Midterm 1 Study Guide (Solutions)

MATH2300 - Calculus II

Spring 2026

Contents

5.5 u-Substitution (Solutions)

1. $\int x\sqrt{1-x^2}\,dx$

Substitution: Let $u = 1-x^2$. Then $du = -2x\,dx \implies -\frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int \sqrt{1-x^2}\cdot(x\,dx) &= \int u^{1/2}\cdot\left(-\frac{1}{2}\,du\right)\\ &= -\frac{1}{2}\int u^{1/2}\,du\\ &= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)+C\\ &= \boxed{-\frac{1}{3}(1-x^2)^{3/2}+C}\end{aligned}$$

2. $\int x\sec^2(x^2)\,dx$

Substitution: Let $u = x^2$. Then $du = 2x\,dx \implies \frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int \sec^2(x^2)\cdot(x\,dx) &= \int \sec^2(u)\cdot\left(\frac{1}{2}\,du\right)\\ &= \frac{1}{2}\int \sec^2(u)\,du\\ &= \frac{1}{2}\tan(u)+C\\ &= \boxed{\frac{1}{2}\tan(x^2)+C}\end{aligned}$$

3. $\int \frac{e^x}{1+e^x}\,dx$

Substitution: Let $u = 1+e^x$. Then $du = e^x\,dx$.

$$\begin{aligned}\int \frac{1}{1+e^x}\cdot(e^x\,dx) &= \int \frac{1}{u}\,du\\ &= \ln|u|+C\\ &= \boxed{\ln(1+e^x)+C}\end{aligned}$$

4. $\int \frac{x}{(x^2+1)^3}\,dx$

Substitution: Let $u = x^2+1$. Then $du = 2x\,dx \implies \frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int (x^2+1)^{-3}\cdot(x\,dx) &= \int u^{-3}\cdot\left(\frac{1}{2}\,du\right)\\ &= \frac{1}{2}\left(\frac{u^{-2}}{-2}\right)+C\\ &= -\frac{1}{4}u^{-2}+C\\ &= \boxed{-\frac{1}{4(x^2+1)^2}+C}\end{aligned}$$

5. $\int \frac{x}{1+x^4}\,dx$

Strategy: Rewrite x^4 *as* $(x^2)^2$.

Substitution: Let $u = x^2$. Then $du = 2x\,dx \implies \frac{1}{2}du = x\,dx$.

$$\begin{aligned}\int \frac{1}{1+(x^2)^2} \cdot (x\,dx) &= \int \frac{1}{1+u^2} \cdot \left(\frac{1}{2}\,du\right) \\ &= \frac{1}{2}\arctan(u) + C \\ &= \boxed{\frac{1}{2}\arctan(x^2) + C}\end{aligned}$$

6. $\int \frac{x}{\sqrt{x+2}}\,dx$

Substitution: Let $u = x+2$. Then $du = dx$.

Critical Step: Solve for $x \implies x = u - 2$.

$$\begin{aligned}\int \frac{u-2}{\sqrt{u}}\,du &= \int \left(\frac{u}{u^{1/2}} - \frac{2}{u^{1/2}}\right)du \\ &= \int \left(u^{1/2} - 2u^{-1/2}\right)du \\ &= \left(\frac{2}{3}u^{3/2}\right) - 2\left(2u^{1/2}\right) + C \\ &= \boxed{\frac{2}{3}(x+2)^{3/2} - 4\sqrt{x+2} + C}\end{aligned}$$

7. $\int_0^1 (5x+1)^3\,dx$

Substitution: Let $u = 5x+1 \implies \frac{1}{5}du = dx$.

New Bounds: $x = 0 \to u = 1$; $x = 1 \to u = 6$.

$$\begin{aligned}\int_1^6 u^3 \cdot \left(\frac{1}{5}\,du\right) &= \frac{1}{5}\left[\frac{u^4}{4}\right]_1^6 \\ &= \frac{1}{20}(6^4 - 1^4) \\ &= \frac{1295}{20} = \boxed{\frac{259}{4}}\end{aligned}$$

8. $\int_0^{\pi/4} \sec^2(x)\tan(x)\,dx$

Substitution: Let $u = \tan(x) \implies du = \sec^2(x)\,dx$.

New Bounds: $x = 0 \to u = 0$; $x = \pi/4 \to u = 1$.

$$\begin{aligned}\int_0^1 u\,du &= \left[\frac{u^2}{2}\right]_0^1 \\ &= \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

9. $\int_0^{\sqrt{7}} x\sqrt{x^2+1}\,dx$

Substitution: Let $u = x^2 + 1 \implies \frac{1}{2}du = x\,dx$.
New Bounds: $x = 0 \to u = 1; \quad x = \sqrt{7} \to u = 8$.

$$\begin{aligned}\int_1^8 u^{1/2} \cdot \left(\frac{1}{2}\,du\right) &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^8 \\ &= \frac{1}{3}(8^{3/2} - 1^{3/2}) \\ &= \frac{1}{3}(16\sqrt{2} - 1) = \boxed{\frac{16\sqrt{2}-1}{3}}\end{aligned}$$

10. $\int_0^{\ln 2} e^{-x}\,dx$

Substitution: Let $u = -x \implies -du = dx$.
New Bounds: $x = 0 \to u = 0; \quad x = \ln 2 \to u = -\ln 2$.

$$\begin{aligned}\int_0^{-\ln 2} e^u(-du) &= -\left[e^u\right]_0^{-\ln 2} \\ &= -(e^{-\ln 2} - e^0) \\ &= -(\tfrac{1}{2} - 1) \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

7.1 Integration by Parts (Solutions)

1. $\int_1^e xe^x\,dx$

$$u = x,\ dv = e^x\,dx \ \Rightarrow\ du = dx,\ v = e^x.$$

$$\int_1^e xe^x\,dx = \Big[xe^x - \int e^x\,dx\Big]_1^e = \Big[e^x(x-1)\Big]_1^e = \boxed{e^e(e-1)}.$$

2. $\int x\ln x\,dx$

$$u = \ln x,\ dv = x\,dx \ \Rightarrow\ du = \frac{1}{x}\,dx,\ v = \frac{x^2}{2}.$$

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}.$$

3. $\int_0^\pi x^2\sin x\,dx$

$$u = x^2,\ dv = \sin x\,dx \ \Rightarrow\ du = 2x\,dx,\ v = -\cos x.$$

$$\int x^2\sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx.$$

For $\int 2x\cos x\,dx$, take $u = 2x$, $dv = \cos x\,dx$, so $du = 2\,dx$, $v = \sin x$:

$$\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx = 2x\sin x + 2\cos x.$$

$$\int_0^\pi x^2\sin x\,dx = \Big[-x^2\cos x + 2x\sin x + 2\cos x\Big]_0^\pi = \boxed{\pi^2 - 4}.$$

4. $\int x\cos x\,dx$

$$u = x,\ dv = \cos x\,dx \ \Rightarrow\ du = dx,\ v = \sin x.$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = \boxed{x\sin x + \cos x + C}.$$

5. $\int_1^2 x^3\ln x\,dx$

$$u = \ln x,\ dv = x^3\,dx \ \Rightarrow\ du = \frac{1}{x}\,dx,\ v = \frac{x^4}{4}.$$

$$\int_1^2 x^3\ln x\,dx = \Big[\frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx\Big]_1^2 = \Big[\frac{x^4}{4}\ln x - \frac{x^4}{16}\Big]_1^2 = \boxed{4\ln 2 - \frac{15}{16}}.$$

6. $\int x^2\ln(x^2)\,dx$

Use $\ln(x^2) = 2\ln|x|$:

$$\int x^2\ln(x^2)\,dx = 2\int x^2\ln|x|\,dx.$$

$$u = \ln|x|,\ dv = x^2\,dx \ \Rightarrow\ du = \frac{1}{x}\,dx,\ v = \frac{x^3}{3}.$$

$$\int x^2\ln(x^2)\,dx = 2\Big(\frac{x^3}{3}\ln|x| - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx\Big) = 2\Big(\frac{x^3}{3}\ln|x| - \int \frac{x^2}{3}\,dx\Big) = \boxed{\frac{2x^3}{3}\ln|x| - \frac{2x^3}{9} + C}.$$

7. $\int x^2\cos(2x)\,dx$

$$u = x^2,\ dv = \cos(2x)\,dx \ \Rightarrow\ du = 2x\,dx,\ v = \frac{1}{2}\sin(2x).$$

$$\int x^2\cos(2x)\,dx = \frac{x^2}{2}\sin(2x) - \int x\sin(2x)\,dx.$$

For $\int x\sin(2x)\,dx$, take $u = x$, $dv = \sin(2x)\,dx$, so $du = dx$, $v = -\frac{1}{2}\cos(2x)$:

$$\int x\sin(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x).$$

$$\int x^2\cos(2x)\,dx = \boxed{\frac{x^2}{2}\sin(2x) + \frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C}.$$

8. $\int \ln(x^2+1)\,dx$

$$u = \ln(x^2+1),\ dv = dx \ \Rightarrow\ du = \frac{2x}{x^2+1}\,dx,\ v = x.$$

$$\int \ln(x^2+1)\,dx = x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx = x\ln(x^2+1) - \int \Big(2 - \frac{2}{x^2+1}\Big)\,dx = \boxed{x\ln(x^2+1) - 2x + 2\arctan x + C}.$$

7.1 Boomerang Integrals (Solutions)

1. $\displaystyle\int e^{3x}\cos(3x)\,dx$

   Let $I = \int e^{3x}\cos(3x)\,dx$.

$$I = \frac{e^{3x}\cos(3x)}{3} + \int e^{3x}\sin(3x)\,dx.$$

   Let $J = \int e^{3x}\sin(3x)\,dx$. Then

$$\begin{aligned} J &= \frac{e^{3x}\sin(3x)}{3} - \int e^{3x}\cos(3x)\,dx \\ &= \frac{e^{3x}\sin(3x)}{3} - I. \end{aligned}$$

$$\begin{aligned} I &= \frac{e^{3x}\cos(3x)}{3} + J \\ &= \frac{e^{3x}\cos(3x)}{3} + \frac{e^{3x}\sin(3x)}{3} - I, \end{aligned}$$

   so

$$\begin{aligned} 2I &= \frac{e^{3x}}{3}\big(\sin(3x)+\cos(3x)\big) \\ I &= \boxed{\frac{e^{3x}\big(\sin(3x)+\cos(3x)\big)}{6} + C}. \end{aligned}$$

2. $\displaystyle\int e^{-x}\sin x\,dx$

   Let $I = \int e^{-x}\sin x\,dx$.

$$I = -e^{-x}\sin x + \int e^{-x}\cos x\,dx.$$

   Let $J = \int e^{-x}\cos x\,dx$. Then

$$\begin{aligned} J &= -e^{-x}\cos x - \int e^{-x}\sin x\,dx \\ &= -e^{-x}\cos x - I. \end{aligned}$$

$$\begin{aligned} I &= -e^{-x}\sin x + J \\ &= -e^{-x}\sin x - e^{-x}\cos x - I, \end{aligned}$$

   so

$$\begin{aligned} 2I &= -e^{-x}\big(\sin x + \cos x\big) \\ I &= \boxed{-\frac{e^{-x}\big(\sin x+\cos x\big)}{2} + C}. \end{aligned}$$

7.2 Trigonometric Integrals (Solutions)

1. $\displaystyle\int \sin^3 x\,dx$

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin x\,(1-\cos^2 x)\,dx\\ &= -\int (1-u^2)\,du \qquad (u=\cos x,\ du=-\sin x\,dx)\\ &= -\left(u-\frac{u^3}{3}\right)+C\\ &= -\cos x+\frac{\cos^3 x}{3}+C.\end{aligned}$$

2. $\displaystyle\int \sin^6 x\,dx$

$$\begin{aligned}\int \sin^6 x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right)^3 dx\\ &= \frac{1}{8}\int \Big(1-3\cos 2x+3\cos^2 2x-\cos^3 2x\Big)\,dx\\ &= \frac{1}{8}\int \Big(1-3\cos 2x+\tfrac{3}{2}(1+\cos 4x)-\tfrac{1}{4}(3\cos 2x+\cos 6x)\Big)\,dx\\ &= \int \left(\frac{5}{16}-\frac{15}{32}\cos 2x+\frac{3}{16}\cos 4x-\frac{1}{32}\cos 6x\right)\,dx\\ &= \frac{5}{16}x-\frac{15}{64}\sin 2x+\frac{3}{64}\sin 4x-\frac{1}{192}\sin 6x+C.\end{aligned}$$

3. $\displaystyle\int \cos^5 x\,dx$

$$\begin{aligned}\int \cos^5 x\,dx &= \int \cos x\,(1-\sin^2 x)^2\,dx\\ &= \int (1-2u^2+u^4)\,du \qquad (u=\sin x,\ du=\cos x\,dx)\\ &= u-\frac{2}{3}u^3+\frac{1}{5}u^5+C\\ &= \sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C.\end{aligned}$$

4. $\displaystyle\int \cos^4 x\,dx$

$$\begin{aligned}\int \cos^4 x\,dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx\\ &= \frac{1}{4}\int \Big(1+2\cos 2x+\cos^2 2x\Big)\,dx\\ &= \frac{1}{4}\int \Big(1+2\cos 2x+\tfrac{1}{2}(1+\cos 4x)\Big)\,dx\\ &= \int \left(\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)\,dx\\ &= \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.\end{aligned}$$

5. $\displaystyle\int \sin^2 x\,\cos x\,dx$

$$\begin{aligned}\int \sin^2 x\,\cos x\,dx &= \int u^2\,du \qquad (u=\sin x,\ du=\cos x\,dx)\\ &= \frac{u^3}{3}+C\\ &= \frac{\sin^3 x}{3}+C.\end{aligned}$$

6. $\int \sin x \, \cos^2 x \, dx$

$$\begin{aligned}\int \sin x \, \cos^2 x \, dx &= -\int u^2 \, du \qquad (u = \cos x, \; du = -\sin x \, dx) \\ &= -\frac{u^3}{3} + C \\ &= -\frac{\cos^3 x}{3} + C.\end{aligned}$$

7. $\int \sin^4 x \, \cos^3 x \, dx$

$$\begin{aligned}\int \sin^4 x \, \cos^3 x \, dx &= \int \sin^4 x \, \cos^2 x \, \cos x \, dx \\ &= \int \sin^4 x \, (1 - \sin^2 x) \, \cos x \, dx \\ &= \int (u^4 - u^6) \, du \qquad (u = \sin x, \; du = \cos x \, dx) \\ &= \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.\end{aligned}$$

8. $\int \sin^3 x \, \cos^4 x \, dx$

$$\begin{aligned}\int \sin^3 x \, \cos^4 x \, dx &= \int \sin^2 x \, \cos^4 x \, \sin x \, dx \\ &= \int (1 - \cos^2 x) \cos^4 x \, \sin x \, dx \\ &= -\int (u^4 - u^6) \, du \qquad (u = \cos x, \; du = -\sin x \, dx) \\ &= -\frac{u^5}{5} + \frac{u^7}{7} + C \\ &= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.\end{aligned}$$

9. $\int \sin^2 x \, \cos^2 x \, dx$

$$\begin{aligned}\int \sin^2 x \, \cos^2 x \, dx &= \int \frac{1}{4} \sin^2(2x) \, dx \\ &= \int \frac{1}{4} \cdot \frac{1 - \cos 4x}{2} \, dx \\ &= \int \frac{1}{8} \big(1 - \cos 4x\big) \, dx \\ &= \frac{x}{8} - \frac{1}{32} \sin 4x + C.\end{aligned}$$

10. $\int \tan^2 x \, dx$

$$\begin{aligned}\int \tan^2 x \, dx &= \int (\sec^2 x - 1) \, dx \\ &= \tan x - x + C.\end{aligned}$$

11. $\int \sec^4 x \, dx$

$$\begin{aligned}\int \sec^4 x \, dx &= \int \sec^2 x (1 + \tan^2 x) \, dx \\ &= \int (1 + u^2) \, du \qquad \big(u = \tan x, \; du = \sec^2 x \, dx\big) \\ &= u + \frac{u^3}{3} + C \\ &= \tan x + \frac{\tan^3 x}{3} + C.\end{aligned}$$

12. $\int \tan^4 x \sec^2 x \, dx$

$$\begin{aligned} \int \tan^4 x \sec^2 x \, dx &= \int u^4 \, du \qquad (u = \tan x,\ du = \sec^2 x \, dx) \\ &= \frac{u^5}{5} + C \\ &= \frac{\tan^5 x}{5} + C. \end{aligned}$$

13. $\int \tan^3 x \sec^3 x \, dx$

$$\begin{aligned} \int \tan^3 x \sec^3 x \, dx &= \int \tan^2 x \sec^2 x \, (\sec x \tan x) \, dx \\ &= \int (\sec^2 x - 1) \sec^2 x \, (\sec x \tan x) \, dx \\ &= \int (u^2 - 1) u^2 \, du \qquad (u = \sec x,\ du = \sec x \tan x \, dx) \\ &= \int (u^4 - u^2) \, du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C. \end{aligned}$$

14. $\int \tan^2 x \sec^3 x \, dx$

$$\begin{aligned} \int \tan^2 x \sec^3 x \, dx &= \int (\sec^2 x - 1) \sec^3 x \, dx \\ &= \int \sec^5 x \, dx - \int \sec^3 x \, dx. \end{aligned}$$

Compute $\int \sec^5 x \, dx$ by integration by parts with $u = \sec^3 x$ and $dv = \sec^2 x \, dx$:

$$\begin{aligned} \int \sec^5 x \, dx &= \int \sec^3 x \sec^2 x \, dx \\ &= \sec^3 x \tan x - \int \tan x \, (3 \sec^3 x \tan x) \, dx \\ &= \sec^3 x \tan x - 3 \int \sec^3 x \tan^2 x \, dx \\ &= \sec^3 x \tan x - 3 \int \sec^3 x (\sec^2 x - 1) \, dx \\ &= \sec^3 x \tan x - 3 \left(\int \sec^5 x \, dx - \int \sec^3 x \, dx \right). \end{aligned}$$

Solve for $\int \sec^5 x \, dx$:

$$\begin{aligned} 4 \int \sec^5 x \, dx &= \sec^3 x \tan x + 3 \int \sec^3 x \, dx \\ \int \sec^5 x \, dx &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx. \end{aligned}$$

Substitute back to finish:

$$\begin{aligned} \int \tan^2 x \sec^3 x \, dx &= \left(\frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx \right) - \int \sec^3 x \, dx \\ &= \frac{1}{4} \sec^3 x \tan x - \frac{1}{4} \int \sec^3 x \, dx + C. \end{aligned}$$

7.3 Trigonometric Substitution (Solutions)

1. $\displaystyle\int \frac{x}{\sqrt{9+3x^2}}\,dx$

   (a) *Choose a substitution.* Let $u = 9+3x^2$. Then $du = 6x\,dx$, so $x\,dx = \frac{1}{6}\,du$.

   (b) *Rewrite and integrate.*

$$\begin{aligned}\int \frac{x}{\sqrt{9+3x^2}}\,dx &= \int \frac{1}{\sqrt{u}}\left(\frac{1}{6}\,du\right)\\ &= \frac{1}{6}\int u^{-1/2}\,du\\ &= \frac{1}{6}\cdot 2u^{1/2} + C\\ &= \frac{1}{3}\sqrt{u} + C.\end{aligned}$$

   (c) *Substitute back.*

$$\boxed{\int \frac{x}{\sqrt{9+3x^2}}\,dx = \frac{1}{3}\sqrt{9+3x^2} + C.}$$

2. $\displaystyle\int \frac{x}{(9+2x^2)^{3/2}}\,dx$

   (a) *Choose a substitution.* Let $u = 9+2x^2$. Then $du = 4x\,dx$, so $x\,dx = \frac{1}{4}\,du$.

   (b) *Rewrite and integrate.*

$$\begin{aligned}\int \frac{x}{(9+2x^2)^{3/2}}\,dx &= \int u^{-3/2}\left(\frac{1}{4}\,du\right)\\ &= \frac{1}{4}\int u^{-3/2}\,du\\ &= \frac{1}{4}\left(\frac{u^{-1/2}}{-1/2}\right) + C\\ &= -\frac{1}{2}u^{-1/2} + C.\end{aligned}$$

   (c) *Substitute back.*

$$\boxed{\int \frac{x}{(9+2x^2)^{3/2}}\,dx = -\frac{1}{2\sqrt{9+2x^2}} + C.}$$

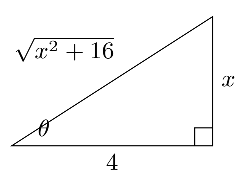
3. $\displaystyle\int \frac{x^3}{\sqrt{16+x^2}}\,dx$

   Quick note. This integral can also be done efficiently by u-sub with $u = 16+x^2$ and by rewriting x^3 as $x^2 \cdot x$.

   (a) *Identify the form.*

$$\sqrt{16+x^2} = \sqrt{x^2+4^2} \quad \Rightarrow \quad x = 4\tan\theta.$$

   (b) *Draw the triangle (for $\tan\theta = \frac{x}{4}$).*



   (c) *Substitute.* Let $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\,d\theta$.

   (d) *Simplify the radical.*

$$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = 4\sec\theta.$$

   (e) *Rewrite and integrate.*

$$\begin{aligned}\int \frac{x^3}{\sqrt{16+x^2}}\,dx &= \int \frac{(4\tan\theta)^3}{4\sec\theta}(4\sec^2\theta)\,d\theta\\ &= 64\int \tan^3\theta\,\sec\theta\,d\theta\\ &= 64\int (\sec^2\theta - 1)\,\sec\theta\tan\theta\,d\theta.\end{aligned}$$

Let $u = \sec\theta$. Then $du = \sec\theta\tan\theta\,d\theta$, and

$$\begin{aligned} 64\int(\sec^2\theta - 1)\,\sec\theta\tan\theta\,d\theta &= 64\int(u^2-1)\,du \\ &= 64\left(\frac{u^3}{3} - u\right) + C \\ &= \frac{64}{3}\sec^3\theta - 64\sec\theta + C. \end{aligned}$$

(f) *Back-substitute.* From the triangle, $\sec\theta = \dfrac{\sqrt{x^2+16}}{4}$. Thus

$$\begin{aligned} \frac{64}{3}\sec^3\theta - 64\sec\theta &= \frac{64}{3}\left(\frac{\sqrt{x^2+16}}{4}\right)^3 - 64\left(\frac{\sqrt{x^2+16}}{4}\right) \\ &= \frac{1}{3}(x^2+16)^{3/2} - 16\sqrt{x^2+16}. \end{aligned}$$

(g) *Final answer.*

$$\boxed{\int \frac{x^3}{\sqrt{16+x^2}}\,dx = \frac{1}{3}(x^2+16)^{3/2} - 16\sqrt{x^2+16} + C.}$$

4. $\displaystyle\int \sqrt{5-2x^2}\,dx$

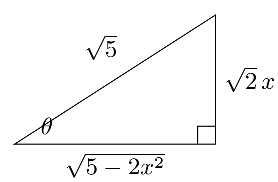
(a) *Identify the form.*

$$\sqrt{5-2x^2} = \sqrt{5}\sqrt{1-\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2} \text{ with } u = \frac{\sqrt{2}\,x}{\sqrt{5}}, \text{ so use } u = \sin\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{5}{2}}\sin\theta, \qquad dx = \sqrt{\frac{5}{2}}\cos\theta\,d\theta.$$

(c) *Reference triangle (for* $\sin\theta = \dfrac{\sqrt{2}\,x}{\sqrt{5}}$*).*



(d) *Rewrite the integral.*

$$\sqrt{5-2x^2} = \sqrt{5-5\sin^2\theta} = \sqrt{5}\cos\theta,$$

so

$$\begin{aligned} \int\sqrt{5-2x^2}\,dx &= \int(\sqrt{5}\cos\theta)\left(\sqrt{\frac{5}{2}}\cos\theta\right)d\theta \\ &= \frac{5}{\sqrt{2}}\int\cos^2\theta\,d\theta. \end{aligned}$$

(e) *Integrate.*

$$\int\cos^2\theta\,d\theta = \int\frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C,$$

so

$$\int\sqrt{5-2x^2}\,dx = \frac{5}{2\sqrt{2}}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C.$$

(f) *Back-substitute.* From the reference triangle,

$$\sin\theta = \frac{\sqrt{2}\,x}{\sqrt{5}}, \qquad \cos\theta = \frac{\sqrt{5-2x^2}}{\sqrt{5}},$$

so

$$\theta = \arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right).$$

Also,

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)\left(\frac{\sqrt{5-2x^2}}{\sqrt{5}}\right) \\ &= \frac{2\sqrt{2}\,x\sqrt{5-2x^2}}{5}.\end{aligned}$$

(g) *Final answer.*

$$\boxed{\int \sqrt{5-2x^2}\,dx = \frac{x}{2}\sqrt{5-2x^2} + \frac{5}{2\sqrt{2}}\arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right) + C.}$$

5. $\displaystyle\int \frac{dx}{\sqrt{5-2x^2}}$

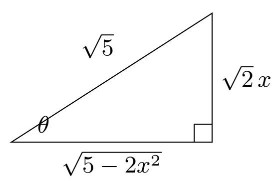
(a) *Identify the form.*

$$\sqrt{5-2x^2} = \sqrt{5}\sqrt{1-\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2} \text{ with } u = \frac{\sqrt{2}\,x}{\sqrt{5}}, \text{ so use } u = \sin\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{5}{2}}\sin\theta, \qquad dx = \sqrt{\frac{5}{2}}\cos\theta\,d\theta.$$

(c) *Reference triangle (for* $\sin\theta = \dfrac{\sqrt{2}\,x}{\sqrt{5}}$*).*



(d) *Rewrite the integral.*

$$\sqrt{5-2x^2} = \sqrt{5}\,\cos\theta,$$

so

$$\int \frac{dx}{\sqrt{5-2x^2}} = \int \frac{\sqrt{\frac{5}{2}}\cos\theta}{\sqrt{5}\,\cos\theta}\,d\theta = \frac{1}{\sqrt{2}}\int d\theta.$$

(e) *Integrate.*

$$\frac{1}{\sqrt{2}}\int d\theta = \frac{\theta}{\sqrt{2}} + C.$$

(f) *Back-substitute.*

$$\theta = \arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right).$$

(g) *Final answer.*

$$\boxed{\int \frac{dx}{\sqrt{5-2x^2}} = \frac{1}{\sqrt{2}}\arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right) + C.}$$

6. $\displaystyle\int \frac{dx}{\sqrt{16+x^2}}$

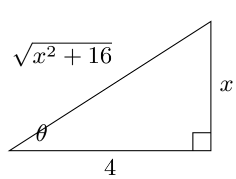
(a) *Identify the form.*

$$\sqrt{16+x^2} = \sqrt{x^2+4^2} \quad\Rightarrow\quad \sqrt{x^2+a^2} \text{ with } a = 4, \text{ so use } x = 4\tan\theta.$$

(b) *Trig substitution.*

$$x = 4\tan\theta, \qquad dx = 4\sec^2\theta\,d\theta.$$

(c) *Reference triangle (for* $\tan\theta = \dfrac{x}{4}$*).*



(d) *Rewrite the integral.*

$$\sqrt{16+x^2} = 4\sec\theta,$$

so

$$\int \frac{dx}{\sqrt{16+x^2}} = \int \frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int \sec\theta\,d\theta.$$

(e) *Integrate.*

$$\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C.$$

(f) *Back-substitute.* From the triangle,

$$\sec\theta = \frac{\sqrt{x^2+16}}{4}, \qquad \tan\theta = \frac{x}{4},$$

so

$$\ln|\sec\theta+\tan\theta| = \ln\left|\frac{\sqrt{x^2+16}+x}{4}\right|.$$

(g) *Final answer.*

$$\boxed{\int \frac{dx}{\sqrt{16+x^2}} = \ln\left|x+\sqrt{x^2+16}\right| + C.}$$

7. $\displaystyle\int \frac{dx}{\sqrt{8x^2-11}}$

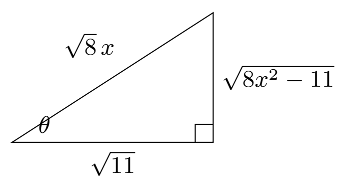
(a) *Identify the form.*

$$\sqrt{8x^2-11} = \sqrt{11}\sqrt{\left(\frac{\sqrt{8}\,x}{\sqrt{11}}\right)^2 - 1} \quad\Rightarrow\quad \sqrt{u^2-a^2} \text{ with } u = \frac{\sqrt{8}\,x}{\sqrt{11}}, \text{ so use } u = \sec\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{11}{8}}\sec\theta, \qquad dx = \sqrt{\frac{11}{8}}\sec\theta\tan\theta\,d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \dfrac{\sqrt{8}\,x}{\sqrt{11}}$*).*



(d) *Rewrite the integral.*

$$\sqrt{8x^2-11} = \sqrt{11}\tan\theta,$$

so

$$\int \frac{dx}{\sqrt{8x^2-11}} = \int \frac{\sqrt{\frac{11}{8}}\sec\theta\tan\theta}{\sqrt{11}\tan\theta}\,d\theta = \frac{1}{\sqrt{8}}\int \sec\theta\,d\theta.$$

(e) *Integrate.*

$$\frac{1}{\sqrt{8}}\int \sec\theta\,d\theta = \frac{1}{\sqrt{8}}\ln|\sec\theta+\tan\theta| + C.$$

(f) *Back-substitute.* From the triangle,

$$\sec\theta = \frac{\sqrt{8}\,x}{\sqrt{11}}, \qquad \tan\theta = \frac{\sqrt{8x^2-11}}{\sqrt{11}},$$

so

$$\ln|\sec\theta+\tan\theta| = \ln\left|\frac{\sqrt{8}\,x+\sqrt{8x^2-11}}{\sqrt{11}}\right|.$$

(g) *Final answer.*

$$\boxed{\int \frac{dx}{\sqrt{8x^2-11}} = \frac{1}{2\sqrt{2}}\ln\left|\sqrt{8}\,x+\sqrt{8x^2-11}\right| + C.}$$

8. $\displaystyle\int \frac{1}{x\sqrt{x^2-16}}\,dx$

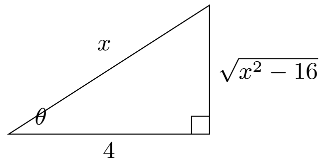
(a) *Identify the form.*

$$\sqrt{x^2-16} = \sqrt{x^2-4^2} \quad\Rightarrow\quad \sqrt{x^2-a^2} \text{ with } a = 4, \text{ so use } x = 4\sec\theta.$$

(b) *Trig substitution.*

$$x = 4\sec\theta, \qquad dx = 4\sec\theta\tan\theta\, d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \dfrac{x}{4}$*).*



(d) *Rewrite the integral.*

$$\sqrt{x^2-16} = \sqrt{16\sec^2\theta - 16} = 4\tan\theta,$$

so

$$\int \frac{1}{x\sqrt{x^2-16}}\, dx = \int \frac{1}{(4\sec\theta)(4\tan\theta)}\,(4\sec\theta\tan\theta)\, d\theta$$
$$= \frac{1}{4}\int d\theta.$$

(e) *Integrate.*

$$\frac{1}{4}\int d\theta = \frac{\theta}{4} + C.$$

(f) *Back-substitute (solve for* θ *using* $\tan\theta$*).* From the triangle,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2-16}}{4},$$

so

$$\theta = \arctan\left(\frac{\sqrt{x^2-16}}{4}\right).$$

(g) *Final answer.*

$$\boxed{\int \frac{1}{x\sqrt{x^2-16}}\, dx = \frac{1}{4}\arctan\left(\frac{\sqrt{x^2-16}}{4}\right) + C.}$$

(h) *Note.* Our book defines $\sec^{-1}$ to return $\theta \in (0,\pi/2)\cup(\pi,3\pi/2)$. With this convention, $\theta = \sec^{-1}\left(\frac{x}{4}\right)$ automatically handles both branches: for $x > 4$ it returns an acute angle, and for $x < -4$ it returns a quadrant-III angle. We can also solve using tangent: the triangle gives the acute reference angle $\alpha = \arctan\left(\frac{\sqrt{x^2-16}}{4}\right) \in (0,\pi/2)$; on the $x < -4$ branch the needed angle is $\theta = \pi + \alpha$ (since $\tan(\pi+\alpha) = \tan\alpha$). But the antiderivative after substitution is $\frac{1}{4}\theta + C$, so

$$\frac{1}{4}\theta + C = \frac{1}{4}(\pi+\alpha) + C = \frac{1}{4}\alpha + \left(C + \frac{\pi}{4}\right),$$

and $C + \frac{\pi}{4}$ is just a new constant. Thus using arctan produces the same antiderivative (up to $+C$).

9. $\displaystyle\int \frac{x^2}{\sqrt{10-3x^2}}\, dx$

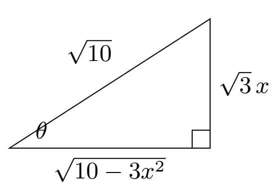
(a) *Identify the form.*

$$\sqrt{10-3x^2} = \sqrt{10}\sqrt{1 - \left(\frac{\sqrt{3}\,x}{\sqrt{10}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2} \text{ with } u = \frac{\sqrt{3}\,x}{\sqrt{10}}, \text{ so use } u = \sin\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{10}{3}}\sin\theta, \qquad dx = \sqrt{\frac{10}{3}}\cos\theta\, d\theta.$$

(c) *Reference triangle (for* $\sin\theta = \dfrac{\sqrt{3}\,x}{\sqrt{10}}$*).*



(d) *Rewrite the integral.*

$$x^2 = \frac{10}{3}\sin^2\theta, \qquad \sqrt{10-3x^2} = \sqrt{10(1-\sin^2\theta)} = \sqrt{10}\cos\theta,$$

so

$$\int \frac{x^2}{\sqrt{10-3x^2}}\,dx = \int \frac{\frac{10}{3}\sin^2\theta}{\sqrt{10}\cos\theta}\left(\sqrt{\frac{10}{3}}\cos\theta\right)d\theta$$
$$= \frac{10}{3\sqrt{3}}\int \sin^2\theta\,d\theta.$$

(e) *Integrate.*

$$\int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C,$$

so

$$\int \frac{x^2}{\sqrt{10-3x^2}}\,dx = \frac{5}{3\sqrt{3}}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C.$$

(f) *Back-substitute.*

$$\theta = \arcsin\left(\frac{\sqrt{3}\,x}{\sqrt{10}}\right), \qquad \sin 2\theta = 2\sin\theta\cos\theta = \frac{\sqrt{3}\,x\sqrt{10-3x^2}}{5}.$$

(g) *Final answer.*

$$\boxed{\int \frac{x^2}{\sqrt{10-3x^2}}\,dx = \frac{5}{3\sqrt{3}}\arcsin\left(\frac{\sqrt{3}\,x}{\sqrt{10}}\right) - \frac{x}{6}\sqrt{10-3x^2} + C.}$$

10. $\displaystyle\int \frac{\sqrt{9x^2-16}}{x^4}\,dx$

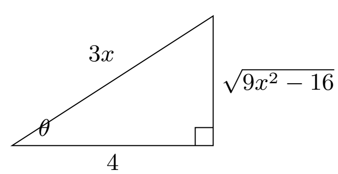
(a) *Identify the form.*

$$\sqrt{9x^2-16} = \sqrt{(3x)^2-4^2} \quad\Rightarrow\quad \sqrt{u^2-a^2} \text{ with } u = 3x,\ a = 4, \text{ so use } u = 4\sec\theta.$$

(b) *Trig substitution.*

$$3x = 4\sec\theta \quad\Rightarrow\quad x = \frac{4}{3}\sec\theta, \qquad dx = \frac{4}{3}\sec\theta\tan\theta\,d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \frac{3x}{4}$*).*



(d) *Rewrite the integral.* From $(3x)^2 - 4^2 = (4\sec\theta)^2 - 4^2$, we get

$$\sqrt{9x^2-16} = 4\tan\theta, \qquad x^4 = \left(\frac{4}{3}\right)^4\sec^4\theta = \frac{256}{81}\sec^4\theta.$$

Therefore,

$$\int \frac{\sqrt{9x^2-16}}{x^4}\,dx = \int \frac{4\tan\theta}{\frac{256}{81}\sec^4\theta}\left(\frac{4}{3}\sec\theta\tan\theta\right)d\theta$$
$$= \frac{27}{16}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{27}{16}\int \sin^2\theta\cos\theta\,d\theta.$$

(e) *Integrate.* Let $u = \sin\theta$, $du = \cos\theta\,d\theta$.

$$\frac{27}{16}\int \sin^2\theta\cos\theta\,d\theta = \frac{27}{16}\int u^2\,du = \frac{9}{16}\sin^3\theta + C.$$

(f) *Back-substitute.* From the triangle,

$$\sin\theta = \frac{\sqrt{9x^2-16}}{3x} \quad\Rightarrow\quad \sin^3\theta = \frac{(9x^2-16)^{3/2}}{27x^3}.$$

Hence

$$\frac{9}{16}\sin^3\theta = \frac{9}{16}\cdot\frac{(9x^2-16)^{3/2}}{27x^3} = \frac{1}{48}\,\frac{(9x^2-16)^{3/2}}{x^3}.$$

(g) *Final answer.*

$$\boxed{\int \frac{\sqrt{9x^2-16}}{x^4}\,dx = \frac{1}{48}\,\frac{(9x^2-16)^{3/2}}{x^3} + C.}$$

11. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$

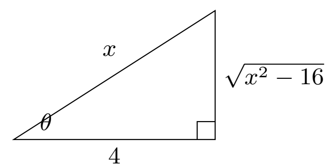
(a) *Identify the form.*

$$\sqrt{x^2-16} = \sqrt{x^2-4^2} \quad\Rightarrow\quad \sqrt{x^2-a^2} \;\text{ with } a=4, \text{ so use } x = 4\sec\theta.$$

(b) *Trig substitution.*

$$x = 4\sec\theta, \qquad dx = 4\sec\theta\tan\theta\,d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \frac{x}{4}$*).*



(d) *Rewrite the integral.*

$$x^2 = 16\sec^2\theta, \qquad \sqrt{x^2-16} = 4\tan\theta,$$

so

$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \int \frac{1}{(16\sec^2\theta)(4\tan\theta)}\,(4\sec\theta\tan\theta)\,d\theta$$
$$= \frac{1}{16}\int \cos\theta\,d\theta.$$

(e) *Integrate.*

$$\frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C.$$

(f) *Back-substitute.* From the triangle,

$$\sin\theta = \frac{\sqrt{x^2-16}}{x}.$$

(g) *Final answer.*

$$\boxed{\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \frac{\sqrt{x^2-16}}{16x} + C.}$$

12. $\displaystyle\int_0^{\sqrt{5/2}} \sqrt{5-2x^2}\,dx$

(a) *Identify the form.*

$$\sqrt{5-2x^2} = \sqrt{5}\sqrt{1-\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2} \;\text{ with } u = \frac{\sqrt{2}\,x}{\sqrt{5}}, \text{ so use } u = \sin\theta.$$

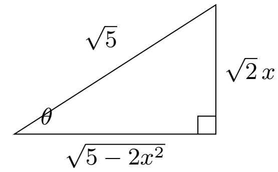
(b) *Trig substitution.*

$$x = \sqrt{\frac{5}{2}}\sin\theta, \qquad dx = \sqrt{\frac{5}{2}}\cos\theta\,d\theta.$$

Change bounds:

$$x = 0 \Rightarrow \theta_1 = 0, \qquad x = \sqrt{\frac{5}{2}} \Rightarrow \theta_2 = \frac{\pi}{2}.$$

(c) *Reference triangle (for* $\sin\theta = \frac{\sqrt{2}\,x}{\sqrt{5}}$*).*



(d) *Rewrite the integral.*

$$\sqrt{5-2x^2} = \sqrt{5}\,\cos\theta,$$

so

$$\int_0^{\sqrt{5/2}} \sqrt{5-2x^2}\,dx = \int_0^{\pi/2} (\sqrt{5}\,\cos\theta)\left(\sqrt{\frac{5}{2}}\cos\theta\right)d\theta = \frac{5}{\sqrt{2}}\int_0^{\pi/2}\cos^2\theta\,d\theta.$$

(e) *Integrate.*

$$\frac{5}{\sqrt{2}}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{5}{\sqrt{2}}\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta = \frac{5}{2\sqrt{2}}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}.$$

(f) *Back-substitute.*

$$\frac{5}{2\sqrt{2}}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{5}{2\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{5\pi}{4\sqrt{2}}.$$

(g) *Final answer.*

$$\boxed{\int_0^{\sqrt{5/2}} \sqrt{5-2x^2}\,dx = \frac{5\pi}{4\sqrt{2}}.}$$

13. $\displaystyle\int_0^2 \frac{dx}{\sqrt{16+x^2}}$

(a) *Identify the form.*

$$\sqrt{16+x^2} = \sqrt{x^2+4^2} \quad\Rightarrow\quad \sqrt{x^2+a^2} \text{ with } a=4, \text{ so use } x = 4\tan\theta.$$

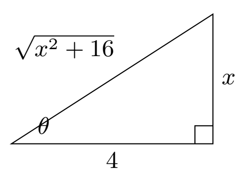
(b) *Trig substitution.*

$$x = 4\tan\theta, \qquad dx = 4\sec^2\theta\,d\theta, \qquad \sqrt{16+x^2} = 4\sec\theta.$$

Change bounds:

$$x = 0 \Rightarrow \theta_1 = 0, \qquad x = 2 \Rightarrow \theta_2 = \arctan\left(\frac{1}{2}\right).$$

(c) *Reference triangle (for* $\tan\theta = \frac{x}{4}$*).*



(d) *Rewrite the integral.*

$$\int_0^2 \frac{dx}{\sqrt{16+x^2}} = \int_0^{\theta_2}\frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int_0^{\theta_2}\sec\theta\,d\theta.$$

(e) *Integrate.*

$$\int_0^{\theta_2}\sec\theta\,d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_0^{\theta_2}.$$

(f) *Back-substitute.* At $\theta = \theta_2$, we have $\tan\theta_2 = \dfrac{1}{2}$, so

$$\sec\theta_2 = \sqrt{1+\tan^2\theta_2} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}, \qquad \sec 0 + \tan 0 = 1.$$

Therefore

$$\left[\ln|\sec\theta + \tan\theta|\right]_0^{\theta_2} = \ln\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right) - \ln(1) = \ln\left(\frac{\sqrt{5}+1}{2}\right).$$

(g) *Final answer.*

$$\boxed{\int_0^2 \frac{dx}{\sqrt{16+x^2}} = \ln\left(\frac{\sqrt{5}+1}{2}\right).}$$

14. $\displaystyle\int \frac{x^2}{(9+2x^2)^{3/2}}\,dx$

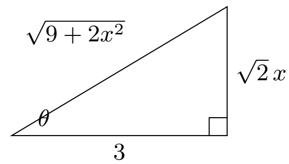
(a) *Identify the form.*

$$(9+2x^2)^{3/2} = \left(3^2 + (\sqrt{2}\,x)^2\right)^{3/2} \quad\Rightarrow\quad \sqrt{a^2+u^2} \text{ with } u = \sqrt{2}\,x,\ a = 3, \text{ so use } u = a\tan\theta.$$

(b) *Trig substitution.*

$$\sqrt{2}\,x = 3\tan\theta \quad \Rightarrow \quad x = \frac{3}{\sqrt{2}}\tan\theta, \qquad dx = \frac{3}{\sqrt{2}}\sec^2\theta\,d\theta.$$

(c) *Reference triangle (for* $\tan\theta = \dfrac{\sqrt{2}\,x}{3}$*).*



(d) *Rewrite the integral.*

$$x^2 = \frac{9}{2}\tan^2\theta, \qquad (9+2x^2)^{3/2} = (9\sec^2\theta)^{3/2} = 27\sec^3\theta,$$

so

$$\int \frac{x^2}{(9+2x^2)^{3/2}}\,dx = \int \frac{\frac{9}{2}\tan^2\theta}{27\sec^3\theta}\left(\frac{3}{\sqrt{2}}\sec^2\theta\right)d\theta$$
$$= \frac{1}{2\sqrt{2}}\int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \frac{1}{2\sqrt{2}}\int (\sec\theta - \cos\theta)\,d\theta.$$

(e) *Integrate.*

$$\frac{1}{2\sqrt{2}}\int (\sec\theta - \cos\theta)\,d\theta = \frac{1}{2\sqrt{2}}\left(\ln|\sec\theta + \tan\theta| - \sin\theta\right) + C.$$

(f) *Back-substitute.* From the triangle,

$$\tan\theta = \frac{\sqrt{2}\,x}{3}, \qquad \sec\theta = \frac{\sqrt{9+2x^2}}{3}, \qquad \sin\theta = \frac{\sqrt{2}\,x}{\sqrt{9+2x^2}},$$

so

$$\ln|\sec\theta + \tan\theta| = \ln\left|\frac{\sqrt{9+2x^2} + \sqrt{2}\,x}{3}\right|.$$

(g) *Final answer.*

$$\boxed{\int \frac{x^2}{(9+2x^2)^{3/2}}\,dx = \frac{1}{2\sqrt{2}}\ln\left|\sqrt{9+2x^2} + \sqrt{2}\,x\right| - \frac{x}{2\sqrt{9+2x^2}} + C.}$$

15. $\displaystyle\int \frac{1}{(25+3x^2)^{3/2}}\,dx$

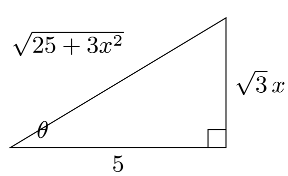
(a) *Identify the form.*

$$(25+3x^2)^{3/2} = \left(5^2 + (\sqrt{3}\,x)^2\right)^{3/2} \quad \Rightarrow \quad \sqrt{a^2+u^2} \text{ with } u = \sqrt{3}\,x,\ a = 5, \text{ so use } u = a\tan\theta.$$

(b) *Trig substitution.*

$$\sqrt{3}\,x = 5\tan\theta \quad \Rightarrow \quad x = \frac{5}{\sqrt{3}}\tan\theta, \qquad dx = \frac{5}{\sqrt{3}}\sec^2\theta\,d\theta.$$

(c) *Reference triangle (for* $\tan\theta = \dfrac{\sqrt{3}\,x}{5}$*).*



(d) *Rewrite the integral.*

$$(25+3x^2)^{3/2} = (25\sec^2\theta)^{3/2} = 125\sec^3\theta,$$

so

$$\int \frac{1}{(25+3x^2)^{3/2}}\,dx = \int \frac{1}{125\sec^3\theta}\left(\frac{5}{\sqrt{3}}\sec^2\theta\right)d\theta = \frac{1}{25\sqrt{3}}\int \cos\theta\,d\theta.$$

(e) *Integrate.*

$$\frac{1}{25\sqrt{3}}\int \cos\theta\,d\theta = \frac{1}{25\sqrt{3}}\sin\theta + C.$$

(f) *Back-substitute.* From the triangle,

$$\sin\theta = \frac{\sqrt{3}\,x}{\sqrt{25+3x^2}}.$$

(g) *Final answer.*

$$\boxed{\int \frac{1}{(25+3x^2)^{3/2}}\,dx = \frac{x}{25\sqrt{25+3x^2}} + C.}$$

7.4 Partial Fractions Solutions

1. $\int \frac{1}{x^2-4}\,dx$

   Factor the denominator:

   $$x^2 - 4 = (x-2)(x+2)$$

   Partial fraction decomposition: Decompose the integrand into partial fractions:

   $$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

   Multiply both sides by $(x-2)(x+2)$ to clear the denominator:

   $$1 = A(x+2) + B(x-2)$$

   Solve for coefficients A and B:

   - Let $x = 2$:
   $$1 = A(2+2) + B(0) \implies 4A = 1 \implies A = \frac{1}{4}$$
   - Let $x = -2$:
   $$1 = A(0) + B(-2-2) \implies -4B = 1 \implies B = -\frac{1}{4}$$

   Integrate: Rewrite the integral using the determined coefficients:

   $$\int \left(\frac{1/4}{x-2} - \frac{1/4}{x+2} \right) dx$$
   $$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

2. $\int \frac{1}{(x-1)(x+2)}\,dx$

   Partial fraction decomposition:

   $$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

   Multiply by $(x-1)(x+2)$:

   $$1 = A(x+2) + B(x-1)$$

   Solve for coefficients:

   - Let $x = 1$:
   $$1 = A(1+2) \implies 3A = 1 \implies A = \frac{1}{3}$$
   - Let $x = -2$:
   $$1 = B(-2-1) \implies -3B = 1 \implies B = -\frac{1}{3}$$

   Integrate:

   $$\int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2} \right) dx$$
   $$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

3. $\int \frac{1}{x^2+3x+2}\,dx$

   Factor the denominator:

   $$x^2 + 3x + 2 = (x+1)(x+2)$$

   Partial fraction decomposition:

   $$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

   Multiply by $(x+1)(x+2)$:

   $$1 = A(x+2) + B(x+1)$$

   Solve for coefficients:

   - Let $x = -1$:
   $$1 = A(-1+2) \implies A = 1$$

- Let $x = -2$:

$$1 = B(-2+1) \implies -B = 1 \implies B = -1$$

Integrate:

$$\int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx$$
$$= \ln|x+1| - \ln|x+2| + C$$

4. $\displaystyle\int \frac{2x+3}{x^2+x-2}\, dx$

Factor the denominator:

$$x^2 + x - 2 = (x-1)(x+2)$$

Partial fraction decomposition:

$$\frac{2x+3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Multiply by $(x-1)(x+2)$:

$$2x + 3 = A(x+2) + B(x-1)$$

Solve for coefficients:

- Let $x = 1$:

$$2(1) + 3 = A(1+2) \implies 5 = 3A \implies A = \frac{5}{3}$$

- Let $x = -2$:

$$2(-2) + 3 = B(-2-1) \implies -1 = -3B \implies B = \frac{1}{3}$$

Integrate:

$$\int \left(\frac{5/3}{x-1} + \frac{1/3}{x+2} \right) dx$$
$$= \frac{5}{3}\ln|x-1| + \frac{1}{3}\ln|x+2| + C$$

5. $\displaystyle\int \frac{1}{x^3 - x}\, dx$

Factor the denominator:

$$x^3 - x = x(x^2-1) = x(x-1)(x+1)$$

Partial fraction decomposition:

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Multiply by $x(x-1)(x+1)$:

$$1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Solve for coefficients:

- Let $x = 0$:

$$1 = A(-1)(1) \implies 1 = -A \implies A = -1$$

- Let $x = 1$:

$$1 = B(1)(2) \implies 1 = 2B \implies B = \frac{1}{2}$$

- Let $x = -1$:

$$1 = C(-1)(-2) \implies 1 = 2C \implies C = \frac{1}{2}$$

Integrate:

$$\int \left(-\frac{1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} \right) dx$$
$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

6. $\displaystyle\int \frac{x^2}{x^3 - 1}\, dx$

Method 1: Substitution

Let $u = x^3 - 1$. Then, $du = 3x^2\, dx$, which implies $x^2\, dx = \frac{1}{3}\, du$. Substituting into the integral:

$$\int \frac{1}{u} \cdot \frac{1}{3}\, du = \frac{1}{3}\ln|u| + C$$

Substituting back $u = x^3 - 1$:

$$= \frac{1}{3} \ln |x^3 - 1| + C$$

Method 2: Partial Fractions
Factor the denominator: $x^3 - 1 = (x-1)(x^2 + x + 1)$.

$$\frac{x^2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Multiply by $(x-1)(x^2+x+1)$:

$$x^2 = A(x^2+x+1) + (Bx+C)(x-1)$$

- Let $x = 1$: $1 = 3A \implies A = \frac{1}{3}$.
- Equate coefficients of x^2: $1 = A + B \implies 1 = \frac{1}{3} + B \implies B = \frac{2}{3}$.
- Equate constant terms: $0 = A - C \implies C = A = \frac{1}{3}$.

The integral becomes:

$$\frac{1}{3}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{2x+1}{x^2+x+1}\,dx$$

Noting that the numerator $(2x+1)$ is the exact derivative of the denominator (x^2+x+1) in the second term:

$$= \frac{1}{3}\ln|x-1| + \frac{1}{3}\ln|x^2+x+1| + C$$

Using log properties $(\ln a + \ln b = \ln ab)$, this confirms the result from Method 1:

$$= \frac{1}{3}\ln|(x-1)(x^2+x+1)| + C = \frac{1}{3}\ln|x^3-1| + C$$

7. $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$

Partial fraction decomposition:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiply by $x(x^2+1)$:

$$1 = A(x^2+1) + x(Bx+C)$$

Expanding terms:

$$1 = (A+B)x^2 + Cx + A$$

Solve for coefficients: Equating coefficients of like powers:

- Constant term: $A = 1$.
- x term: $C = 0$.
- x^2 term: $A + B = 0 \implies 1 + B = 0 \implies B = -1$.

Integrate:

$$\int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$

For the second term, let $u = x^2+1$, so $du = 2x\,dx$, or $\frac{1}{2}du = x\,dx$.

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

(Note: $|x^2+1|$ can be written as (x^2+1) since it is always positive.)

8. $\displaystyle\int \frac{x}{x^3+x^2}\,dx$

Simplify the integrand: Before applying partial fractions, reduce the fraction:

$$\frac{x}{x^3+x^2} = \frac{x}{x^2(x+1)} = \frac{1}{x(x+1)}$$

Partial fraction decomposition:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Multiply by $x(x+1)$:

$$1 = A(x+1) + Bx$$

Solve for coefficients:

- Let $x = 0$: $1 = A(1) \implies A = 1$.
- Let $x = -1$: $1 = B(-1) \implies B = -1$.

Integrate:

$$\int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$
$$= \ln|x| - \ln|x+1| + C$$

9. $\displaystyle\int \frac{1}{(x-1)^2(x+2)}\, dx$

Partial fraction decomposition: Include a term for each power of the repeated factor $(x-1)$:

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Multiply by $(x-1)^2(x+2)$:

$$1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Solve for coefficients:

- Let $x = 1$:
$$1 = B(1+2) \implies 3B = 1 \implies B = \frac{1}{3}$$
- Let $x = -2$:
$$1 = C(-2-1)^2 \implies 1 = C(-3)^2 \implies 9C = 1 \implies C = \frac{1}{9}$$
- Equate coefficients of x^2: Expansion gives $Ax^2 + Cx^2 = (A+C)x^2$. Since there is no x^2 term on the left side:
$$A + C = 0 \implies A = -C \implies A = -\frac{1}{9}$$

Integrate:

$$\int \left(-\frac{1/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{1/9}{x+2}\right) dx$$
$$= -\frac{1}{9}\ln|x-1| + \frac{1}{3}\int (x-1)^{-2}\, dx + \frac{1}{9}\ln|x+2|$$
$$= -\frac{1}{9}\ln|x-1| - \frac{1}{3(x-1)} + \frac{1}{9}\ln|x+2| + C$$

10. $\displaystyle\int \frac{x^4 + 2x^3 + 3x^2 + 4x + 5}{(x-1)(x^2+x+1)}\, dx$

Step 1: Long Division

The denominator expands to $x^3 - 1$. Since the degree of the numerator (4) is greater than the degree of the denominator (3), perform polynomial long division.

Dividing $x^4 + 2x^3 + 3x^2 + 4x + 5$ by $x^3 - 1$ yields:

$$\text{Quotient} = x + 2, \quad \text{Remainder} = 3x^2 + 5x + 7$$

Thus, the integral becomes:

$$\int (x+2)\, dx + \int \frac{3x^2 + 5x + 7}{(x-1)(x^2+x+1)}\, dx$$

The first part integrates to $\frac{1}{2}x^2 + 2x$. We focus now on the rational function.

Step 2: Partial Fraction Decomposition

$$\frac{3x^2 + 5x + 7}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+x+1}$$

Multiply by $(x-1)(x^2+x+1)$:

$$3x^2 + 5x + 7 = A(x^2 + x + 1) + (Bx + C)(x-1)$$

Solve for coefficients:

- Let $x = 1$:
$$3(1) + 5(1) + 7 = A(1+1+1) \implies 15 = 3A \implies A = 5$$
- Equate coefficients of x^2:
$$3 = A + B \implies 3 = 5 + B \implies B = -2$$

- Equate constant terms:

$$7 = A - C \implies 7 = 5 - C \implies C = -2$$

Step 3: Integrate the Rational Part

$$\int \frac{5}{x-1}\,dx + \int \frac{-2x-2}{x^2+x+1}\,dx$$

The first term is $5\ln|x-1|$. For the second term, factor out -1 and split the numerator to match the derivative of the denominator $(2x+1)$:

$$-\int \frac{2x+2}{x^2+x+1}\,dx = -\int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{1}{x^2+x+1}\,dx$$

(a) $\int \frac{2x+1}{x^2+x+1}\,dx = \ln|x^2+x+1|$ (using u-substitution).

(b) $\int \frac{1}{x^2+x+1}\,dx$: Complete the square in the denominator.

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

Using the standard integral $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{u}{a})$, where $a = \frac{\sqrt{3}}{2}$:

$$\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Final Answer: Combining all parts:

$$\frac{1}{2}x^2 + 2x + 5\ln|x-1| - \ln(x^2+x+1) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

7.8 Improper Integrals Solutions

1. $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

   This is an integral over an infinite interval. We evaluate the limit:

$$\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\Big[\arctan x\Big]_0^b$$

$$= \lim_{b\to\infty}(\arctan b - \arctan 0) = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

2. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

   The integrand has an infinite discontinuity at $x = 0$.

$$\lim_{a\to 0^+}\int_a^1 x^{-1/2}\,dx = \lim_{a\to 0^+}\Big[2x^{1/2}\Big]_a^1$$

$$= \lim_{a\to 0^+}(2(1) - 2\sqrt{a}) = 2 - 0 = \boxed{2}$$

3. $\displaystyle\int_0^\infty e^{-x}\,dx$

   Evaluate as a limit to infinity:

$$\lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b$$

$$= \lim_{b\to\infty}(-e^{-b} - (-e^0)) = 0 + 1 = \boxed{1}$$

4. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

$$\lim_{b\to\infty}\int_1^b x^{-2}\,dx = \lim_{b\to\infty}\Big[-\frac{1}{x}\Big]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b} - (-1)\right) = 0 + 1 = \boxed{1}$$

5. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

   The integrand is discontinuous at $x = 1$. Let $u = 1 - x$, so $du = -dx$.

   - Limits: As $x \to 0, u \to 1$. As $x \to 1^-, u \to 0^+$.

$$\int_0^1 (1-x)^{-1/2}\,dx = \int_0^1 u^{-1/2}\,du$$

   The new integral is improper at $u = 0$:

$$\lim_{a\to 0^+}\Big[2u^{1/2}\Big]_a^1 = 2(1) - 0 = \boxed{2}$$

6. $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

   Discontinuous at $x = -1$ and $x = 1$. Technically, we split the integral at 0.

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

   Evaluating the limits at the endpoints:

$$\lim_{x\to 1^-}\arcsin x - \lim_{x\to -1^+}\arcsin x$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

7. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

Rewrite as an improper integral:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx.$$

Use integration by parts with

$$u=\ln x \quad\Rightarrow\quad du=\frac{1}{x}\,dx, \qquad dv=x^{-2}\,dx \quad\Rightarrow\quad v=-\frac{1}{x}.$$

Then

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2}\,dx &= [uv]_1^b - \int_1^b v\,du \\ &= \left[-\frac{\ln x}{x}\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\,dx\right) \\ &= \left[-\frac{\ln x}{x}\right]_1^b + \int_1^b \frac{1}{x^2}\,dx \\ &= \left[-\frac{\ln x}{x}\right]_1^b + \left[-\frac{1}{x}\right]_1^b.\end{aligned}$$

Now take the limit. Since $\dfrac{\ln b}{b}\to 0$ as $b\to\infty$ and $\ln 1 = 0$,

$$\lim_{b\to\infty}\left[-\frac{\ln x}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - 0\right) = 0,$$

and

$$\lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b} - (-1)\right) = 1.$$

Therefore,

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

8. $\displaystyle\int_0^1 \ln(x)\,dx$

Discontinuous at $x=0$. $\int \ln x\,dx = x\ln x - x$.

$$\Big[x\ln x - x\Big]_1 - \lim_{a\to 0^+}(a\ln a - a)$$

We use L'Hôpital's Rule for the limit: $\lim_{a\to 0^+} \frac{\ln a}{1/a} = 0$.

$$= (0-1) - (0-0) = \boxed{-1}$$

9. $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$

Let $u = 1+x^2$, $du = 2x\,dx$.

- Limits: $x=0 \to u=1$, $x\to\infty \to u\to\infty$.

$$\frac{1}{2}\lim_{b\to\infty}\int_1^b u^{-2}\,du = \frac{1}{2}\lim_{b\to\infty}\left[-\frac{1}{u}\right]_1^b$$

$$= \frac{1}{2}(0-(-1)) = \boxed{\frac{1}{2}}$$

10. $\displaystyle\int_0^2 \frac{dx}{(2-x)^{1/3}}$

Discontinuous at $x=2$. Let $u = 2-x$.

- Limits: $x = 0 \to u = 2$, $x \to 2^- \to u \to 0^+$.

$$\int_0^2 u^{-1/3}\,du = \lim_{a\to 0^+}\left[\frac{3}{2}u^{2/3}\right]_a^2$$
$$= \frac{3}{2}(2^{2/3}) - 0 = \boxed{3\cdot 2^{-1/3}}$$

11. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$
Discontinuous at $x = 1$. Trig substitution $x = \sin\theta$ removes the singularity.

- Limits: $x = 0 \to \theta = 0$, $x = 1 \to \theta = \pi/2$.

$$\int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \int_0^{\pi/2} \sin^2\theta\,d\theta$$

The new integral is proper. Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$:

$$\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

12. $\displaystyle\int_0^{\pi/2} \tan x\,dx$
Discontinuous at $x = \pi/2$. Antiderivative is $-\ln|\cos x|$.

$$\lim_{b\to\pi/2^-}\Big[-\ln(\cos x)\Big]_0^b$$

As $b \to \pi/2^-$, $\cos b \to 0^+$, so $\ln(\cos b) \to -\infty$.

$$-(-\infty) - (-\ln 1) = \infty. \quad \boxed{\text{Diverges}}$$

13. $\displaystyle\int_1^\infty \frac{1}{x(\ln x)^2}\,dx$
Let $u = \ln x$, $du = dx/x$.

- Limits: $x = 1 \to u = 0$, $x \to \infty \to u \to \infty$.

This transforms into $\int_0^\infty u^{-2}\,du$. This integral must be split because it is improper at both 0 and ∞.

$$\int_0^1 u^{-2}du + \int_1^\infty u^{-2}du$$

Looking at the first part: $\lim_{a\to 0^+}[-1/u]_a^1 = -1 - (-\infty) = \infty$. Since one part diverges, the whole integral diverges.

$$\boxed{\text{Diverges}}$$

14. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$
Improper at 0 and ∞. Let $x = t^2$, $dx = 2t\,dt$.

- Limits: $x = 0 \to t = 0$, $x \to \infty \to t \to \infty$.

$$\int_0^\infty \frac{2t\,dt}{t(1+t^2)} = 2\int_0^\infty \frac{1}{1+t^2}\,dt$$

The substitution removed the singularity at 0. We only evaluate the limit at ∞:

$$2\lim_{b\to\infty}\Big[\arctan t\Big]_0^b = 2\left(\frac{\pi}{2} - 0\right) = \boxed{\pi}$$

9.1 Differential Equations (Solutions)

Multiple Choice Practice

1. Verify whether $y = e^{2x}$ is a solution to $y' = 2y$.

$$y = e^{2x}, \quad y' = 2e^{2x}$$
$$2y = 2e^{2x} = y'$$

(A) Yes

2. Is $y = x^2 + 1$ a solution to $y' = 2x$?

$$y = x^2 + 1, \quad y' = 2x$$

(A) Yes

3. Which of the following functions satisfy $y' = 3y$? (Select all that apply). **(A) and (B)**

4. Which of the following satisfy $y'' + y = 0$? (Select all that apply). **(A), (B), and (D)**

5. Solve $y' = 6x$.

$$\int dy = \int 6x\,dx$$
$$y = 3x^2 + C$$

(D) $y = 3x^2 + C$

6. Solve $\dfrac{dy}{dx} = \cos(x)$.

$$y = \int \cos(x)\,dx = \sin(x) + C$$

(B) $y = \sin(x) + C$

7. Solve $y' = 2x$, with $y(1) = 6$.

$$\int dy = \int 2x\,dx$$
$$y = x^2 + C$$
$$y(1) = 6 \Rightarrow 1 + C = 6 \Rightarrow C = 5$$
$$\boxed{y = x^2 + 4}$$

(D)

8. Find the particular solution to $\dfrac{dy}{dx} = e^x$, $y(0) = 3$.

$$y = \int e^x\,dx = e^x + C$$
$$y(0) = 3 \Rightarrow 1 + C = 3 \Rightarrow C = 2$$
$$\boxed{y = e^x + 2}$$

(B)

9. True or False: If $y' = 2y$, then any solution graph must be increasing wherever $y > 0$.
True: If $y > 0$, then $y' = 2y > 0$, so the graph increases.

10. True or False: If $y' = -y$, then solution curves are always decreasing when $y > 0$.
True: If $y > 0$, then $y' = -y < 0$, so the graph decreases.

Free Response Practice

1. Verify that $y = e^{3x}$ is a solution to $y' = 3y$.

$$\begin{aligned} y &= e^{3x} \\ y' &= 3e^{3x} = 3y \end{aligned}$$

2. Verify that $y = \sin(2x)$ is a solution to $y'' + 4y = 0$.

$$\begin{aligned} y &= \sin(2x) \\ y' &= 2\cos(2x) \\ y'' &= -4\sin(2x) \\ y'' + 4y &= -4\sin(2x) + 4\sin(2x) = 0 \end{aligned}$$

3. Verify that $y = x^2 + 1$ satisfies $y' = 2x$.

$$\begin{aligned} y &= x^2 + 1 \\ y' &= 2x \end{aligned}$$

4. Solve $y' = 2x$ given $y(1) = 5$.

$$\begin{aligned} \frac{dy}{dx} &= 2x \\ \int dy &= \int 2x\,dx \\ y &= x^2 + C \\ \text{Use } y(1) = 5: \quad 5 &= 1^2 + C \Rightarrow C = 4 \\ &\boxed{y = x^2 + 4} \end{aligned}$$

5. Find the particular solution to $\frac{dy}{dx} = 3e^x$ with $y(0) = 2$.

$$\begin{aligned} \frac{dy}{dx} &= 3e^x \\ \int dy &= \int 3e^x\,dx \\ y &= 3e^x + C \\ \text{Use } y(0) = 2: \quad 2 = 3e^0 + C &= 3 + C \Rightarrow C = -1 \\ &\boxed{y = 3e^x - 1} \end{aligned}$$

6. Solve $\frac{dy}{dx} = \sin(x)$, with $y\left(\frac{\pi}{2}\right) = 0$.

$$\begin{aligned} \frac{dy}{dx} &= \sin(x) \\ \int dy &= \int \sin(x)\,dx \\ y &= -\cos(x) + C \\ \text{Use } y\left(\frac{\pi}{2}\right) = 0: \quad 0 = -\cos\left(\frac{\pi}{2}\right) &+ C \Rightarrow C = 0 \\ &\boxed{y = -\cos(x)} \end{aligned}$$

9.3 Separable Differential Equations (Solutions)

Multiple Choice Practice

1. **(C)** $\dfrac{dy}{dx} = xy$
   Separable: $\frac{1}{y}dy = x\,dx$

2. **(C)** $\dfrac{dy}{dx} = x + y$
   Not separable — cannot write as a product $g(x)h(y)$

3. **(C)** $y = Ce^{x^3/3}$
   From $\frac{dy}{dx} = x^2y \Rightarrow \frac{1}{y}dy = x^2dx \Rightarrow \ln|y| = \frac{1}{3}x^3 + C$

4. **(A)** $y = Cx^3$
   From $\frac{dy}{dx} = \frac{3y}{x} \Rightarrow \frac{1}{y}dy = \frac{3}{x}dx \Rightarrow \ln|y| = 3\ln|x| + C$

5. **(A)** $y = \dfrac{1}{2-x}$
   From $\frac{dy}{dx} = y^2 \Rightarrow -\frac{1}{y} = x + C$, use $y(0) = 2 \Rightarrow C = -\frac{1}{2}$

6. **(B)** $y = 3e^{x - x^3/3}$
   Separable: $\frac{1}{y}dy = (1 - x^2)dx \Rightarrow \ln|y| = x - \frac{1}{3}x^3 + C$, then apply initial condition

7. **(B)** $y = Cx^2$
   From $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{1}{y}dy = \frac{2}{x}dx \Rightarrow \ln|y| = 2\ln|x| + C$

8. **(A)** $y^2 = \dfrac{2x^3}{3} + C$
   From $\frac{dy}{dx} = \frac{x^2}{y} \Rightarrow y\,dy = x^2dx \Rightarrow \frac{1}{2}y^2 = \frac{2}{3}x^3 + C$

Free Response Practice

1. Solve $y' = xy$

Note: $y = 0$ is a contant solution. If $y \neq 0$, then

$$\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{1}{y}\,dy &= x\,dx \\
\int \frac{1}{y}\,dy &= \int x\,dx \\
\ln|y| &= \frac{1}{2}x^2 + C \\
|y| &= e^{\frac{1}{2}x^2 + C} = e^C \cdot e^{x^2/2} \\
y &= \pm e^C \cdot e^{x^2/2} \\
y &= Ae^{x^2/2}
\end{aligned}$$

where A is any constant, including 0.

2. Solve $y' = \frac{2y}{x}$

Note: $y = 0$ is a contant solution. If $y \neq 0$, then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2y}{x} \\
\frac{1}{y}\,dy &= \frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx \\
\ln|y| &= 2\ln|x| + C \\
|y| &= e^C \cdot |x|^2 \\
y &= \pm e^C \cdot x^2 \\
y &= Ax^2
\end{aligned}$$

where A is any constant, including 0.

3. Solve $y' = x(1 + y^2)$

$$\begin{aligned}
\frac{dy}{dx} &= x(1 + y^2) \\
\frac{1}{1+y^2}\,dy &= x\,dx \\
\int \frac{1}{1+y^2}\,dy &= \int x\,dx \\
\arctan y &= \frac{1}{2}x^2 + C \\
y &= \tan\left(\frac{1}{2}x^2 + C\right)
\end{aligned}$$

4. Solve $y' = y^2$, with $y(0) = 1$

$$\begin{aligned}
\frac{dy}{dx} &= y^2 \\
\frac{1}{y^2}\,dy &= dx \\
\int y^{-2}\,dy &= \int dx \\
-\frac{1}{y} &= x + C \Rightarrow y = \frac{-1}{x+C} \\
y(0) = 1 \Rightarrow \frac{-1}{C} &= 1 \Rightarrow C = -1 \\
&\boxed{y = \frac{-1}{x-1}}
\end{aligned}$$

5. Solve $y' = (1-x^2)y$, with $y(0) = 2$

$$\begin{aligned}\frac{dy}{dx} &= (1-x^2)y \\ \frac{1}{y}\,dy &= (1-x^2)\,dx \\ \int \frac{1}{y}\,dy &= \int (1-x^2)\,dx \\ \ln|y| &= x - \frac{1}{3}x^3 + C \\ |y| &= e^C \cdot e^{x-x^3/3} \\ y &= \pm e^C \cdot e^{x-x^3/3} = Ae^{x-x^3/3}\end{aligned}$$

$$y(0) = 2 \Rightarrow A = 2$$

$$\boxed{y = 2e^{x-x^3/3}}$$

6. Solve $y' = x^2(1+y)$, with $y(1) = 0$

$$\begin{aligned}\frac{dy}{dx} &= x^2(1+y) \\ \frac{1}{1+y}\,dy &= x^2\,dx \\ \int \frac{1}{1+y}\,dy &= \int x^2\,dx \\ \ln|1+y| &= \frac{1}{3}x^3 + C \\ |1+y| &= e^C \cdot e^{x^3/3} \\ 1+y &= \pm e^C \cdot e^{x^3/3} = Ae^{x^3/3} \\ y &= Ae^{x^3/3} - 1 \\ y(1) &= 0 \Rightarrow Ae^{1/3} = 1 \Rightarrow A = e^{-1/3}\end{aligned}$$

$$\boxed{y = e^{(x^3-1)/3} - 1}$$

7. Solve $y' = \frac{y}{x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} \\ \frac{1}{y}\,dy &= \frac{1}{x}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\ \ln|y| &= \ln|x| + C \\ |y| &= e^C|x| \\ y &= \pm e^C x = Ax\end{aligned}$$

$$\boxed{y = Ax}$$

8. Solve $y' = \frac{x}{y}$

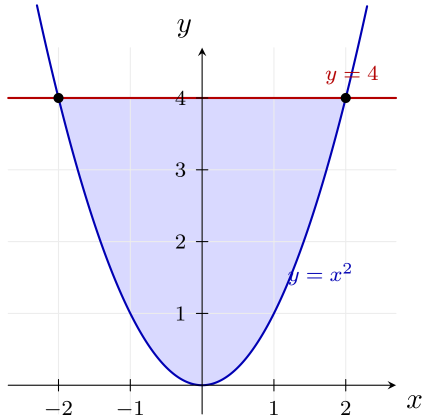
$$\frac{dy}{dx} = \frac{x}{y}$$
$$y\,dy = x\,dx$$
$$\int y\,dy = \int x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
$$\boxed{y^2 = x^2 + C}$$

9. Solve $y' = \frac{2y}{x+1}$

$$\frac{dy}{dx} = \frac{2y}{x+1}$$
$$\frac{1}{y}\,dy = \frac{2}{x+1}\,dx$$
$$\int \frac{1}{y}\,dy = \int \frac{2}{x+1}\,dx$$
$$\ln|y| = 2\ln|x+1| + C$$
$$|y| = e^C|x+1|^2$$
$$y = \pm e^C(x+1)^2 = A(x+1)^2$$
$$\boxed{y = A(x+1)^2}$$

6.1 Areas Between Curves (Solutions)

1. **Find the area enclosed by the curves $y = x^2$ and $y = 4$.**

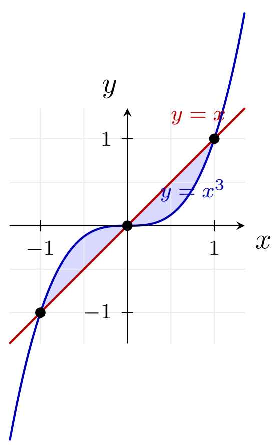


Solve $x^2 = 4$ to get $x = \pm 2$. On $[-2, 2]$, the top curve is $y = 4$ and the bottom curve is $y = x^2$, so

$$A = \int_{-2}^{2} (4 - x^2)\, dx = 2 \int_0^2 (4 - x^2)\, dx = 2 \left[4x - \frac{x^3}{3} \right]_0^2 = \frac{32}{3}.$$

$$\boxed{A = \frac{32}{3}}$$

2. **Find the area between the curves $y = x^3$ and $y = x$.**

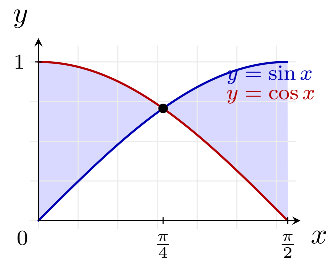


Solve $x^3 = x$: $x(x^2 - 1) = 0$, so $x = -1, 0, 1$. On $[0, 1]$, $x \geq x^3$, and symmetry gives

$$A = 2 \int_0^1 (x - x^3)\, dx = 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2}.$$

$$\boxed{A = \frac{1}{2}}$$

3. **Compute the area enclosed by $y = \sin x$ and $y = \cos x$ on $0 \le x \le \frac{\pi}{2}$.**

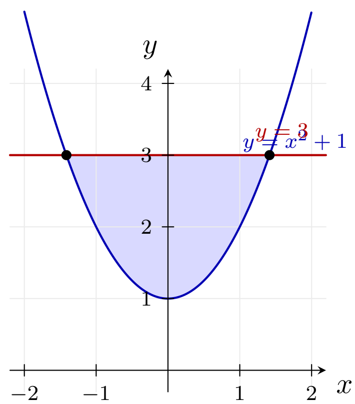


The curves meet when $\sin x = \cos x$, which occurs at $x = \frac{\pi}{4}$. On $\left[0, \frac{\pi}{4}\right]$, $\cos x \ge \sin x$, and the two pieces have equal area, so

$$A = 2\int_0^{\pi/4} \big(\cos x - \sin x\big)\, dx = 2\left[\sin x + \cos x\right]_0^{\pi/4} = 2(\sqrt{2} - 1).$$

$$\boxed{A = 2(\sqrt{2} - 1)}$$

4. **Determine the area of the region bounded by $y = x^2 + 1$ and $y = 3$.**



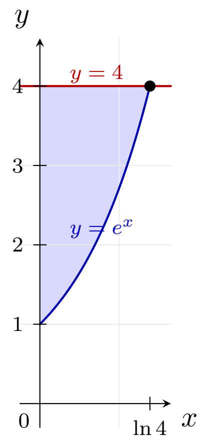
Intersect where $x^2 + 1 = 3$, so $x = \pm\sqrt{2}$. The top curve is $y = 3$ and the bottom curve is $y = x^2 + 1$, hence

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \Big(3 - (x^2 + 1)\Big)\, dx = \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)\, dx = 2\int_0^{\sqrt{2}} (2 - x^2)\, dx.$$

$$A = 2\left[2x - \frac{x^3}{3}\right]_0^{\sqrt{2}} = \frac{8\sqrt{2}}{3}.$$

$$\boxed{A = \frac{8\sqrt{2}}{3}}$$

5. **Find the area enclosed by $y = e^x$ and $y = 4$ for $0 \le x \le \ln 4$.**



On $[0, \ln 4]$, the top curve is $y = 4$ and the bottom curve is $y = e^x$, so

$$A = \int_0^{\ln 4} \left(4 - e^x\right) dx = [4x - e^x]_0^{\ln 4} = 4 \ln 4 - 3.$$

$$\boxed{A = 4 \ln 4 - 3}$$

6.2 Volume by Known Cross-Sections (Solutions)

1. **Circular Base, Square Cross-Sections.**

- **Base:** The base is the circle
$$x^2 + y^2 \leq 4.$$

- **Slicing:** A slice perpendicular to the x-axis at a given x meets the circle at
$$y = \pm\sqrt{4 - x^2},$$
so the width is
$$2\sqrt{4 - x^2}.$$

- **Cross-Section:** Each slice is a square with side length
$$s(x) = 2\sqrt{4 - x^2},$$
hence its area is
$$A(x) = \left(2\sqrt{4 - x^2}\right)^2 = 16 - 4x^2.$$

- **Volume:** The solid extends from $x = -2$ to $x = 2$, so
$$V = \int_{-2}^{2} (16 - 4x^2)\, dx.$$

- **Computation:**
$$\begin{aligned} V &= \int_{-2}^{2} 16\, dx - 4\int_{-2}^{2} x^2\, dx \\ &= 16\, [x]_{-2}^{2} - 4\left[\frac{x^3}{3}\right]_{-2}^{2} \\ &= 16(2 - (-2)) - 4\left(\frac{8}{3} - \left(-\frac{8}{3}\right)\right) \\ &= 16 \cdot 4 - 4\left(\frac{16}{3}\right) \\ &= 64 - \frac{64}{3} \\ &= \frac{192 - 64}{3} = \frac{128}{3}. \end{aligned}$$

- **Answer:** $V = \frac{128}{3}$ cubic units.

2. **Triangular Base, Semicircular Cross-Sections.**

- **Base:** The triangular base is bounded by

$$x = 0, \quad y = 0, \quad \text{and} \quad y = 4 - x.$$

- **Slicing:** For a fixed x (with $0 \leq x \leq 4$), the vertical slice extends from $y = 0$ to $y = 4 - x$. Thus, the diameter of the semicircular cross-section is

$$D(x) = 4 - x.$$

- **Cross-Section:**
  - The radius is

$$r(x) = \frac{4 - x}{2}.$$

  - The area of a semicircle is half that of a full circle:

$$A(x) = \frac{1}{2}\pi\,[r(x)]^2 = \frac{\pi(4 - x)^2}{8}.$$

- **Volume:** The volume is

$$V = \int_0^4 A(x)\,dx = \frac{\pi}{8}\int_0^4 (4 - x)^2\,dx.$$

- **Substitution:** Let $u = 4 - x$ so that $du = -dx$. When $x = 0$, $u = 4$; when $x = 4$, $u = 0$. Then,

$$V = \frac{\pi}{8}\int_4^0 u^2\,(-du) = \frac{\pi}{8}\int_0^4 u^2\,du.$$

- **Compute the Integral:**

$$\int_0^4 u^2\,du = \left.\frac{u^3}{3}\right|_0^4 = \frac{64}{3}.$$

- **Result:**

$$V = \frac{\pi}{8} \cdot \frac{64}{3} = \frac{8\pi}{3}.$$

- **Answer:** $V = \frac{8\pi}{3}$ cubic units.

3. **Square Base, Equilateral Triangular Cross-Sections.**

- **Slicing:** Slicing perpendicular to the x-axis, each slice has a base (in the y-direction) from $y = 0$ to $y = 3$ (length 3).
- **Cross-Section:** For an equilateral triangle of side s, the area is

$$A = \frac{\sqrt{3}}{4} s^2.$$

  With $s = 3$, the area becomes

$$A(x) = \frac{\sqrt{3}}{4}(3^2) = \frac{9\sqrt{3}}{4}.$$

- **Volume:**

$$V = \int_0^3 A(x)\,dx = \frac{9\sqrt{3}}{4} \cdot 3 = \frac{27\sqrt{3}}{4}.$$

- **Answer:** $V = \frac{27\sqrt{3}}{4}$ cubic units.

4. **Rectangular Base, Right Triangular Cross-Sections.**

- **Slicing:** We slice perpendicular to the y-axis. At a fixed y, the slice has a length of 5 in the x-direction.
- **Cross-Section:** Each slice is a right triangle with base 5 and height y. Thus, the area is

$$A(y) = \frac{1}{2} \cdot 5 \cdot y = \frac{5y}{2}.$$

- **Volume:**

$$V = \int_0^2 \frac{5y}{2}\, dy.$$

- **Compute:**

$$\begin{aligned} V &= \frac{5}{2} \int_0^2 y\, dy \\ &= \frac{5}{2} \left[\frac{y^2}{2} \right]_0^2 \\ &= \frac{5}{2} \cdot \frac{4}{2} = 5. \end{aligned}$$

- **Answer:** $V = 5$ cubic units.

5. **Circular Base, Equilateral Triangles.**

- **Base:** The base is the circle
$$x^2 + y^2 \le 1.$$

- **Slicing:** At a fixed y, the chord in the x-direction runs from
$$x = -\sqrt{1-y^2} \quad \text{to} \quad x = \sqrt{1-y^2},$$
with length
$$2\sqrt{1-y^2}.$$

- **Cross-Section:** This chord forms the base of an equilateral triangle, so its area is
$$A(y) = \frac{\sqrt{3}}{4}\left(2\sqrt{1-y^2}\right)^2 = \sqrt{3}(1-y^2).$$

- **Volume:**
$$V = \int_{-1}^{1} \sqrt{3}(1-y^2)\,dy.$$

- **Compute the inner integral:**
$$\begin{aligned}\int_{-1}^{1}(1-y^2)\,dy &= \left[y - \frac{y^3}{3}\right]_{-1}^{1} \\ &= \left[(1-\tfrac{1}{3}) - (-1+\tfrac{1}{3})\right] \\ &= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.\end{aligned}$$

- **Result:**
$$V = \sqrt{3}\cdot\frac{4}{3} = \frac{4\sqrt{3}}{3}.$$

- **Answer:** $V = \frac{4\sqrt{3}}{3}$ cubic units.

6. **Triangular Base, Rectangular Cross-Sections.**

- **Base:** The triangular base is bounded by

$$x = 0, \quad y = 0, \quad x + y = 6.$$

For a fixed $x \in [0, 6]$, y runs from 0 to $6 - x$. Define

$$L(x) = 6 - x.$$

- **Cross-Section:** The rectangle at x has:
  - **Base (in the xy-plane):** $L(x) = 6 - x$,
  - **Height (in the z-direction):** $2(6 - x)$.

  Thus, its area is

$$A(x) = (6 - x) \cdot [2(6 - x)] = 2(6 - x)^2.$$

- **Volume:**

$$V = \int_0^6 2(6 - x)^2 \, dx.$$

- **Substitution:** Let $u = 6 - x$ so that $du = -dx$. Then,

$$\begin{aligned} V &= \int_{u=6}^{0} 2u^2 \, (-du) \\ &= \int_0^6 2u^2 \, du. \end{aligned}$$

- **Compute:**

$$\begin{aligned} \int_0^6 2u^2 \, du &= 2 \left[\frac{u^3}{3} \right]_0^6 \\ &= 2 \cdot \frac{216}{3} = 144. \end{aligned}$$

- **Answer:** $V = 144$ cubic units.

7. **Elliptical Base, Square Cross-Sections.**

- **Base:** The ellipse is given by
$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

- **Slicing:** For a fixed y, solve for x^2:
$$x^2 = 9\left(1 - \frac{y^2}{4}\right) = 9 - \frac{9y^2}{4}.$$
Hence,
$$x = \pm\sqrt{9 - \frac{9y^2}{4}} = \pm\frac{3}{2}\sqrt{4 - y^2}.$$
The total width is
$$2 \cdot \frac{3}{2}\sqrt{4 - y^2} = 3\sqrt{4 - y^2}.$$

- **Cross-Section:** Since the cross-sections are squares,
$$A(y) = \left[3\sqrt{4 - y^2}\right]^2 = 9(4 - y^2) = 36 - 9y^2.$$

- **Volume:** Integrate with respect to y (from -2 to 2):
$$V = \int_{-2}^{2} (36 - 9y^2)\,dy.$$

- **Computation:**
$$\int_{-2}^{2} 36\,dy = 36\,[y]_{-2}^{2} = 36(4) = 144,$$
$$\int_{-2}^{2} y^2\,dy = \left[\frac{y^3}{3}\right]_{-2}^{2} = \frac{8}{3} - \left(-\frac{8}{3}\right) = \frac{16}{3}.$$
Therefore,
$$V = 144 - 9\left(\frac{16}{3}\right) = 144 - 48 = 96.$$

- **Answer:** $V = 96$ cubic units.

8. **Isosceles Trapezoidal Cross-Sections over a Square Base.**

- **Slicing:** For a fixed x, the cross-section extends in y from 0 to 4. Thus, the longer base is 4 and the shorter base is 2; the trapezoid's height (perpendicular to the xy-plane) is 1.
- **Cross-Section:** Its area is
$$A = \frac{4+2}{2} \times 1 = 3.$$
- **Volume:** Since the area is constant,
$$V = \int_0^4 3\,dx = 12.$$
- **Answer:** $V = 12$ cubic units.

9. **Parabolic Region, Rectangular Cross-Sections (Height Proportional to y).**

- **Base:** For a given x, the region in y goes from 0 to $4 - x^2$; thus the width is
$$4 - x^2.$$
- **Cross-Section:** The rectangle has width $4 - x^2$ and (constant for the slice) height
$$2(4 - x^2).$$
Its area is
$$A(x) = (4 - x^2) \cdot 2(4 - x^2) = 2(4 - x^2)^2.$$
- **Volume:** The region in x is from 0 to 2, so
$$V = \int_0^2 2(4 - x^2)^2 \, dx.$$
- **Expand and Compute:** Note that
$$(4 - x^2)^2 = 16 - 8x^2 + x^4.$$
Then,
$$\begin{aligned} V &= 2 \int_0^2 \left(16 - 8x^2 + x^4\right) dx \\ &= 2 \left[\int_0^2 16 \, dx - 8 \int_0^2 x^2 \, dx + \int_0^2 x^4 \, dx \right]. \end{aligned}$$
- **Individual Integrals:**
$$\int_0^2 16 \, dx = 32, \quad \int_0^2 x^2 \, dx = \frac{8}{3}, \quad \int_0^2 x^4 \, dx = \frac{32}{5}.$$
- **Thus,**
$$\begin{aligned} V &= 2 \left(32 - 8 \cdot \frac{8}{3} + \frac{32}{5} \right) \\ &= 2 \left(32 - \frac{64}{3} + \frac{32}{5} \right) \\ &= 2 \left[\frac{32 \cdot 15 - 64 \cdot 5 + 32 \cdot 3}{15} \right] \\ &= 2 \left[\frac{480 - 320 + 96}{15} \right] \\ &= 2 \left[\frac{256}{15} \right] = \frac{512}{15}. \end{aligned}$$
- **Answer:** $V = \frac{512}{15}$ cubic units.