Midterm 1 Study Guide

MATH2300 - Calculus II

Spring 2026

Contents

Overview

Exam information:

- **Date:** Monday, Feb 9
- **Time:** 5:45pm – 7:15pm
- **Location:** CHEM 140

Topics covered:

- §5.5: u-Substitution
- §7.1: Integration by Parts
- §7.2: Trig Integrals
- §7.3: Trig Substitution
- §7.4: Partial Fractions
- §7.8: Improper Integrals
- §9.1: Differential Equations
- §9.3: Separable Differential Equations
- §6.1: Areas Between Curves
- §6.2: Volumes by Known Cross Sections

Not on the exam: Approximate integration (§7.7).

Suggested things to study:

- Fall 2022 practice exam on Canvas; Spring 2023 practice exam on Canvas
- Quizzes (Quiz 1, Quiz 2, Quiz 3, Quiz 4)
- WebAssign
- Written homeworks
- The problems in this study guide

A good way to prepare is to practice actively: try problems without notes first, check your work, and then redo similar problems until each method feels routine. Focus on choosing the right method, writing clean setups, and finishing with a correct final answer.

Formulas to Know

Integrals

Integral	Antiderivative
$\int x^n \, dx$	$\frac{x^{n+1}}{n+1} + C, \ (n \neq -1)$
$\int \frac{1}{x} \, dx$	$\ln\lvert x \rvert + C$
$\int e^x \, dx$	$e^x + C$
$\int \sin x \, dx$	$-\cos x + C$
$\int \cos x \, dx$	$\sin x + C$
$\int \tan x \, dx$	$-\ln\lvert \cos x \rvert + C$
$\int \cot x \, dx$	$\ln\lvert \sin x \rvert + C$

Integral	Antiderivative
$\int \sec x \, dx$	$\ln\lvert \sec x + \tan x \rvert + C$
$\int \csc x \, dx$	$-\ln\lvert \csc x + \cot x \rvert + C$
$\int \sec^2 x \, dx$	$\tan x + C$
$\int \csc^2 x \, dx$	$-\cot x + C$
$\int \sec x \, \tan x \, dx$	$\sec x + C$
$\int \csc x \, \cot x \, dx$	$-\csc x + C$
$\int \frac{1}{x^2 + a^2} \, dx$	$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$	$\sin^{-1}\left(\frac{x}{a}\right) + C$

Trigonometric Identities

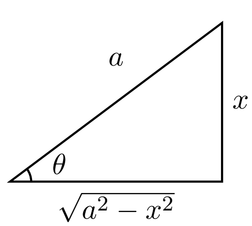
Identity	Formula
Pythagorean	$\sin^2 x + \cos^2 x = 1$
Tangent Identity	$\tan^2 x + 1 = \sec^2 x$
Power-Reducing Formulas	$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$
Product-to-Sum	$\sin x \cos x = \frac{1}{2} \sin 2x$
Double-Angle Formulas	$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Trigonometric Substitutions

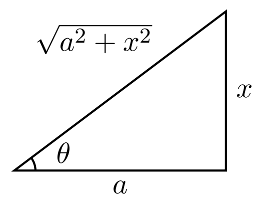
The table below summarizes the three standard trigonometric substitutions:

Expression	Substitution	Interval
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$

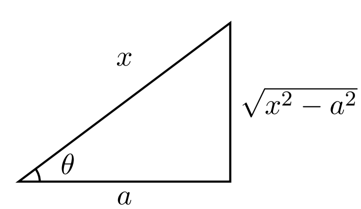
Each substitution corresponds to the sides of a right triangle:



Substitution: $x = a \sin\theta$



Substitution: $x = a \tan\theta$



Substitution: $x = a \sec\theta$

5.5 u-Substitution

This goal is to rewrite an integral in a simpler variable by spotting an "inside function" whose derivative (or a constant multiple of it) also appears in the integrand. General procedure:

1. **Choose u:** pick a function built from the complicated part of the integrand—often what is *inside* a power, root, exponential, or logarithm.
2. **Differentiate:** compute $du = u'(x)\,dx$.
3. **Match du:** rewrite the integral so that a factor of $u'(x)\,dx$ is present, then replace it by du.
4. **Substitute:** replace every remaining x-expression with u, producing an integral in u.
5. **Integrate and back-substitute:** integrate with respect to u, then substitute $u = g(x)$ back in.
6. **If bounds are given:** either convert the bounds to u-values (recommended), or back-substitute first and then evaluate in x.

Notes

1. $\int x^3 \cos(x^4 + 2)\ dx$
2. $\int \frac{x}{\sqrt{1-4x^2}}\ dx$
3. $\int \tan(x)\,dx$
4. $\int e^{5x}\ dx$
5. $\int \frac{e^{1/x}}{x^2}\ dx$
6. $\int x\sqrt{x-1}\,dx$
7. $\int x^5 \sqrt{1+x^2}\ dx$
8. $\int_0^4 \sqrt{2x+1}\ dx$
9. $\int_0^2 xe^{x^2}\ dx$
10. $\int_1^e \frac{\ln x}{x}\ dx$

WebAssign

1. $\int \frac{x^3}{x^4-4}\,dx$
2. $\int \frac{\cos(\sqrt{t})}{\sqrt{t}}\ dt$
3. $\int (3-8x)^{10}\,dx$
4. $\int \frac{(\ln(x))^{30}}{x}\ dx$
5. $\int y^2\left(5-y^3\right)^{2/3}\ dy$
6. $\int_0^1 \sqrt[3]{1+7x}\,dx$
7. $\int_1^6 \frac{e^{1/x}}{x^2}\ dx$
8. $\int_0^{\pi/6} \frac{\sin(t)}{\cos^2(t)}\ dt$

Practice

1. $\int x\sqrt{1-x^2}\,dx$
2. $\int x\sec^2(x^2)\,dx$
3. $\int \frac{e^x}{1+e^x}\ dx$
4. $\int \frac{x}{(x^2+1)^3}\ dx$
5. $\int \frac{x}{1+x^4}\ dx$
6. $\int \frac{x}{\sqrt{x+2}}\ dx$
7. $\int_0^1 (5x+1)^3\,dx$
8. $\int_0^{\pi/4} \sec^2(x)\tan(x)\,dx$
9. $\int_0^{\sqrt{7}} x\sqrt{x^2+1}\,dx$
10. $\int_0^{\ln 2} e^{-x}\ dx$

7.1 Integration by Parts

Integration by Parts is the *product rule in reverse*. Use it when an integrand is a *product* of two functions and differentiating one factor makes the problem simpler.

Indefinite formula.

$$\int u\,dv \;=\; uv \;-\; \int v\,du.$$

(For indefinite integrals, we usually add a single +C at the end of the final antiderivative.)

Definite formula. If the integral has bounds $[a, b]$, then

$$\int_a^b u\,dv \;=\; \big[uv\big]_a^b \;-\; \int_a^b v\,du, \qquad \text{where} \qquad \big[uv\big]_a^b = u(b)v(b) - u(a)v(a).$$

Choose u so that du is simpler than u, and choose dv so that you can compute $v = \int dv$ easily. The goal is that the new integral $\int v\,du$ is simpler than the original. A common heuristic for choosing u is **LIATE**, listed from *highest* to *lowest* priority:

$$\boxed{\mathbf{L} > \mathbf{I} > \mathbf{A} > \mathbf{T} > \mathbf{E}}$$

- **L**ogarithmic: $\ln x$
- **I**nverse trig: $\arctan x$, $\arcsin x$, etc.
- **A**lgebraic: powers of x, polynomials, roots
- **T**rig: $\sin x$, $\cos x$, etc.
- **E**xponential: e^x, a^x

Notes

1. $\int x \sin x\,dx$.
2. $\int x^2 \ln x\,dx$.
3. $\int \ln x\,dx$.
4. $\int t^2 e^t\,dt$.
5. $\int_0^1 \tan^{-1}(x)\,dx$.

WebAssign

1. $\int xe^{7x}\,dx$.
2. $\int \arctan(5t)\,dt$.
3. $\int 6 \arcsin x\,dx$.
4. $\int (x^2 + 4x)\cos(x)\,dx$.

Practice

1. $\int xe^x\,dx$
2. $\int x\ln(x)\,dx$
3. $\int x^2 \sin(x)\,dx$
4. $\int x\cos(x)\,dx$
5. $\int x^3 \ln(x)\,dx$
6. $\int x^2 \ln(x^2)\,dx$
7. $\int x^2 \cos(2x)\,dx$
8. $\int \ln(x^2 + 1)\,dx$

7.1 Boomerang Integrals

Sometimes Integration by Parts produces an integral that is essentially the same as the one you started with. When that happens, the integral "boomerangs" back. The key is to recognize the repeat, apply IBP enough times to make the original integral reappear, and then solve algebraically.

Boomerang integrals often occur with products involving trig and exponential functions, such as

$$\int e^{ax}\sin(bx)\,dx, \qquad \int e^{ax}\cos(bx)\,dx,$$

because two rounds of IBP typically bring you back to a constant multiple of the original integral. They can also appear in repeated IBP patterns like $\int (\ln x)^n\,dx$.

Notes

1. $\displaystyle\int e^{x}\sin(x)\,dx$.
2. $\displaystyle\int \sec^3(x)\,dx$.†

Practice

1. $\displaystyle\int e^{3x}\cos(3x)\,dx$
2. $\displaystyle\int e^{-x}\sin(x)\,dx$

WebAssign

1. $\displaystyle\int e^{2\theta}\sin(3\theta)\,d\theta$

†We did this in Section 7.2.

7.2 Trigonometric Integrals

Idea: In trig integrals, your goal is usually to *create a useful derivative factor* and then use identities to rewrite whatever is left.

- $\sin^m x \cos^n x$**:** If *one power is odd*, save one factor of that function and use

$$\sin^2 x = 1 - \cos^2 x \qquad \text{or} \qquad \cos^2 x = 1 - \sin^2 x$$

to convert the remaining even power. Then use a u-sub:

$$\text{save } \cos x \Rightarrow u = \sin x, \qquad \text{save } \sin x \Rightarrow u = \cos x.$$

If *both powers are even*, use half-angle identities (often repeatedly) to rewrite everything in terms of $\cos(2x)$ and constants.

- $\tan^m x \sec^n x$**:** Try to save a factor of $\sec^2 x$ (so $u = \tan x$), or save a factor of $\sec x \tan x$ (so $u = \sec x$). Then rewrite what remains using

$$\tan^2 x = \sec^2 x - 1 \qquad \text{or} \qquad \sec^2 x = 1 + \tan^2 x.$$

In practice: if n is even, save $\sec^2 x$; if m is odd, save $\sec x \tan x$.

Notes

1. $\int \sin^5(x)\cos^2(x)\,dx$
2. $\int \sin^4(x)\,dx$
3. $\int \tan^6(x)\sec^4(x)\,dx$
4. $\int \tan^5(\theta)\sec^7(\theta)\,d\theta$
5. $\int \tan^3(x)\,dx$
6. $\int \sec^3(x)\,dx$

WebAssign

1. $\int 2\sin^2(x)\cos^3(x)\,dx$
2. $\int_0^{\pi/4} \sin^5(x)\,dx$
3. $\int_0^{\pi/2} 3\cos^2(\theta)\,d\theta$
4. $\int_0^{\pi/2} 5\sin^2(x)\cos^2(x)\,dx$
5. $\int 4\tan^3(x)\sec(x)\,dx$
6. $\int 13\tan^4(x)\sec^6(x)\,dx$

Practice

1. $\int \sin^3 x\,dx$
2. $\int \sin^6 x\,dx$
3. $\int \cos^5 x\,dx$
4. $\int \cos^4 x\,dx$
5. $\int \sin^2 x\,\cos x\,dx$
6. $\int \sin x\,\cos^2 x\,dx$
7. $\int \sin^4 x\,\cos^3 x\,dx$
8. $\int \sin^3 x\,\cos^4 x\,dx$
9. $\int \sin^2 x\,\cos^2 x\,dx$
10. $\int \tan^2 x\,dx$
11. $\int \sec^4 x\,dx$
12. $\int \tan^4 x\,\sec^2 x\,dx$
13. $\int \tan^3 x\,\sec^3 x\,dx$
14. $\int \tan^2 x\,\sec^3 x\,dx$

7.3 Trigonometric Substitution

Idea: Trig substitution is for integrals involving $\sqrt{\text{quadratic}}$. Choose a substitution that turns the radical into *one trig function* using a Pythagorean identity.

- $\sqrt{a^2 - x^2}$: let $x = a\sin\theta \Rightarrow \sqrt{a^2 - x^2} = a\cos\theta$.
- $\sqrt{a^2 + x^2}$: let $x = a\tan\theta \Rightarrow \sqrt{a^2 + x^2} = a\sec\theta$.
- $\sqrt{x^2 - a^2}$: let $x = a\sec\theta \Rightarrow \sqrt{x^2 - a^2} = a\tan\theta$.

Rewrite the entire integrand in θ, integrate, then convert back to x (using a right triangle or identities).

Notes

1. $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\,dx.$
2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2 + 4}}\,dx.$
3. $\displaystyle\int \frac{x}{\sqrt{x^2 + 4}}\,dx.$
4. $\displaystyle\int \frac{1}{\sqrt{x^2 - 5}}\,dx.$
5. $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\,dx.$

WebAssign

1. $\displaystyle\int \frac{x^2}{\sqrt{x^2 - 3}}\,dx$
2. $\displaystyle\int \frac{x^3}{\sqrt{25 + x^2}}\,dx$
3. $\displaystyle\int_0^7 \frac{dt}{\sqrt{49 + t^2}}$
4. $\displaystyle\int \frac{x}{\sqrt{x^2 - 7}}\,dx$
5. $\displaystyle\int \frac{\sqrt{x^2 - 81}}{x^4}\,dx$

Practice

1. $\displaystyle\int \frac{x}{\sqrt{9 + 3x^2}}\,dx$
2. $\displaystyle\int \frac{x}{(9 + 2x^2)^{3/2}}\,dx$
3. $\displaystyle\int \frac{x^3}{\sqrt{16 + x^2}}\,dx$
4. $\displaystyle\int \sqrt{5 - 2x^2}\,dx$
5. $\displaystyle\int \frac{dx}{\sqrt{5 - 2x^2}}$
6. $\displaystyle\int \frac{dx}{\sqrt{16 + x^2}}$
7. $\displaystyle\int \frac{dx}{\sqrt{8x^2 - 11}}$
8. $\displaystyle\int \frac{1}{x\sqrt{x^2 - 16}}\,dx$
9. $\displaystyle\int \frac{x^2}{\sqrt{10 - 3x^2}}\,dx$
10. $\displaystyle\int \frac{\sqrt{9x^2 - 16}}{x^4}\,dx$
11. $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - 16}}\,dx$
12. $\displaystyle\int_0^{\sqrt{5/2}} \sqrt{5 - 2x^2}\,dx$
13. $\displaystyle\int_0^2 \frac{dx}{\sqrt{16 + x^2}}$
14. $\displaystyle\int \frac{x^2}{(9 + 2x^2)^{3/2}}\,dx$
15. $\displaystyle\int \frac{1}{(25 + 3x^2)^{3/2}}\,dx$

7.4 Partial Fractions

Idea: Partial fractions is for integrals of *rational functions* $\dfrac{P(x)}{Q(x)}$. Rewrite the integrand as a *sum of simpler fractions* and integrate term-by-term.

- If $\deg P \geq \deg Q$: do long division first.
- Factor $Q(x)$ completely.
- Use the templates:

$$\frac{A}{x-a}, \quad \frac{A_1}{x-a}+\cdots+\frac{A_m}{(x-a)^m}, \quad \frac{Ax+B}{x^2+bx+c}, \quad \frac{A_1x+B_1}{x^2+bx+c}+\cdots+\frac{A_mx+B_m}{(x^2+bx+c)^m}.$$

- Solve for constants by clearing denominators, then integrate each term (logs / arctan).

Notes

1. $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$
2. $\displaystyle\int \frac{1}{x^2-9}\,dx.$
3. $\displaystyle\int \frac{x^4+x^3+6x^2+3x+4}{x^3+4x}\,dx.$

WebAssign

1. $\dfrac{x-72}{x^2+x-72}.$
2. $\dfrac{1}{x^2+x^4}.$
3. $\displaystyle\int \frac{37}{(x-1)(x^2+36)}\,dx.$
4. $\dfrac{4y^2-6y-12}{y(y+2)(y-3)}.$
5. $\displaystyle\int \frac{9r^2}{r+2}\,dr.$

Practice

1. $\displaystyle\int \frac{1}{x^2-4}\,dx$
2. $\displaystyle\int \frac{1}{(x-1)(x+2)}\,dx$
3. $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$
4. $\displaystyle\int \frac{2x+3}{x^2+x-2}\,dx$
5. $\displaystyle\int \frac{1}{x^3-x}\,dx$
6. $\displaystyle\int \frac{x^2}{x^3-1}\,dx$
7. $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$
8. $\displaystyle\int \frac{x}{x^3+x^2}\,dx$
9. $\displaystyle\int \frac{x+1}{(x^2-1)(x+3)}\,dx$
10. $\displaystyle\int \frac{x}{(x^2-4)(x+1)}\,dx$

7.8 Improper Integrals

A definite integral is called **improper** if the interval is unbounded ($a = -\infty$ or $b = \infty$) or if the integrand becomes infinite at an endpoint (or at a point inside the interval).

Strategy: rewrite the integral using limits.

- **Infinite interval:** $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx.$
- **Infinite discontinuity:** if $f(x) \to \infty$ as $x \to a^+$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$. If f blows up at $c \in (a,b)$, then *split the integral*:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and convert *each* piece to a limit.

Convergence: the improper integral **converges** if the relevant limit(s) exist and are finite; otherwise it **diverges**. If you split at c, both limits must converge.

Notes

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx.$
2. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx.$
3. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx.$
4. $\displaystyle\int_{-\infty}^0 xe^x\,dx.$
5. $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$
6. $\displaystyle\int_0^3 \frac{dx}{x-1}.$

WebAssign

1. $\displaystyle\int_8^9 \frac{1}{9x-1}\,dx.$
2. $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx.$
3. $\displaystyle\int_{-\infty}^\infty \frac{\sin(x)}{1+2x^2}\,dx.$
4. $\displaystyle\int_1^3 \ln(x-1)\,dx.$
5. $\displaystyle\int_0^\infty e^{-6x}\,dx.$
6. $\displaystyle\int_{-\infty}^0 \frac{x}{(x^2+2)^4}\,dx.$
7. $\displaystyle\int_{-\infty}^\infty \frac{x^5}{x^6+1}\,dx.$
8. $\displaystyle\int_e^\infty \frac{11}{x(\ln(x))^3}\,dx.$
9. $\displaystyle\int_0^1 \frac{5}{x^5}\,dx.$
10. $\displaystyle\int_3^{11} \frac{1}{\sqrt{11-x}}\,dx.$
11. $\displaystyle\int_2^\infty \frac{1}{x-\ln(x)}\,dx.$
12. $\displaystyle\int_1^\infty \frac{8+\cos(x)}{\sqrt{x^4+x^2}}\,dx.$

Practice

1. $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$
2. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
3. $\displaystyle\int_0^\infty e^{-x}\,dx$
4. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$
6. $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$
7. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$
8. $\displaystyle\int_0^1 \ln(x)\,dx$
9. $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$
10. $\displaystyle\int_0^2 \frac{dx}{(2-x)^{1/3}}$
11. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$
12. $\displaystyle\int_0^{\pi/2} \tan(x)\,dx$
13. $\displaystyle\int_1^\infty \frac{1}{x\,(\ln x)^2}\,dx$
14. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$

9.1 Differential Equations

Definition. A **differential equation** is an equation that relates an unknown function to one or more of its derivatives.

Key vocabulary.

- **Order:** the highest-order derivative that appears.
- **General solution:** the full family of solutions (typically containing one or more arbitrary constants).
- **Particular solution:** one specific member of that family.
- **Initial-value problem (IVP):** a differential equation together with an initial condition such as $y(t_0) = y_0$, used to select the particular solution.

Remark. To verify that $y = f(x)$ satisfies a differential equation:

1. Compute the derivatives that appear in the equation.
2. Substitute $y = f(x)$ and those derivatives into the differential equation.
3. Simplify. If the left and right sides match for all x in the interval, then f is a solution.

Multiple Choice Practice

1. Verify whether $y = e^{2x}$ is a solution to $y' = 2y$.

   (A) Yes, it satisfies the differential equation.
   (B) No, it does not satisfy the differential equation.

2. Is $y = x^2 + 1$ a solution to the differential equation $y' = 2x$?

   (A) Yes
   (B) No

3. Which of the following functions satisfy $y' = 3y$? (Select all that apply.)

   (a) $y = e^{3x}$
   (b) $y = 2e^{3x}$
   (c) $y = e^{-3x}$
   (d) $y = 3e^{x}$

4. Which of the following satisfy $y'' + y = 0$? (Select all that apply).

   (a) $y = \sin(x)$
   (b) $y = \cos(x)$
   (c) $y = e^{x}$
   (d) $y = \sin(x) + \cos(x)$

5. Solve the differential equation $y' = 6x$.

   (A) $y = 6x + C$
   (B) $y = 3x + C$
   (C) $y = 12x + C$
   (D) $y = 3x^2 + C$
   (E) $y = 6x^2 + C$

6. Solve the differential equation $\dfrac{dy}{dx} = \cos(x)$.

   (A) $y = \cos(x) + C$
   (B) $y = \sin(x) + C$
   (C) $y = -\sin(x) + C$
   (D) $y = -\cos(x) + C$
   (E) $y = x\cos(x) + C$

7. Solve the differential equation $y' = 2x$ given that $y(1) = 6$.

   (A) $y = x^2 + 2$
   (B) $y = x^2 + 3$
   (C) $y = x^2 + 4$
   (D) $y = x^2 + 5$
   (E) $y = 2x^2 + 5$

8. Find the particular solution to $\dfrac{dy}{dx} = e^x$ satisfying $y(0) = 3$.

   (A) $y = e^x + 3$
   (B) $y = e^x + 2$
   (C) $y = e^x + 1$
   (D) $y = e^x - 3$
   (E) $y = 3e^x$

9. True or False: If $y' = 2y$, then any solution graph must be increasing wherever $y > 0$.

10. True or False: If $y' = -y$, then the solution curves are always decreasing when $y > 0$.

Free Response Practice

1. Verify that $y = e^{3x}$ is a solution to the differential equation:

$$y' = 3y$$

2. Verify that $y = \sin(2x)$ is a solution to the differential equation:

$$y'' + 4y = 0$$

3. Verify that $y = x^2 + 1$ satisfies the differential equation:

$$y' = 2x$$

4. Solve the differential equation $y' = 2x$ given that $y(1) = 5$.

5. Find the particular solution to $\frac{dy}{dx} = 3e^x$ satisfying $y(0) = 2$.

6. Solve $\frac{dy}{dx} = \sin(x)$ with the initial condition $y\left(\frac{\pi}{2}\right) = 0$.

9.3 Separable Differential Equations

Definition. A differential equation is **separable** if it can be written in the form

$$\frac{dy}{dx} = g(x)h(y)$$

This allows separation of variables:

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

Steps for Solving a Separable Differential Equation

1. Rewrite in the form $\frac{dy}{dx} = g(x)h(y)$.
2. Separate variables: $\frac{1}{h(y)}\,dy = g(x)\,dx$.
3. Integrate both sides.
4. Solve explicitly for y, if possible.
5. Apply any initial condition to determine the constant C.

Remark. If the integral involves $\int \frac{1}{y}\,dy$, the solution includes a logarithm:

$$\int \frac{1}{y}\,dy = \ln|y| + C$$

Be careful with absolute values when solving for y.

Multiple Choice Practice

1. Which of the following differential equations is separable?

   (A) $\dfrac{dy}{dx} = x^2 + y^2$

   (B) $\dfrac{dy}{dx} = e^x + y$

   (C) $\dfrac{dy}{dx} = xy$

   (D) $\dfrac{dy}{dx} = \tan(x + y)$

   (E) $\dfrac{dy}{dx} = \ln(x + y)$

2. Which of the following is NOT separable?

   (A) $\dfrac{dy}{dx} = \dfrac{x}{1 + y^2}$

   (B) $\dfrac{dy}{dx} = x(1 + y^2)$

   (C) $\dfrac{dy}{dx} = x + y$

   (D) $\dfrac{dy}{dx} = \dfrac{y}{x}$

   (E) $\dfrac{dy}{dx} = \dfrac{y^2}{x}$

3. Solve for the general solution to $\frac{dy}{dx} = x^2 y$.

(A) $y = Ce^{x^2}$
(B) $y = Cx^2$
(C) $y = Ce^{x^3/3}$
(D) $y = Cxe^x$
(E) $y = Cx^3 + 1$

4. Solve for the general solution to $\frac{dy}{dx} = \frac{3y}{x}$.

(A) $y = Cx^3$
(B) $y = Cx$
(C) $y = Cx^2$
(D) $y = Ce^{3x}$
(E) $y = Cx^{-3}$

5. Find the particular solution to $\frac{dy}{dx} = y^2$ with initial condition $y(0) = 2$.

(A) $y = \frac{1}{2 - x}$
(B) $y = \frac{1}{x + 2}$
(C) $y = 2 + x$
(D) $y = 2 - x$
(E) $y = \frac{1}{x - 2}$

6. Find the particular solution to $\frac{dy}{dx} = (1 - x^2)y$ with $y(0) = 3$.

(A) $y = 3e^{-x+x^3/3}$
(B) $y = 3e^{x-x^3/3}$
(C) $y = 3e^{x^2/2}$
(D) $y = 3e^{x-x^2}$
(E) $y = 3e^{x^3-x}$

7. Solve the separable differential equation $\frac{dy}{dx} = \frac{2y}{x}$.

(A) $y = Cx$
(B) $y = Cx^2$
(C) $y = Cx^3$
(D) $y = Cx^4$
(E) $y = C\ln(x)$

8. Solve the separable differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y}$.

(A) $y^2 = \dfrac{2x^3}{3} + C$

(B) $y = Cx^2$

(C) $y = \dfrac{x^3}{3}$

(D) $y^2 = x^2 + C$

(E) $y = e^{x^2}$

Free Response Practice

1. Solve the differential equation $y' = xy$ by finding the general solution.

2. Solve the differential equation $y' = \dfrac{2y}{x}$ by finding the general solution.

3. Solve the differential equation $y' = x(1 + y^2)$ by finding the general solution.

4. Solve the differential equation: $y' = y^2$ with the initial condition $y(0) = 1$.

5. Solve the differential equation: $y' = (1 - x^2)y$ with $y(0) = 2$.

6. Solve the differential equation: $y' = x^2(1 + y)$ given that $y(1) = 0$.

7. Solve the differential equation: $y' = \dfrac{y}{x}$ and leave your answer in implicit form if necessary.

8. Solve the differential equation: $y' = \dfrac{x}{y}$ and leave your answer in implicit form if necessary.

9. Solve the differential equation: $y' = \dfrac{2y}{x+1}$ and leave your answer in implicit form if necessary.

6.1 Areas Between Curves

Area is always *(bigger boundary)* − *(smaller boundary)* integrated over the correct interval. You can slice the region two ways:

- **Vertical slices (dx):** write the boundaries as y-functions of x. If the region runs from $x = a$ to $x = b$, then
$$A = \int_a^b \big[y_T(x) - y_B(x)\big]\,dx \qquad \text{(top minus bottom).}$$
- **Horizontal slices (dy):** write the boundaries as x-functions of y. If the region runs from $y = c$ to $y = d$, then
$$A = \int_c^d \big[x_R(y) - x_L(y)\big]\,dy \qquad \text{(right minus left).}$$

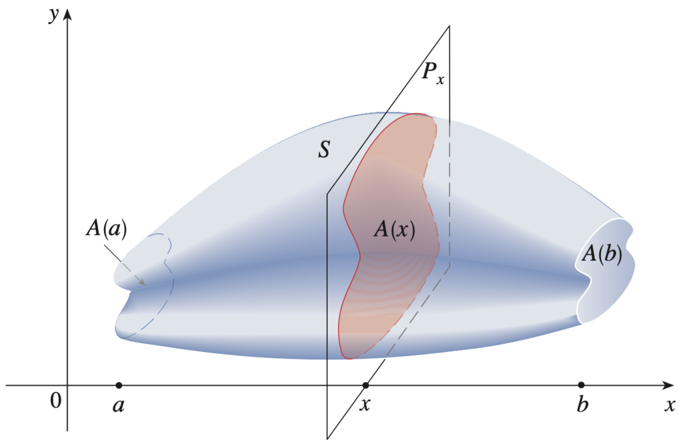
Important: if the curves cross inside the interval, the "top/bottom" (or "right/left") ordering changes. Find the intersection value(s) and *split the integral* so the ordering stays consistent on each piece (or use absolute value with the same split points).

Practice

1. Find the area enclosed by the curves $y = x^2$ and $y = 4$.
2. Find the area between the curves $y = x^3$ and $y = x$.
3. Compute the area enclosed by the curves $y = \sin x$ and $y = \cos x$ over the interval $0 \le x \le \frac{\pi}{2}$.
4. Determine the area of the region bounded by $y = x^2 + 1$ and $y = 3$.
5. Find the area enclosed by the curves $y = e^x$ and $y = 4$ for $0 \le x \le \ln 4$.

6.2 Volume by Known Cross-Sections

When a solid has cross-sectional areas that are easy to compute, we can find its volume by "stacking" these slices.



Let S be a solid lying between $x = a$ and $x = b$. Suppose that for each x in $[a, b]$ the cross-sectional area perpendicular to the x-axis is given by $A(x)$. Then the volume V of S is defined by

$$V = \int_a^b A(x)\,dx.$$

Remark. This definition generalizes the familiar formula for a cylinder. For a cylinder with constant base area A and height h, we have $V = Ah$.

Common Cross-Section Shapes

1. Square Cross Sections. If the cross section perpendicular to the x-axis is a square with side length $s(x)$, then the area is

$$A(x) = \big[s(x)\big]^2.$$

2. Rectangular Cross Sections. For a rectangle with base $b(x)$ and height $h(x)$, the area is

$$A(x) = b(x)\,h(x).$$

3. Isosceles Right Triangle Cross Sections. If each cross section is an isosceles right triangle with leg length $s(x)$, then its area is given by

$$A(x) = \frac{1}{2}\big[s(x)\big]^2.$$

4. Equilateral Triangle Cross Sections. For an equilateral triangle with side length $s(x)$, the area is

$$A(x) = \frac{\sqrt{3}}{4}\big[s(x)\big]^2.$$

5. Circular Cross Sections (Disks). If the cross section is a circle (or disk) with radius $r(x)$, then

$$A(x) = \pi\big[r(x)\big]^2.$$

6. Semicircular Cross Sections. For a semicircular cross section with radius $r(x)$, the area is

$$A(x) = \frac{1}{2}\pi\big[r(x)\big]^2.$$

Practice

1. A solid has a circular base of radius 2 centered at the origin in the xy-plane. The cross-sections perpendicular to the x-axis (i.e., vertical slices) are squares. If the side of each square lies in the plane of the cross-section, find the volume of this solid.

2. A solid has a right isosceles triangular base in the xy-plane with vertices at $(0,0)$, $(4,0)$, and $(0,4)$. The cross-sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

3. A solid has a square base in the xy-plane with corners at $(0,0)$, $(3,0)$, $(3,3)$, and $(0,3)$. The cross-sections perpendicular to the x-axis are equilateral triangles. Each triangle's base runs from $y=0$ to $y=3$. Find the volume of the solid.

4. A solid has a rectangular base in the xy-plane: $0 \le x \le 5$, $0 \le y \le 2$. Cross-sections perpendicular to the y-axis are right triangles whose legs lie along the base plane. Specifically, for each fixed y, the base of the right triangle is 5 (the full length in the x direction) and the height of the triangle is y. Find the volume of the solid.

5. A solid has a circular base of radius 1 (centered at the origin). Cross-sections perpendicular to the y-axis are equilateral triangles whose base is the chord of the circle at that y. Find the volume of this solid.

6. A solid has a right isosceles triangular base in the xy-plane with vertices at $(0,0)$, $(6,0)$, and $(0,6)$. For each fixed x in the interval $[0,6]$, the vertical line through the base (parallel to the y-axis) intersects the triangle from $y=0$ up to $y=6-x$. At each such x, a rectangle is erected whose:

   - Base (lying in the xy-plane) is the segment from $y=0$ to $y=6-x$, of length $6-x$, and
   - Height (in the z-direction) is defined to be twice the length of that segment, i.e., $2(6-x)$.

   Find the volume of the solid.

7. A solid has an elliptical base given by $\frac{x^2}{9}+\frac{y^2}{4} \le 1$. Cross-sections perpendicular to the y-axis are squares whose side length equals the x-span of the ellipse at that y. Find the volume.

8. A solid sits over a square base $[0,4]\times[0,4]$ in the xy-plane. The cross-sections perpendicular to the x-axis are isosceles trapezoids with:

   - One base along the y-range from 0 to 4 (i.e., length 4).
   - The other base has length 2, centered with respect to the first base.
   - The height (thickness in the z-direction) of each trapezoid is $h=1$ (constant).

   Find the volume of the solid.

9. A solid has a base in the xy-plane bounded by $y=0$, $x=2$, and $y=4-x^2$ (which intersects $x=2$ at $y=0$). The cross-sections perpendicular to the x-axis are rectangles. The width of each rectangle is from $y=0$ to $y=4-x^2$, and its height (out of the plane) is twice the y-value at that slice. Find the volume.