

$\square$

12. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5n + \sin(n)}{n + 10}$$

converges or diverges. If it converges, find its sum.

Solution.

Let

$$a_n = \frac{5n + \sin(n)}{n + 10}.$$

To test the series $\sum a_n$, we first check whether

$$\lim_{n\to\infty} a_n = 0.$$

Divide numerator and denominator by n:

$$a_n = \frac{5 + \frac{\sin(n)}{n}}{1 + \frac{10}{n}}.$$

As $n \to \infty$,

$$\frac{\sin(n)}{n} \to 0 \qquad \text{and} \qquad \frac{10}{n} \to 0,$$

so

$$\lim_{n\to\infty} a_n = \frac{5+0}{1+0} = 5.$$

Since

$$\lim_{n\to\infty} a_n \neq 0,$$

the series diverges by the Test for Divergence.

$$\boxed{\text{The series diverges.}}$$

□

13. Suppose a series $\sum_{k=1}^{\infty} a_k$ has nth partial sum

$$S_n = \frac{6n^2 + 3n + 2}{2n^2 + 5n + 1}.$$

Solution.

A series converges if and only if the sequence of partial sums S_n converges. Thus we compute

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{6n^2 + 3n + 2}{2n^2 + 5n + 1}.$$

Since the numerator and denominator have the same degree, the limit is the ratio of the leading coefficients:

$$\lim_{n\to\infty} S_n = \frac{6}{2} = 3.$$

Therefore, the series $\sum_{k=1}^{\infty} a_k$ converges, and its sum is

$$\boxed{3}.$$

□