Midterm 1 Study Guide

MATH2300 - Calculus II

Spring 2025

Contents

Overview of Topics

§5.5 u-Substitution

- **When to use:** For integrals of composite functions where one part is the derivative of another.
- **Steps:**
  1. Choose $u = g(x)$ where $g(x)$ is inside another function.
  2. Compute $du = g'(x)\,dx$.
  3. Rewrite the integral in terms of u and du.
  4. For definite integrals, update the limits: if $x = a$, then $u = g(a)$; if $x = b$, then $u = g(b)$.
  5. Integrate with respect to u and substitute back (if needed).

§7.1 Integration by Parts and Boomerang Integrals

- **Formula:**
$$\int u\,dv = uv - \int v\,du.$$
- **Choosing u and dv:** Use the LIATE rule (Logarithmic, Inverse trig, Algebraic, Trigonometric, Exponential).
- **Boomerang Integrals:** In some cases (e.g., $\int e^{ax}\cos(bx)\,dx$ or $\int e^{ax}\sin(bx)\,dx$), applying integration by parts twice brings you back to a multiple of the original integral. Solve for the original integral algebraically.

§7.2 Trigonometric Integrals Using Identities and u-Substitution

- **Key Identities:**
  - Pythagorean: $\sin^2 x + \cos^2 x = 1$.
  - Tangent: $\tan^2 x + 1 = \sec^2 x$
  - Power-Reducing: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.
  - Product-to-Sum: $\sin x \cos x = \frac{1}{2}\sin 2x$.
- **When to use u-substitution:** If the integral involves sine and cosine, reserve one factor for du if one power of sine or cosine is odd. If the integral involves secant and tangent, reserve either $\sec x \tan x$ or $\sec^2 x$ for du.
- **Strategy:** Simplify the integrand with identities, then apply u-substitution as appropriate.

§7.3 Trigonometric Substitution

- **When to use:** For integrals containing radicals such as $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$.
- **Substitutions:**
  - For $\sqrt{a^2 - x^2}$: let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, and $\sqrt{a^2 - x^2} = a\cos\theta$.
  - For $\sqrt{a^2 + x^2}$: let $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$, and $\sqrt{a^2 + x^2} = a\sec\theta$.
  - For $\sqrt{x^2 - a^2}$: let $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$, and $\sqrt{x^2 - a^2} = a\tan\theta$.
- **After Substitution:** Use trigonometric identities to simplify the integrand before integrating.

§7.4 Partial Fractions

- **When to use:** For rational functions $\frac{P(x)}{Q(x)}$ where the degree of $P(x)$ is less than the degree of $Q(x)$.
- **Steps:**
  1. Factor the denominator $Q(x)$ into linear and/or irreducible quadratic factors.
  2. Express $\frac{P(x)}{Q(x)}$ as a sum of simpler fractions.
  3. Solve for the unknown coefficients.
  4. Integrate each term separately.

§7.7 Approximating Definite Integrals

Midpoint Rule

- **Method:** Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$.
- **Midpoints:** $m_i = a + \left(i - \frac{1}{2}\right) \Delta x$ for $i = 1, 2, \ldots, n$.
- **Approximation:**

$$M_n = \Delta x \sum_{i=1}^{n} f(m_i).$$

Trapezoidal Rule

- **Method:** Divide $[a, b]$ into n subintervals, with $\Delta x = \frac{b-a}{n}$.
- **Approximation:**

$$T_n = \frac{\Delta x}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(a + i\Delta x) + f(b) \right].$$

§7.8 Improper Integrals and the Comparison Theorem

- **Improper Integrals:**
  - **Infinite Limits:** Replace ∞ with a limit:

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

  - **Discontinuous Integrands:** Break the integral at the point of discontinuity and take limits.
- **Comparison Theorem:**
  - If $0 \leq f(x) \leq g(x)$ for $x \geq a$ and $\int_a^{\infty} g(x)\,dx$ converges, then $\int_a^{\infty} f(x)\,dx$ also converges.
  - Conversely, if $\int_a^{\infty} f(x)\,dx$ diverges, so does $\int_a^{\infty} g(x)\,dx$.

Formulas to Know

Integrals

Integral	Antiderivative
$\int x^n\,dx$	$\dfrac{x^{n+1}}{n+1} + C,\ (n \neq -1)$
$\int \dfrac{1}{x}\,dx$	$\ln\lvert x\rvert + C$
$\int e^x\,dx$	$e^x + C$
$\int \sin x\,dx$	$-\cos x + C$
$\int \cos x\,dx$	$\sin x + C$
$\int \tan x\,dx$	$-\ln\lvert\cos x\rvert + C$
$\int \cot x\,dx$	$\ln\lvert\sin x\rvert + C$

Integral	Antiderivative
$\int \sec x\,dx$	$\ln\lvert\sec x + \tan x\rvert + C$
$\int \csc x\,dx$	$-\ln\lvert\csc x + \cot x\rvert + C$
$\int \sec^2 x\,dx$	$\tan x + C$
$\int \csc^2 x\,dx$	$-\cot x + C$
$\int \sec x\,\tan x\,dx$	$\sec x + C$
$\int \csc x\,\cot x\,dx$	$-\csc x + C$
$\int \dfrac{1}{x^2+a^2}\,dx$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
$\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx$	$\sin^{-1}\left(\dfrac{x}{a}\right) + C$

Trigonometric Identities

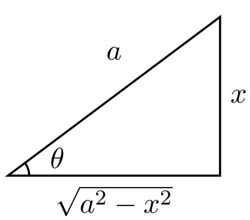
Identity	Formula
Pythagorean	$\sin^2 x + \cos^2 x = 1$
Tangent Identity	$\tan^2 x + 1 = \sec^2 x$
Power-Reducing Formulas	$\sin^2 x = \dfrac{1}{2}(1-\cos 2x), \quad \cos^2 x = \dfrac{1}{2}(1+\cos 2x)$
Product-to-Sum	$\sin x \cos x = \frac{1}{2}\sin 2x$
Double-Angle Formulas	$\sin 2x = 2\sin x\cos x, \quad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Trigonometric Substitutions

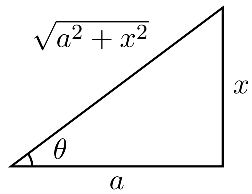
The table below summarizes the three standard trigonometric substitutions:

Expression	Substitution	Interval	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

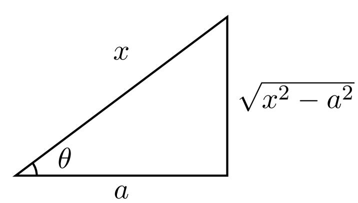
Each substitution corresponds to the sides of a right triangle:



Substitution: $x = a\sin\theta$



Substitution: $x = a\tan\theta$

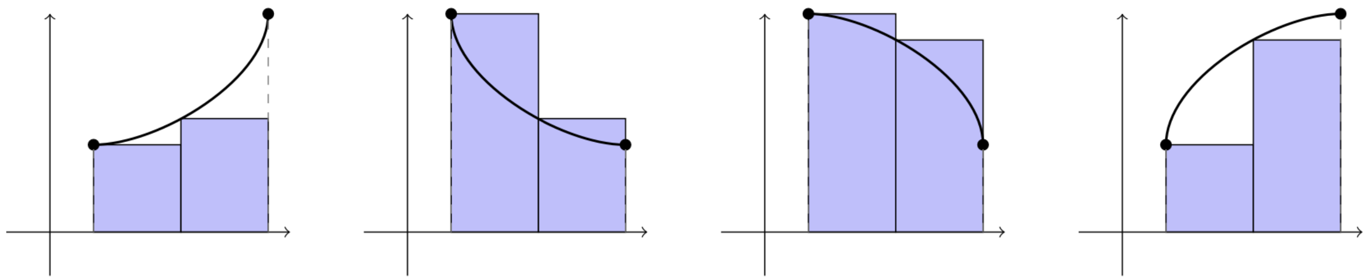


Substitution: $x = a\sec\theta$

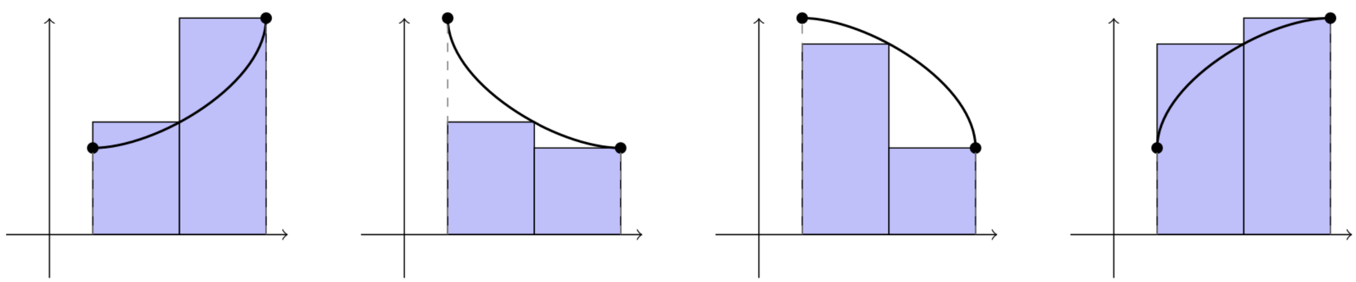
Approximate Integration

We approximate the integral $I = \int_a^b f(x)\,dx$ using four common methods:

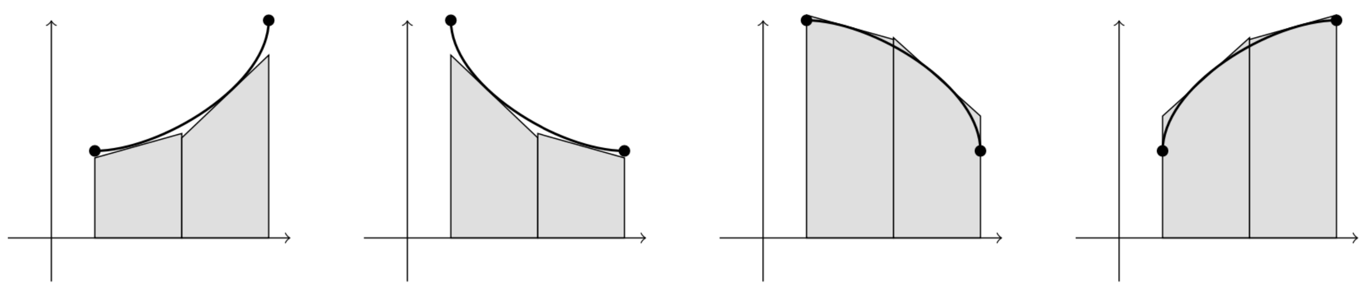
- Left-hand sum (L_n)



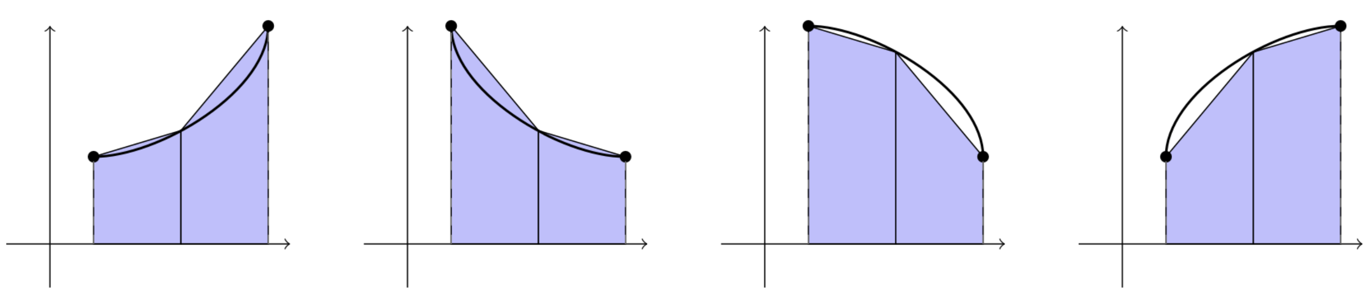
- Right-hand sum (R_n)



- Midpoint rule (M_n), (here, the tops of the rectangles have been "rotated")



- Trapezoidal rule (T_n)



Relation of Each Method to the Integral

The behavior of each method depends on two key features of f:

1. **Increasing/Decreasing:** For an *increasing* function, the left-hand sum underestimates and the right-hand sum overestimates I. For a decreasing function, the roles are reversed.

2. **Concavity:**

   - If f is *concave up*, then the graph of f lies *below* the chord joining any two points on it. Consequently, the **trapezoidal rule** *overestimates* the true area for an increasing function. In contrast, the **midpoint rule** uses the value at the center of the interval; for a concave up function this value is lower than the average of the endpoints, so the midpoint rule *underestimates* the area.
   - If f is *concave down*, the graph lies *above* the chord. Thus, for an increasing function the trapezoidal rule *underestimates* I while the midpoint rule *overestimates* it.

Function Type	Concave Up	Concave Down
Increasing	L_n : Underestimate R_n : Overestimate M_n : Underestimate T_n : Overestimate	L_n : Underestimate R_n : Overestimate M_n : Overestimate T_n : Underestimate
Decreasing	L_n : Overestimate R_n : Underestimate M_n : Underestimate T_n : Overestimate	L_n : Overestimate R_n : Underestimate M_n : Overestimate T_n : Underestimate

Comparison of Integral Approximations

1. Increasing, Concave Up: Since the function is increasing, the left Riemann sum L_n underestimates the integral, while the right Riemann sum R_n overestimates it. Therefore:

$$L_n < I < R_n$$

Now, imagine constructing a rectangle whose height is determined using midpoints. The function is increasing, so the function value at the left endpoint is lower than at the midpoint. Since the function is also concave up, the secant line between the endpoints of the subinterval lies above the function's graph. This means that if we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would need to raise its top. In other words, the midpoint rectangle determines a trapezoid that lies below the curve, and the secant line determines a trapezoid that lies above the curve. Therefore:

$$M_n < I < T_n$$

Comparing L_n and M_n, the left endpoints are lower than the midpoints, so:

$$L_n < M_n.$$

Each trapezoid lies within the rectangle obtained from using right endpoints, so:

$$T_n < R_n.$$

Thus, the overall ordering is:

$$L_n < M_n < \int_a^b f(x)dx < T_n < R_n.$$

2. Increasing, Concave Down: Since the function is increasing, L_n underestimates the integral, while R_n overestimates it. Therefore:

$$L_n < I < R_n$$

Now, consider the midpoint rule. Since the function is concave down, the secant line between the endpoints lies below the function's graph. This means that if we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would then need to lower its top. In other words, the midpoint rectangle determines a trapezoid that lies above the curve, while the secant line determines a trapezoid that lies below the curve. Therefore:

$$T_n < I < M_n$$

Comparing L_n and T_n, each rectangle determined by the left endpoint lies within a trapezoid determined by the secant line, so:

$$L_n < T_n.$$

Since the midpoints are lower than the right endpoints:

$$M_n < R_n.$$

Thus, the overall ordering is:

$$L_n < T_n < \int_a^b f(x)dx < M_n < R_n.$$

3. Decreasing, Concave Up: Since the function is decreasing, L_n overestimates the integral, while R_n underestimates it. Therefore:

$$R_n < I < L_n$$

For the midpoint rule, since the function is concave up, the secant line between the endpoints lies above the function's graph. If we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would then need to raise its top. This means the midpoint rectangle determines a trapezoid that lies below the curve, while the secant line determines a trapezoid that lies above the curve. Therefore:

$$M_n < I < T_n$$

Comparing R_n and M_n, the right endpoints are lower than the midpoints, so:

$$R_n < M_n.$$

Each trapezoid lies within the rectangle obtained from using left endpoints, so:

$$T_n < L_n.$$

Thus, the overall ordering is:

$$R_n < M_n < \int_a^b f(x)dx < T_n < L_n.$$

4. Decreasing, Concave Down: Since the function is decreasing, L_n overestimates the integral, while R_n underestimates it. Therefore:

$$R_n < I < L_n$$

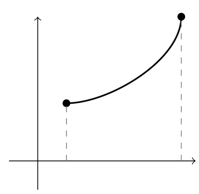
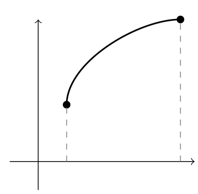
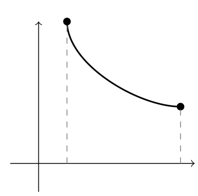
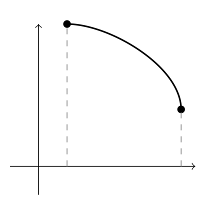
For the midpoint rule, since the function is concave down, the secant line between the endpoints lies below the function's graph. If we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would then need to lower its top. This means the midpoint rectangle determines a trapezoid that lies above the curve, while the secant line determines a trapezoid that lies below the curve. Therefore:

$$T_n < I < M_n$$

Comparing R_n and T_n, each rectangle determined by the right endpoint lies within a trapezoid determined by the secant line, so $R_n < T_n$. Since the midpoints are lower than the left endpoints, we have $M_n < L_n$. Thus, the overall ordering is:

$$R_n < T_n < \int_a^b f(x)dx < M_n < L_n.$$

Summary of Orderings:

Case	**Graph**	**Order**
Increasing, Concave Up		$L_n < M_n < \int f(x)dx < T_n < R_n$
Increasing, Concave Down		$L_n < T_n < \int f(x)dx < M_n < R_n$
Decreasing, Concave Up		$R_n < M_n < \int f(x)dx < T_n < L_n$
Decreasing, Concave Down		$R_n < T_n < \int f(x)dx < M_n < L_n$

Integral Practice

5.5 u-Substitution

Notes

1. $\int x^3 \cos(x^4+2)\,dx$
2. $\int \sqrt{2x+1}\,dx$
3. $\int \frac{x}{\sqrt{1-4x^2}}\,dx$
4. $\int e^{5x}\,dx$
5. $\int \tan(x)\,dx$
6. $\int_0^4 \sqrt{2x+1}\,dx$
7. $\int_1^2 \frac{dx}{(3-5x)^2}$
8. $\int_1^e \frac{\ln x}{x}\,dx$

WebAssign

1. $\int \frac{x^3}{x^4-4}\,dx$
2. $\int \frac{\cos(\sqrt{t})}{\sqrt{t}}\,dt$
3. $\int (3-8x)^{10}\,dx$
4. $\int \frac{(\ln(x))^{30}}{x}\,dx$
5. $\int y^2\left(5-y^3\right)^{2/3}\,dy$
6. $\int_0^1 \sqrt[3]{1+7x}\,dx$
7. $\int_1^6 \frac{e^{1/x}}{x^2}\,dx$
8. $\int_0^{\pi/6} \frac{\sin(t)}{\cos^2(t)}\,dt$

Practice

1. $\int xe^{x^2}\,dx$
2. $\int \cos(3x)\,dx$
3. $\int x\sqrt{x^2+1}\,dx$
4. $\int \sin(2x)\cos(2x)\,dx$
5. $\int \frac{\ln(x)}{x}\,dx$
6. $\int \frac{x}{x^2+1}\,dx$
7. $\int x^3 e^{x^4}\,dx$
8. $\int x\sqrt{x^2-1}\,dx$
9. $\int \frac{x}{\sqrt{x^2+1}}\,dx$
10. $\int \sin^2(x)\cos(x)\,dx$
11. $\int \sin^2(3x)\cos(3x)\,dx$
12. $\int \tan(2x)\sec^2(2x)\,dx$
13. $\int \sqrt{x+1}\,dx$
14. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$
15. $\int \frac{1}{x\sqrt{x+1}}\,dx$
16. $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$
17. $\int 2x\sqrt{1-x^2}\,dx$

7.1 Integration by Parts

Notes

1. $\int x \sin x \, dx$.
2. $\int \ln x \, dx$.
3. $\int t^2 e^t \, dt$.
4. $\int e^x \sin x \, dx$.
5. $\int_0^1 \tan^{-1}(x) \, dx$.

WebAssign

1. $\int x e^{7x} \, dx$.
2. $\int \arctan(5t) \, dt$.
3. $\int 6 \arcsin x \, dx$.
4. $\int (x^2 + 4x) \cos(x) \, dx$.
5. $\int e^{2\theta} \sin(3\theta) \, d\theta$.

Practice

1. $\int x e^x \, dx$
2. $\int x \ln(x) \, dx$
3. $\int x^2 \sin(x) \, dx$
4. $\int x \cos(x) \, dx$
5. $\int x^3 \ln(x) \, dx$
6. $\int x^2 \ln(x^2) \, dx$
7. $\int x^2 \cos(2x) \, dx$
8. $\int \ln(x^2 + 1) \, dx$

7.1 Boomerang Integrals

Notes

1. $\int e^x \sin(x)\, dx.$
2. $\int \sec^3(x)\, dx.$

Practice

1. $\int e^{3x} \cos(3x)\, dx$
2. $\int e^{-x} \sin(x)\, dx$

WebAssign

1. $\int e^{2\theta} \sin(3\theta)\, d\theta$

7.2 Trigonometric Integrals

Notes

1. $\int \sin^5(x)\cos^2(x)\,dx$
2. $\int_0^{\pi} \sin^2(x)\,dx$
3. $\int \sin^4(x)\,dx$
4. $\int \tan^6(x)\sec^4(x)\,dx$
5. $\int \tan^5(\theta)\sec^7(\theta)\,d\theta$
6. $\int \tan^3(x)\,dx$
7. $\int \sec^3(x)\,dx$

WebAssign

1. $\int 2\sin^2(x)\cos^3(x)\,dx$
2. $\int_0^{\pi/4} \sin^5(x)\,dx$
3. $\int_0^{\pi/2} 3\cos^2(\theta)\,d\theta$
4. $\int_0^{\pi/2} 5\sin^2(x)\cos^2(x)\,dx$
5. $\int 4\tan^3(x)\sec(x)\,dx$
6. $\int 13\tan^4(x)\sec^6(x)\,dx$

Practice

1. $\int \sin^3(x)\,dx$
2. $\int \sin^2(x)\cos(x)\,dx$
3. $\int \sin^4(x)\cos(x)\,dx$
4. $\int \cos^4(x)\sin(x)\,dx$
5. $\int \sec^2(x)\,dx$
6. $\int \sec^4(x)\,dx$
7. $\int \sec^2(x)\tan^2(x)\,dx$
8. $\int \tan^3(x)\sec^2(x)\,dx$
9. $\int \sec^2(x)\tan(x)\,dx$
10. $\int \tan^2(x)\,dx$
11. $\int \tan^4(x)\sec^2(x)\,dx$
12. $\int \tan^3(x)\sec^3(x)\,dx$
13. $\int \sin^2(x)\ \cos^3(x)\,dx$
14. $\int \sin^3(x)\ \cos^2(x)\,dx$
15. $\int \sin^4(x)\ \cos^3(x)\,dx$

7.3 Trigonometric Substitution

Notes

1. $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$
2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$
3. $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx.$
4. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, where $a>0$.
5. $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$
6. $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx.$

WebAssign

1. $\displaystyle\int \frac{x^2}{\sqrt{x^2-3}}\,dx$
2. $\displaystyle\int \frac{x^3}{\sqrt{25+x^2}}\,dx$
3. $\displaystyle\int_0^7 \frac{dt}{\sqrt{49+t^2}}$
4. $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$
5. $\displaystyle\int \frac{\sqrt{x^2-81}}{x^4}\,dx$

Practice

1. $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$
2. $\displaystyle\int \frac{x}{\sqrt{x^2-16}}\,dx$
3. $\displaystyle\int_{8\sqrt{2}}^{16} \frac{x}{\sqrt{x^2-64}}\,dx$
4. $\displaystyle\int \frac{\sqrt{x^2+4}}{x^4}\,dx$
5. $\displaystyle\int \frac{1}{x\sqrt{x^2-25}}\,dx$
6. $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{x\sqrt{x^2+9}}\,dx$
7. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$
8. $\displaystyle\int_{2/\sqrt{3}}^{2\sqrt{3}} \frac{x^3}{\sqrt{x^2+4}}\,dx$
9. $\displaystyle\int \frac{\sqrt{36-x^2}}{x^3}\,dx$
10. $\displaystyle\int_{10/\sqrt{2}}^{10} \frac{1}{x^2\sqrt{x^2-25}}\,dx$

7.4 Partial Fractions

Notes

1. $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.
2. $\int \frac{1}{x^2-9}\,dx$.
3. $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.
4. $\int \frac{2x^2-x+4}{x^3+4x}\,dx$.
5. $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$.
6. $\int \frac{x^2+1}{x(x^2+3)}\,dx$?
7. $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$.

WebAssign

- $\frac{x-72}{x^2+x-72}$.
- $\frac{1}{x^2+x^4}$.
- $\int \frac{37}{(x-1)(x^2+36)}\,dx$.
- $\frac{4y^2-6y-12}{y(y+2)(y-3)}$.
- $\int \frac{9r^2}{r+2}\,dr$.

Practice

1. $\int \frac{1}{x^2-4}\,dx$
2. $\int \frac{1}{(x-1)(x+2)}\,dx$
3. $\int \frac{1}{x^2+3x+2}\,dx$
4. $\int \frac{2x+3}{x^2+x-2}\,dx$
5. $\int \frac{1}{x^3-x}\,dx$
6. $\int \frac{x^2}{x^3-1}\,dx$
7. $\int \frac{1}{x(x^2+1)}\,dx$
8. $\int \frac{x}{x^3+x^2}\,dx$
9. $\int \frac{x+1}{(x^2-1)(x+3)}\,dx$
10. $\int \frac{x}{(x^2-4)(x+1)}\,dx$

7.8 Improper Integrals

Notes

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx.$
2. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx.$
3. $\displaystyle\int_{-\infty}^0 xe^x\,dx.$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$
5. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx.$
6. $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$
7. $\displaystyle\int_0^{\pi/2} \sec x\,dx.$
8. $\displaystyle\int_0^3 \frac{dx}{x-1}.$
9. $\displaystyle\int_0^1 \ln(x)\,dx.$
10. Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ is convergent.
11. Show that $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.

WebAssign

1. $\displaystyle\int_8^9 \frac{1}{9x-1}\,dx.$
2. $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx.$
3. $\displaystyle\int_{-\infty}^\infty \frac{\sin(x)}{1+2x^2}\,dx.$
4. $\displaystyle\int_1^3 \ln(x-1)\,dx.$
5. $\displaystyle\int_0^\infty e^{-6x}\,dx.$
6. $\displaystyle\int_{-\infty}^0 \frac{x}{(x^2+2)^4}\,dx.$
7. $\displaystyle\int_{-\infty}^\infty \frac{x^5}{x^6+1}\,dx.$
8. $\displaystyle\int_e^\infty \frac{11}{x(\ln(x))^3}\,dx.$
9. $\displaystyle\int_0^1 \frac{5}{x^5}\,dx.$
10. $\displaystyle\int_3^{11} \frac{1}{\sqrt{11-x}}\,dx.$
11. $\displaystyle\int_2^\infty \frac{1}{x-\ln(x)}\,dx.$
12. $\displaystyle\int_1^\infty \frac{8+\cos(x)}{\sqrt{x^4+x^2}}\,dx.$

Practice

1. $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$
2. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
3. $\displaystyle\int_0^\infty e^{-x}\,dx$
4. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$
6. $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$
7. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$
8. $\displaystyle\int_0^1 \ln(x)\,dx$
9. $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$
10. $\displaystyle\int_0^2 \frac{dx}{(2-x)^{1/3}}$
11. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$
12. $\displaystyle\int_0^{\pi/2} \tan(x)\,dx$
13. $\displaystyle\int_1^\infty \frac{1}{x\,(\ln x)^2}\,dx$
14. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$

Integral Practice Solutions

5.5 u-Substitution Solutions

1. $\int xe^{x^2}\,dx$
   - Let $u = x^2$, then $du = 2x\,dx$.
   - Rewritten integral: $\frac{1}{2}\int e^u\,du$.
   - Solution: $\frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$.
2. $\int \cos(3x)\,dx$
   - Let $u = 3x$, then $du = 3\,dx$.
   - Rewritten integral: $\frac{1}{3}\int \cos(u)\,du$.
   - Solution: $\frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(3x) + C$.
3. $\int x\sqrt{x^2+1}\,dx$
   - Let $u = x^2+1$, then $du = 2x\,dx$.
   - Rewritten integral: $\frac{1}{2}\int \sqrt{u}\,du$.
   - Solution: $\frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C$.
4. $\int \sin(2x)\cos(2x)\,dx$
   - Let $u = \sin(2x)$, then $du = 2\cos(2x)\,dx$.
   - Rewritten integral: $\frac{1}{2}\int u\,du$.
   - Solution: $\frac{1}{4}u^2 + C = \frac{1}{4}\sin^2(2x) + C$.
5. $\int \frac{\ln(x)}{x}\,dx$
   - Let $u = \ln(x)$, then $du = \frac{1}{x}\,dx$.
   - Rewritten integral: $\int u\,du$.
   - Solution: $\frac{1}{2}u^2 + C = \frac{1}{2}(\ln(x))^2 + C$.
6. $\int \frac{x}{x^2+1}\,dx$
   - Let $u = x^2+1$, then $du = 2x\,dx$.
   - Rewritten integral: $\frac{1}{2}\int \frac{1}{u}\,du$.
   - Solution: $\frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$.
7. $\int x^3 e^{x^4}\,dx$
   - Let $u = x^4$, then $du = 4x^3\,dx$.
   - Rewritten integral: $\frac{1}{4}\int e^u\,du$.
   - Solution: $\frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$.
8. $\int x\sqrt{x^2-1}\,dx$
   - Let $u = x^2-1$, then $du = 2x\,dx$.
   - Rewritten integral: $\frac{1}{2}\int \sqrt{u}\,du$.
   - Solution: $\frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C$.
9. $\int \frac{x}{\sqrt{x^2+1}}\,dx$
   - Let $u = x^2+1$, then $du = 2x\,dx$.
   - Rewritten integral: $\frac{1}{2}\int u^{-1/2}\,du$.
   - Solution: $\sqrt{u} + C = \sqrt{x^2+1} + C$.
10. $\int \sin^2(x)\cos(x)\,dx$
    - Let $u = \sin(x)$, then $du = \cos(x)\,dx$.
    - Rewritten integral: $\int u^2\,du$.
    - Solution: $\frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C$.
11. $\int \sin^2(3x)\cos(3x)\,dx$
    - Let $u = \sin(3x)$, then $du = 3\cos(3x)\,dx$.
    - Rewritten integral: $\frac{1}{3}\int u^2\,du$.
    - Solution: $\frac{1}{9}u^3 + C = \frac{1}{9}\sin^3(3x) + C$.
12. $\int \tan(2x)\sec^2(2x)\,dx$
    - Let $u = \tan(2x)$, then $du = 2\sec^2(2x)\,dx$.
    - Rewritten integral: $\frac{1}{2}\int u\,du$.
    - Solution: $\frac{1}{4}u^2 + C = \frac{1}{4}\tan^2(2x) + C$.
13. $\int \sqrt{x+1}\,dx$
    - Let $u = x+1$, then $du = dx$.
    - Rewritten integral: $\int u^{1/2}\,du$.
    - Solution: $\frac{2}{3}u^{3/2} + C = \frac{2}{3}(x+1)^{3/2} + C$.
14. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$
    - Let $u = 1+x^4$, then $du = 4x^3\,dx$.
    - Rewritten integral: $\frac{1}{4}\int u^{-1/2}\,du$.
    - Solution: $\frac{1}{2}u^{1/2} + C = \frac{1}{2}\sqrt{1+x^4} + C$.
15. $\int \frac{1}{x\sqrt{x+1}}\,dx$
    - Let $u = \sqrt{x+1}$, then $u^2 = x+1$, $du = \frac{1}{2\sqrt{x+1}}\,dx$.
    - Rewritten integral: $\int \frac{1}{u^2}\,du$.
    - Solution: $-\frac{1}{u} + C = -\frac{1}{\sqrt{x+1}} + C$.
16. $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$
    - Let $u = x^3+1$, then $du = 3x^2\,dx$.
    - Rewritten integral: $\frac{1}{3}\int u^{-1/2}\,du$.
    - Solution: $\frac{2}{3}u^{1/2} + C = \frac{2}{3}\sqrt{x^3+1} + C$.
17. $\int 2x\sqrt{1-x^2}\,dx$
    - Let $u = 1-x^2$, then $du = -2x\,dx$.
    - Rewritten integral: $-\int \sqrt{u}\,du$.
    - Solution: $-\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1-x^2)^{3/2} + C$.

7.1 Integration by Parts Solutions

1. $\int_1^e xe^x \, dx$
   - Identify parts:
     - Let $u = x$, so that $du = dx$.
     - Let $dv = e^x dx$, so that $v = e^x$.
   - Apply integration by parts:
   $$\begin{aligned}\int u \, dv &= uv - \int v \, du \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x.\end{aligned}$$
   - Evaluate from 1 to e:
   $$\begin{aligned}[xe^x - e^x]_1^e &= (e \cdot e^e - e^e) - \left(1 \cdot e^1 - e^1\right) \\ &= e^{e+1} - e^e - (e - e) \\ &= e^{e+1} - e^e.\end{aligned}$$

2. $\int x \ln(x) \, dx$
   - Identify parts:
     - Let $u = \ln(x)$, so that $du = \frac{1}{x} dx$.
     - Let $dv = x dx$, so that $v = \frac{x^2}{2}$.
   - Apply integration by parts:
   $$\begin{aligned}\int u \, dv &= uv - \int v \, du \\ &= \frac{x^2}{2} \ln(x) - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2}{2} \ln(x) - \int \frac{x}{2} dx \\ &= \frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C.\end{aligned}$$

3. $\int_0^\pi x^2 \sin(x) \, dx$
   - Identify parts:
     - Let $u = x^2$, so that $du = 2x dx$.
     - Let $dv = \sin(x) dx$, so that $v = -\cos(x)$.
   - Apply integration by parts:
   $$\begin{aligned}\int u \, dv &= uv - \int v \, du \\ &= -x^2 \cos(x) + \int 2x \cos(x) dx.\end{aligned}$$
   - Use integration by parts again on $\int 2x \cos(x) dx$.
     - Let $u = 2x$, so $du = 2dx$.
     - Let $dv = \cos(x) dx$, so $v = \sin(x)$.
   - Apply:
   $$\begin{aligned}\int 2x \cos(x) dx &= 2x \sin(x) - \int 2 \sin(x) dx \\ &= 2x \sin(x) + 2 \cos(x).\end{aligned}$$
   - Combine results:
   $$\int x^2 \sin(x) dx = -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x).$$
   - Evaluate from 0 to π:
   $$\left[-x^2 \cos(x) + 2x \sin(x) + 2 \cos(x)\right]_0^\pi.$$
   - Compute:
   $$\begin{aligned}&(-\pi^2 \cos(\pi) + 2\pi \sin(\pi) + 2 \cos(\pi)) \\ &\quad - (-0^2 \cos(0) + 2(0) \sin(0) + 2 \cos(0)).\end{aligned}$$

     Since $\cos(\pi) = -1$, $\sin(\pi) = 0$, $\cos(0) = 1$, and $\sin(0) = 0$:
   $$\begin{aligned}(-\pi^2(-1) + 0 + 2(-1)) - (0 + 0 + 2(1)) \\ = (\pi^2 - 2) - (2) \\ = \pi^2 - 4.\end{aligned}$$

4. $\int x \cos(x) \, dx$
   - Identify the parts:
     - Let $u = x$, so that $du = dx$.
     - Let $dv = \cos(x) dx$, so that $v = \sin(x)$.
   - Apply integration by parts:
   $$\begin{aligned}\int u \, dv &= uv - \int v \, du \\ &= x \sin(x) - \int \sin(x) dx.\end{aligned}$$
   - Compute the remaining integral:
   $$\int \sin(x) dx = -\cos(x).$$
   - Final result:
   $$\int x \cos(x) dx = x \sin(x) + \cos(x) + C.$$

5. $\displaystyle\int_1^2 x^3 \ln(x)\, dx$

- Identify the parts:
  - Let $u = \ln(x)$, so that $du = \frac{1}{x}dx$.
  - Let $dv = x^3dx$, so that $v = \frac{x^4}{4}$.
- Apply integration by parts:

$$\begin{aligned}\int u\, dv &= uv - \int v\, du \\ &= \frac{x^4}{4}\ln(x) - \int \frac{x^4}{4}\cdot\frac{1}{x}dx \\ &= \frac{x^4}{4}\ln(x) - \int \frac{x^3}{4}dx.\end{aligned}$$

- Compute the remaining integral:

$$\int \frac{x^3}{4}dx = \frac{1}{4}\cdot\frac{x^4}{4} = \frac{x^4}{16}.$$

- Final result before evaluation:

$$\int x^3\ln(x)dx = \frac{x^4}{4}\ln(x) - \frac{x^4}{16}.$$

- Evaluate from 1 to 2:

$$\left[\frac{x^4}{4}\ln(x) - \frac{x^4}{16}\right]_1^2.$$

  Substituting values:

$$\left(\frac{16}{4}\ln(2) - \frac{16}{16}\right) - \left(\frac{1}{4}\ln(1) - \frac{1}{16}\right).$$

  Since $\ln(1) = 0$:

$$(4\ln(2) - 1) - \left(0 - \frac{1}{16}\right) = 4\ln(2) - 1 + \frac{1}{16}.$$

- Final evaluated result:

$$4\ln(2) - \frac{15}{16}.$$

6. $\displaystyle\int x^2\ln(x^2)\, dx$

- Identify parts:
  - Rewrite $\ln(x^2)$ as $2\ln(x)$.
  - Let $u = \ln(x)$, so that $du = \frac{1}{x}dx$.
  - Let $dv = x^2dx$, so that $v = \frac{x^3}{3}$.
- Apply integration by parts:

$$\begin{aligned}\int u\, dv &= uv - \int v\, du \\ &= \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\cdot\frac{1}{x}dx \\ &= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}dx.\end{aligned}$$

- Compute the remaining integral:

$$\int \frac{x^2}{3}dx = \frac{1}{3}\cdot\frac{x^3}{3} = \frac{x^3}{9}.$$

- Final result:

$$\int x^2\ln(x^2)dx = \frac{2x^3}{3}\ln(x) - \frac{2x^3}{9} + C.$$

7. $\displaystyle\int x^2\cos(2x)\, dx$

- Identify parts:
  - Let $u = x^2$, so that $du = 2xdx$.
  - Let $dv = \cos(2x)dx$, so that $v = \frac{1}{2}\sin(2x)$.
- Apply integration by parts:

$$\begin{aligned}\int u\, dv &= uv - \int v\, du \\ &= \frac{x^2}{2}\sin(2x) - \int \frac{2x}{2}\sin(2x)dx.\end{aligned}$$

- Use integration by parts again for $\int x\sin(2x)dx$.
  - Let $u = x$, so $du = dx$.
  - Let $dv = \sin(2x)dx$, so $v = -\frac{1}{2}\cos(2x)$.
- Compute:

$$\begin{aligned}\int x\sin(2x)dx &= -\frac{x}{2}\cos(2x) + \frac{1}{2}\int\cos(2x)dx \\ &= -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x).\end{aligned}$$

- Final result:

$$\frac{x^2}{2}\sin(2x) + \frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C.$$

8. $\displaystyle\int \ln(x^2+1)\, dx$

- Identify parts:
  - Let $u = \ln(x^2+1)$, so that $du = \frac{2x}{x^2+1}dx$.
  - Let $dv = dx$, so that $v = x$.
- Apply integration by parts:

$$\begin{aligned}\int u\, dv &= uv - \int v\, du \\ &= x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}dx.\end{aligned}$$

- Compute the remaining integral:

$$\begin{aligned}\int \frac{2x^2}{x^2+1}dx &= \int\left(2 - \frac{2}{x^2+1}\right)dx \\ &= 2x - 2\tan^{-1}(x).\end{aligned}$$

- Final result:

$$x\ln(x^2+1) - 2x + 2\tan^{-1}(x) + C.$$

7.1 Boomerang Integrals Solutions

1. $\int e^{3x}\cos(3x)\,dx$

   - Define
   $$I=\int e^{3x}\cos(3x)\,dx.$$
   - **First Integration by Parts:**
     - Let
     $$u=\cos(3x)\quad\text{and}\quad dv=e^{3x}\,dx.$$
     - Then,
     $$du=-3\sin(3x)\,dx\quad\text{and}\quad v=\frac{e^{3x}}{3}.$$
     - Hence,
     $$\begin{aligned}I&=u\,v-\int v\,du\\&=\frac{e^{3x}\cos(3x)}{3}-\int\frac{e^{3x}}{3}\big[-3\sin(3x)\big]\,dx\\&=\frac{e^{3x}\cos(3x)}{3}+\int e^{3x}\sin(3x)\,dx.\end{aligned}$$
   - **Second Integration by Parts:** Denote
   $$J=\int e^{3x}\sin(3x)\,dx.$$
     - Let
     $$u=\sin(3x)\quad\text{and}\quad dv=e^{3x}\,dx.$$
     - Then,
     $$du=3\cos(3x)\,dx\quad\text{and}\quad v=\frac{e^{3x}}{3}.$$
     - Thus,
     $$\begin{aligned}J&=u\,v-\int v\,du\\&=\frac{e^{3x}\sin(3x)}{3}-\int\frac{e^{3x}}{3}\cdot3\cos(3x)\,dx\\&=\frac{e^{3x}\sin(3x)}{3}-\int e^{3x}\cos(3x)\,dx\\&=\frac{e^{3x}\sin(3x)}{3}-I.\end{aligned}$$
   - **Combine and Solve for I:**
   $$\begin{aligned}I&=\frac{e^{3x}\cos(3x)}{3}+J\\&=\frac{e^{3x}\cos(3x)}{3}+\frac{e^{3x}\sin(3x)}{3}-I.\end{aligned}$$
   $$\begin{aligned}2I&=\frac{e^{3x}}{3}\Big(\cos(3x)+\sin(3x)\Big)\\I&=\frac{e^{3x}\Big(\cos(3x)+\sin(3x)\Big)}{6}+C.\end{aligned}$$
   - **Answer:**
   $$\frac{e^{3x}(\cos(3x)+\sin(3x))}{6}+C.$$

2. $\int e^{-x}\sin(x)\,dx$

   - Define
   $$I=\int e^{-x}\sin x\,dx.$$
   - **First Integration by Parts:**
     - Let
     $$u=\sin x\quad\text{and}\quad dv=e^{-x}\,dx.$$
     - Then,
     $$du=\cos x\,dx\quad\text{and}\quad v=-e^{-x}.$$
     - Thus,
     $$\begin{aligned}I&=u\,v-\int v\,du\\&=-e^{-x}\sin x-\int\big[-e^{-x}\big]\cos x\,dx\\&=-e^{-x}\sin x+\int e^{-x}\cos x\,dx.\end{aligned}$$
   - **Second Integration by Parts:** Denote
   $$J=\int e^{-x}\cos x\,dx.$$
     - Let
     $$u=\cos x\quad\text{and}\quad dv=e^{-x}\,dx.$$
     - Then,
     $$du=-\sin x\,dx\quad\text{and}\quad v=-e^{-x}.$$
     - Therefore,
     $$\begin{aligned}J&=u\,v-\int v\,du\\&=-e^{-x}\cos x-\int\big[-e^{-x}\big]\big[-\sin x\big]\,dx\\&=-e^{-x}\cos x-\int e^{-x}\sin x\,dx\\&=-e^{-x}\cos x-I.\end{aligned}$$
   - **Combine and Solve for I:**
   $$\begin{aligned}I&=-e^{-x}\sin x+J\\&=-e^{-x}\sin x-e^{-x}\cos x-I.\end{aligned}$$
   $$\begin{aligned}2I&=-e^{-x}\Big(\sin x+\cos x\Big)\\I&=-\frac{e^{-x}\Big(\sin x+\cos x\Big)}{2}+C.\end{aligned}$$
   - **Answer:**
   $$-\frac{e^{-x}(\sin x+\cos x)}{2}+C.$$

7.2 Trigonometric Integrals Solutions

1. $\displaystyle\int \sin^3(x)\,dx$

   - Write
   $$\sin^3 x = \sin x\,(1-\cos^2 x),$$
   so that
   $$\int \sin^3 x\,dx = \int \sin x\,(1-\cos^2 x)\,dx.$$
   - Let $u=\cos x$ (thus $du = -\sin x\,dx$).
   - Change variables and integrate:
   $$\begin{aligned}\int \sin^3 x\,dx &= -\int (1-u^2)\,du\\ &= -\left(u-\frac{u^3}{3}\right)+C\\ &= -\cos x + \frac{\cos^3 x}{3}+C.\end{aligned}$$

2. $\displaystyle\int \sin^2(x)\cos(x)\,dx$

   - Let $u=\sin x$ so that $du=\cos x\,dx$.
   - Then,
   $$\begin{aligned}\int \sin^2 x\,\cos x\,dx &= \int u^2\,du\\ &= \frac{u^3}{3}+C\\ &= \frac{\sin^3 x}{3}+C.\end{aligned}$$

3. $\displaystyle\int \sin^4(x)\cos(x)\,dx$

   - Let $u=\sin x$ (so that $du=\cos x\,dx$).
   - Then,
   $$\begin{aligned}\int \sin^4 x\,\cos x\,dx &= \int u^4\,du\\ &= \frac{u^5}{5}+C\\ &= \frac{\sin^5 x}{5}+C.\end{aligned}$$

4. $\displaystyle\int \cos^4(x)\sin(x)\,dx$

   - Let $u=\cos x$ so that $du=-\sin x\,dx$.
   - Then,
   $$\begin{aligned}\int \cos^4 x\,\sin x\,dx &= -\int u^4\,du\\ &= -\frac{u^5}{5}+C\\ &= -\frac{\cos^5 x}{5}+C.\end{aligned}$$

5. $\displaystyle\int \sec^2(x)\,dx$

   - Since
   $$\frac{d}{dx}\tan x = \sec^2 x,$$
   it follows immediately that
   $$\int \sec^2 x\,dx = \tan x + C.$$

6. $\displaystyle\int \sec^4(x)\,dx$

   - Write
   $$\sec^4 x = \sec^2 x\big(\tan^2 x+1\big).$$
   - Let $u=\tan x$ so that $du=\sec^2 x\,dx$.
   - Then,
   $$\begin{aligned}\int \sec^4 x\,dx &= \int \big(u^2+1\big)\,du\\ &= \frac{u^3}{3}+u+C\\ &= \frac{\tan^3 x}{3}+\tan x + C.\end{aligned}$$

7. $\displaystyle\int \sec^2(x)\tan^2(x)\,dx$

   - Again, use $\tan^2 x = \sec^2 x - 1$ so that
   $$\int \sec^2 x\,\tan^2 x\,dx = \int \sec^2 x\big(\sec^2 x - 1\big)\,dx.$$
   - This splits into:
   $$\int \sec^4 x\,dx - \int \sec^2 x\,dx.$$
   - From previous results,
   $$\begin{aligned}\int \sec^4 x\,dx &= \frac{\tan^3 x}{3}+\tan x + C,\\ \int \sec^2 x\,dx &= \tan x + C.\end{aligned}$$
   - Therefore,
   $$\begin{aligned}\int \sec^2 x\,\tan^2 x\,dx &= \left(\frac{\tan^3 x}{3}+\tan x\right) - \tan x + C\\ &= \frac{\tan^3 x}{3}+C.\end{aligned}$$

8. $\displaystyle\int \tan^3(x)\sec^2(x)\,dx$

   - Let $u=\tan x$ so that $du=\sec^2 x\,dx$.
   - Then,
   $$\begin{aligned}\int \tan^3 x\,\sec^2 x\,dx &= \int u^3\,du\\ &= \frac{u^4}{4}+C\\ &= \frac{\tan^4 x}{4}+C.\end{aligned}$$

9. $\int \sec^2(x)\tan(x)\,dx$

- Let $u = \tan x$ (so that $du = \sec^2 x\,dx$).
- Then,

$$\int \sec^2 x\,\tan x\,dx = \int u\,du$$
$$= \frac{u^2}{2} + C$$
$$= \frac{\tan^2 x}{2} + C.$$

10. $\int \tan^2(x)\,dx$

- Write $\tan^2 x = \sec^2 x - 1$ so that:

$$\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)dx$$
$$= \int \sec^2 x\,dx - \int 1\,dx$$
$$= \tan x - x + C.$$

11. $\int \tan^4(x)\sec^2(x)\,dx$

- Let $u = \tan x$ so that $du = \sec^2 x\,dx$.
- Then,

$$\int \tan^4 x\,\sec^2 x\,dx = \int u^4\,du$$
$$= \frac{u^5}{5} + C$$
$$= \frac{\tan^5 x}{5} + C.$$

12. $\int \tan^3(x)\sec^3(x)\,dx$

- Rewrite the integrand as

$$\tan^3 x\,\sec^3 x = \left(\tan^2 x\,\sec^2 x\right)\left(\tan x\,\sec x\right).$$

- Use $\tan^2 x = \sec^2 x - 1$ to get:

$$\tan^2 x\,\sec^2 x = \sec^4 x - \sec^2 x.$$

- Let $u = \sec x$ so that $du = \sec x\,\tan x\,dx$.
- Then,

$$\int \tan^3 x\,\sec^3 x\,dx = \int \left(\sec^4 x - \sec^2 x\right)du$$
$$= \int \left(u^4 - u^2\right)du$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$
$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.$$

13. $\int \sin^2 x\,\cos^3 x\,dx$

- Rewrite $\cos^3 x$ as:

$$\cos^3 x = \cos^2 x\,\cos x = \left(1 - \sin^2 x\right)\cos x.$$

- Substitute into the integral:

$$\int \sin^2 x\,\cos^3 x\,dx = \int \sin^2 x\left(1 - \sin^2 x\right)\cos x\,dx.$$

- Let $u = \sin x$, so that $du = \cos x\,dx$.
- Then the integral becomes:

$$\int \sin^2 x\left(1 - \sin^2 x\right)\cos x\,dx = \int u^2\left(1 - u^2\right)du$$
$$= \int \left(u^2 - u^4\right)du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + C.$$

- Substitute back $u = \sin x$:

$$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

14. $\int \sin^3 x\,\cos^2 x\,dx$

- Express $\sin^3 x$ as:

$$\sin^3 x = \sin x\,\sin^2 x,$$

and use the identity:

$$\sin^2 x = 1 - \cos^2 x.$$

- Thus, the integral becomes:

$$\int \sin^3 x\,\cos^2 x\,dx = \int \sin x\left(1 - \cos^2 x\right)\cos^2 x\,dx.$$

- Rewrite the integrand:

$$= \int \sin x\left(\cos^2 x - \cos^4 x\right)dx.$$

- Let $u = \cos x$, so that $du = -\sin x\,dx$.
- Then, rewriting the integral:

$$\int \sin x\left(\cos^2 x - \cos^4 x\right)dx = -\int \left(u^2 - u^4\right)du$$
$$= -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$
$$= -\frac{u^3}{3} + \frac{u^5}{5} + C.$$

- Substitute back $u = \cos x$:

$$-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

15. $\int \sin^4 x\,\cos^3 x\,dx$

- Rewrite $\cos^3 x$ as:

$$\cos^3 x = \cos^2 x\,\cos x = \left(1 - \sin^2 x\right)\cos x.$$

- Substitute into the integral:

$$\int \sin^4 x\,\cos^3 x\,dx = \int \sin^4 x\left(1 - \sin^2 x\right)\cos x\,dx.$$

- Let $u = \sin x$, so that $du = \cos x\,dx$.
- Then the integral becomes:

$$\int \sin^4 x\left(1 - \sin^2 x\right)\cos x\,dx = \int u^4\left(1 - u^2\right)du$$
$$= \int \left(u^4 - u^6\right)du$$
$$= \frac{u^5}{5} - \frac{u^7}{7} + C.$$

- Substitute back $u = \sin x$:

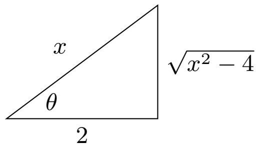
$$\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

7.3 Trigonometric Substitution Solutions

1. $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$

   (a) **Identify the form:** $\sqrt{x^2-a^2}$ suggests $x = a\sec\theta$.

   (b) **Draw the triangle:**

   (c) **Substitute:** Let $x = 2\sec\theta$, so $dx = 2\sec\theta\tan\theta d\theta$.

   (d) **Simplify:**
   $$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\tan\theta.$$

   (e) **Recompute bounds:**
   $$x = 2 \Rightarrow \theta_1 = \sec^{-1}(1) = 0, \quad x = 4 \Rightarrow \theta_2 = \sec^{-1}(2) = \frac{\pi}{3}.$$

   (f) **Rewrite the integral:**
   $$\int_0^{\frac{\pi}{3}} \frac{2\tan\theta}{2\sec\theta}\cdot 2\sec\theta\tan\theta d\theta.$$

   (g) **Simplify:** The expression reduces to:
   $$\int_0^{\frac{\pi}{3}} 2\tan^2\theta\, d\theta.$$
   Using $\tan^2\theta = \sec^2\theta - 1$, we rewrite the integral:
   $$2\int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta.$$

   (h) **Evaluate:**
   $$2(\tan\theta - \theta)\Big|_0^{\frac{\pi}{3}}.$$
   Compute:
   $$2\left(\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - (\tan 0 - 0)\right) = 2\left(\sqrt{3} - \frac{\pi}{3}\right) - 0.$$

   (i) **Final Answer:** $2\sqrt{3} - \frac{2\pi}{3}$.

2. $\displaystyle\int \frac{x}{\sqrt{x^2-16}}\,dx$

   (a) **Identify the form:** This is a u-substitution problem.

   (b) **Substitute:** Let $u = x^2 - 16$, so that $du = 2x\,dx$.

   (c) **Rewrite the integral:**
   $$\int \frac{x}{\sqrt{x^2-16}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}.$$

   (d) **Simplify and integrate:**
   $$\frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} = \sqrt{u}.$$

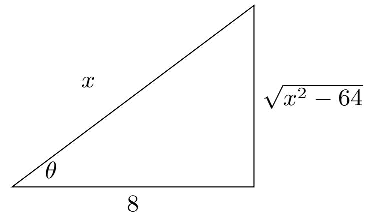
   (e) **Substitute back in terms of x:**
   $$\sqrt{x^2-16} + C.$$

   (f) **Final Answer:** $\sqrt{x^2-16} + C$.

3. $\displaystyle\int_{8\sqrt{2}}^{16} \frac{x}{\sqrt{x^2-64}}\,dx$

   (a) **Identify the form:** $\sqrt{x^2-a^2}$ suggests $x = a\sec\theta$.

   (b) **Draw the triangle:**

   (c) **Substitute:** Let $x = 8\sec\theta$, so $dx = 8\sec\theta\tan\theta d\theta$.

   (d) **Simplify:**
   $$\sqrt{x^2-64} = \sqrt{64\sec^2\theta - 64} = 8\tan\theta.$$

   (e) **Recompute bounds:**
   $$x = 8\sqrt{2} \Rightarrow \theta_1 = \sec^{-1}(\sqrt{2}) = \frac{\pi}{4},$$
   $$x = 16 \Rightarrow \theta_2 = \sec^{-1}(2) = \frac{\pi}{3}.$$

   (f) **Rewrite the integral:**
   Expressing everything in terms of θ:
   $$\int_{8\sqrt{2}}^{16} \frac{x}{\sqrt{x^2-64}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{8\sec\theta}{8\tan\theta}\cdot 8\sec\theta\tan\theta\, d\theta$$
   $$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 8\sec^2\theta\, d\theta$$

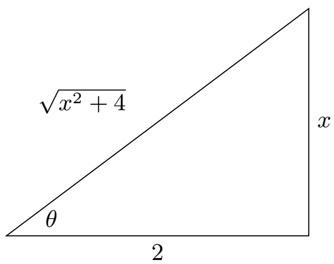
   (g) **Evaluate:**
   $$8\int \sec^2\theta\, d\theta = 8\tan\theta.$$
   Computing over the bounds:
   $$8\tan\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}.$$
   Using values from the unit circle,
   $$\tan\frac{\pi}{3} = \sqrt{3}, \quad \tan\frac{\pi}{4} = 1.$$
   Substituting:
   $$8(\sqrt{3} - 1).$$

   (h) **Final Answer:** $8(\sqrt{3} - 1)$

4. $\int \frac{\sqrt{x^2+4}}{x^4}\,dx$

(a) **Identify the form:** $\sqrt{x^2+a^2}$ suggests $x = a\tan\theta$.

(b) **Draw the triangle:**



(c) **Substitute:** Let $x = 2\tan\theta$, so $dx = 2\sec^2\theta d\theta$.

(d) **Simplify:**

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = 2\sec\theta.$$

(e) **Rewrite the integral:**

$$\int \frac{2\sec\theta}{(2\tan\theta)^4}\cdot 2\sec^2\theta\,d\theta.$$

Simplifying:

$$\int \frac{4\sec^3\theta}{16\tan^4\theta}\,d\theta = \int \frac{\sec^3\theta}{4\tan^4\theta}\,d\theta.$$

Expressing in terms of sine and cosine:

$$\int \frac{(1/\cos^3\theta)}{4(\sin^4\theta/\cos^4\theta)}\,d\theta = \int \frac{\cos\theta}{8\sin^4\theta}\,d\theta.$$

Using substitution, let $u = \sin\theta$, so $du = \cos\theta d\theta$:

$$\int \frac{du}{4u^4}.$$

Evaluating:

$$\frac{-1}{12u^3} + C = \frac{-1}{12\sin^3\theta} + C.$$

Substituting back $\sin\theta = \frac{x}{\sqrt{x^2+4}}$:

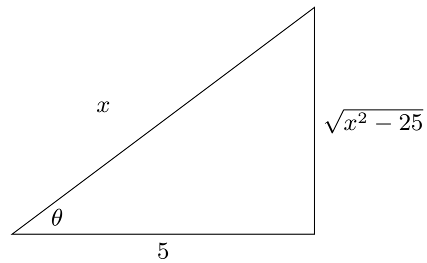
$$-\frac{1}{12}\cdot\frac{(x^2+4)^{3/2}}{x^3} + C.$$

(f) **Final Answer:** $-\frac{(x^2+4)^{3/2}}{12x^3} + C.$

5. $\int \frac{1}{x\sqrt{x^2-25}}\,dx$

(a) **Identify the form:** The expression $\sqrt{x^2-a^2}$ suggests the substitution $x = a\sec\theta$.

(b) **Draw the triangle:**



(c) **Substitute:** Let $x = 5\sec\theta$, so $dx = 5\sec\theta\tan\theta d\theta$.

(d) **Simplify:**

$$\sqrt{x^2-25} = \sqrt{25\sec^2\theta-25} = 5\tan\theta.$$

(e) **Rewrite the integral:**

$$\int \frac{1}{(5\sec\theta)(5\tan\theta)}\cdot(5\sec\theta\tan\theta)\,d\theta.$$

Canceling terms:

$$\int \frac{1}{5}\,d\theta.$$

(f) **Evaluate:**

$$\frac{1}{5}\theta + C.$$

(g) **Express in terms of x:** Since we know

$$\tan\theta = \frac{\sqrt{x^2-25}}{5},$$

we substitute:

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2-25}}{5}\right).$$

Thus, the final answer is

$$\frac{1}{5}\tan^{-1}\left(\frac{\sqrt{x^2-25}}{5}\right) + C.$$

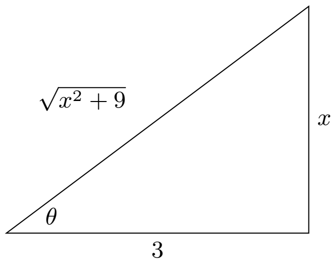
(h) **Final Answer:** $\frac{1}{5}\tan^{-1}\left(\frac{\sqrt{x^2-25}}{5}\right) + C.$

Note: While $\theta = \sec^{-1}(x/5)$ is valid, we typically prefer $\tan^{-1}$ since it also follows directly from the triangle, is more standard in integration tables, and avoids domain restrictions present in $\sec^{-1}$. Additionally, $\sin^{-1}$ and $\cos^{-1}$ could be used, but they have restricted ranges ($[-\pi/2, \pi/2]$ for $\sin^{-1}$ and $[0,\pi]$ for $\cos^{-1}$), which do not always align naturally with the function's domain. Specifically, while $\cos^{-1}(5/x)$ is valid for $|x| > 5$, it does not always give the correct quadrant when $x < -5$. Since we set $x = 5\sec\theta$, when $x < -5$, we know that $\sec\theta < 0$. This means θ must be in Quadrant III, where secant is negative. However, $\cos^{-1}(5/x)$ only returns angles in Quadrants I and II, so it incorrectly places θ in Quadrant II instead of III. To correct this, we would need to adjust using $\theta = \pi + \cos^{-1}(5/x)$ when $x < -5$. By contrast, $\tan^{-1}$ smoothly handles both positive and negative values without requiring quadrant corrections, making it the preferred choice.

6. $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{x\sqrt{x^2+9}}\,dx$

(a) **Identify the form:** The expression $\sqrt{x^2+a^2}$ suggests the substitution $x = a\tan\theta$.

(b) **Draw the triangle:**



(c) **Substitute:** Let $x = 3\tan\theta$, so $dx = 3\sec^2\theta d\theta$.

(d) **Simplify:**

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta.$$

(e) **Change of bounds:**

- When $x = 3$, $\tan\theta = 1 \Rightarrow \theta = \pi/4$.
- When $x = 3\sqrt{3}$, $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$.

(f) **Change of bounds:** Since we substitute $x = 3\tan\theta$, we need to convert the limits: - When $x = 3$, we solve $3 = 3\tan\theta$ to get $\tan\theta = 1 \Rightarrow \theta = \pi/4$. - When $x = 3\sqrt{3}$, we solve $3\sqrt{3} = 3\tan\theta$ to get $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$.

So the new integral limits are from $\theta = \pi/4$ to $\theta = \pi/3$.

(g) **Rewrite the integral:**

$$\int_{\pi/4}^{\pi/3} \frac{1}{(3\tan\theta)(3\sec\theta)} \cdot (3\sec^2\theta)\,d\theta.$$

Simplifying:

$$\int_{\pi/4}^{\pi/3} \frac{1}{3} \cdot \csc\theta\,d\theta.$$

(h) **Evaluate:** Using the standard integral result:

$$\int \csc\theta\,d\theta = -\ln|\csc\theta + \cot\theta|,$$

we compute:

$$-\frac{1}{3}\ln|\csc\theta + \cot\theta|\Big|_{\pi/4}^{\pi/3}.$$

(i) **Compute:** Using known values:

$$\csc(\pi/3) = \frac{2}{\sqrt{3}}, \quad \cot(\pi/3) = \frac{1}{\sqrt{3}},$$

$$\csc(\pi/4) = \sqrt{2}, \quad \cot(\pi/4) = 1,$$

we obtain:

$$-\frac{1}{3}\left(\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln\left|\sqrt{2}+1\right|\right).$$

Simplifying:

$$-\frac{1}{3}\left(\ln\left|\frac{3}{\sqrt{3}}\right| - \ln\left|\sqrt{2}+1\right|\right).$$

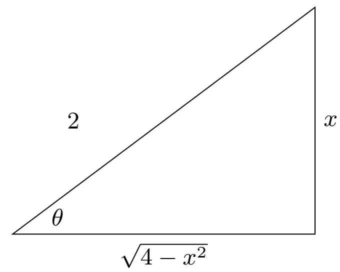
(j) **Final Answer:**

$$\frac{1}{3}\ln\left(\frac{\sqrt{3}}{3}\cdot(\sqrt{2}+1)\right).$$

7. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

(a) **Identify the form:** The expression $\sqrt{4-x^2}$ suggests the substitution $x = 2\sin\theta$.

(b) **Draw the triangle:**



(c) **Substitute:** Let $x = 2\sin\theta$, so $dx = 2\cos\theta d\theta$.

(d) **Simplify:**

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta.$$

(e) **Rewrite the integral:**

$$\int \frac{(4\sin^2\theta)}{2\cos\theta} \cdot (2\cos\theta)\,d\theta.$$

Simplifying:

$$\int 4\sin^2\theta\,d\theta.$$

(f) **Use power reduction for** $\sin^2\theta$: Using the identity:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2},$$

we rewrite the integral:

$$\int 4 \times \frac{1-\cos 2\theta}{2}\,d\theta.$$

(g) **Integrate term by term:**

$$\int (2 - 2\cos 2\theta)\,d\theta.$$

(h) **Final Answer in terms of** x:

$$2\theta - \sin 2\theta + C.$$

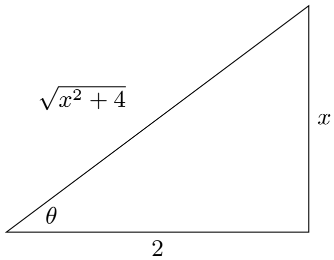
Using $\theta = \sin^{-1}(x/2)$ and $\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2}$,

$$2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C.$$

8. $\displaystyle\int_{2/\sqrt{3}}^{2\sqrt{3}} \frac{x^3}{\sqrt{x^2+4}}\,dx$

(a) **Identify the form:** The expression $\sqrt{x^2+4}$ suggests the substitution $x = 2\tan\theta$.

(b) **Draw the triangle:**



(c) **Substitute:** Let $x = 2\tan\theta$, so $dx = 2\sec^2\theta d\theta$.

(d) **Simplify:**

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sec\theta.$$

(e) **Change of bounds:** Since $x = 2\tan\theta$, we convert the limits:

- When $x = 2/\sqrt{3}$, solving $2/\sqrt{3} = 2\tan\theta$ gives $\tan\theta = 1/\sqrt{3}$, so $\theta = \pi/6$.
- When $x = 2\sqrt{3}$, solving $2\sqrt{3} = 2\tan\theta$ gives $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

(f) **Rewrite the integral:**

$$\int_{\pi/6}^{\pi/3} \frac{x^3}{\sqrt{x^2+4}}\,dx = \int_{\pi/6}^{\pi/3} \frac{(8\tan^3\theta)}{2\sec\theta}\cdot(2\sec^2\theta)\,d\theta.$$

Simplifying:

$$\int_{\pi/6}^{\pi/3} 8\tan^3\theta\sec\theta\,d\theta.$$

(g) **Evaluate the integral:** Using the identity $\tan^3\theta = (\sec^2\theta - 1)\tan\theta$, perform substitution with $u = \sec\theta$:

$$du = \sec\theta\tan\theta\,d\theta.$$

The integral transforms into:

$$8\int_{2/\sqrt{3}}^{2} (u^2-1)\,du.$$

Computing:

$$8\left[\frac{1}{3}u^3 - u\right]_{\frac{2}{\sqrt{3}}}^{2}.$$

Expanding each term:

$$8\left[\left(\frac{8}{3}-2\right) - \left(\frac{8}{9\sqrt{3}} - \frac{2}{\sqrt{3}}\right)\right].$$

Simplifying:

$$\frac{16}{3} + \frac{80}{9\sqrt{3}}.$$

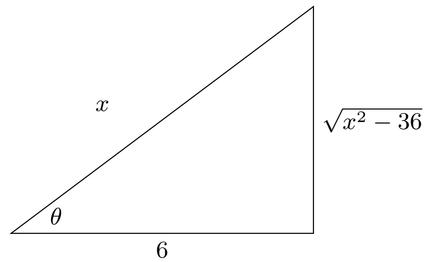
(h) **Final Answer:**

$$\frac{16}{3} + \frac{80}{9\sqrt{3}}$$

9. $\displaystyle\int \frac{\sqrt{x^2-36}}{x^3}\,dx$

(a) **Identify the form:** The expression $\sqrt{x^2-36}$ suggests the substitution $x = 6\sec\theta$.

(b) **Draw the triangle:**



(c) **Substitute:** Let $x = 6\sec\theta$, so $dx = 6\sec\theta\tan\theta d\theta$.

(d) **Simplify:**

$$\sqrt{x^2-36} = \sqrt{36\sec^2\theta - 36} = 6\tan\theta.$$

(e) **Rewrite the integral:**

$$\int \frac{6\tan\theta}{(6\sec\theta)^3}\cdot 6\sec\theta\tan\theta\,d\theta.$$

Simplifying:

$$\int \frac{36\tan^2\theta}{216\sec^2\theta}\,d\theta = \int \frac{1}{6}\cdot\frac{\tan^2\theta}{\sec^2\theta}\,d\theta.$$

Rewriting in terms of sine and cosine:

$$\int \frac{1}{6}\cdot\frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}\,d\theta.$$

Simplifying:

$$\int \frac{1}{6}\cdot\sin^2\theta\,d\theta.$$

Using the identity $\sin^2\theta = \frac{1-\cos 2\theta}{2}$:

$$\int \frac{1}{6}\cdot\frac{1-\cos 2\theta}{2}\,d\theta.$$

Splitting the integral:

$$\frac{1}{12}\int (1-\cos 2\theta)\,d\theta.$$

Evaluating:

$$\frac{1}{12}\left[\theta - \frac{\sin 2\theta}{2}\right] + C.$$

(f) **Expressing $\sin 2\theta$:** Using the double-angle identity,

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

From the right triangle, we note:

$$\sin\theta = \frac{\sqrt{x^2-36}}{x}, \quad \cos\theta = \frac{6}{x}.$$

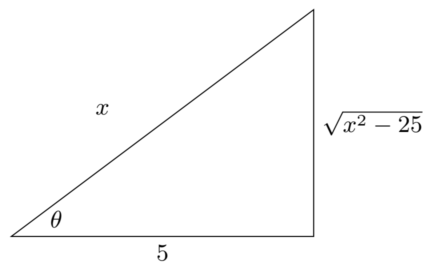
Finally, $\theta = \tan^{-1}\left(\frac{\sqrt{x^2-36}}{6}\right)$:

$$\frac{1}{12}\left[\tan^{-1}\left(\frac{\sqrt{x^2-36}}{6}\right) - \frac{6\sqrt{x^2-36}}{x^2}\right] + C.$$

10. $\displaystyle\int_{5\sqrt{2}}^{10} \frac{1}{x^2\sqrt{x^2-25}}\,dx$

(a) **Identify the form:** The expression $\sqrt{x^2-25}$ suggests the substitution $x = 5\sec\theta$.

(b) **Draw the triangle:**



(c) **Substitute:** Let $x = 5\sec\theta$, so $dx = 5\sec\theta\tan\theta d\theta$.

(d) **Simplify:**

$$\sqrt{x^2-25} = \sqrt{25\sec^2\theta - 25} = 5\tan\theta.$$

(e) **Change of bounds:** Since $x = 5\sec\theta$, we convert the limits:

- When $x = 5\sqrt{2}$, solving $5\sqrt{2} = 5\sec\theta$ gives $\sec\theta = \sqrt{2}$, so $\theta = \pi/4$.
- When $x = 10$, solving $10 = 5\sec\theta$ gives $\sec\theta = 2$, so $\theta = \pi/3$.

(f) **Rewrite the integral:**

$$\int_{\pi/4}^{\pi/3} \frac{1}{(5\sec\theta)^2(5\tan\theta)} \cdot 5\sec\theta\tan\theta\,d\theta.$$

Simplifying:

$$\int_{\pi/4}^{\pi/3} \frac{1}{25\sec^2\theta \cdot 5\tan\theta} \cdot 5\sec\theta\tan\theta\,d\theta.$$

$$\int_{\pi/4}^{\pi/3} \frac{1}{25\sec\theta}\,d\theta.$$

$$\frac{1}{25}\int_{\pi/4}^{\pi/3} \cos\theta\,d\theta.$$

(g) **Evaluate the integral:**

$$\frac{1}{25}\left[\sin\theta\right]_{\pi/4}^{\pi/3}.$$

Computing:

$$\frac{1}{25}\left(\sin\frac{\pi}{3} - \sin\frac{\pi}{4}\right).$$

$$\frac{1}{25}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right).$$

$$\frac{\sqrt{3}-\sqrt{2}}{50}.$$

(h) **Final Answer:**

$$\frac{\sqrt{3}-\sqrt{2}}{50}.$$

7.4 Partial Fractions Solutions

1. $\displaystyle\int \frac{1}{x^2-4}\,dx$

   (a) **Factor the denominator:**
   $$x^2-4=(x-2)(x+2)$$

   (b) **Set up the partial fraction decomposition:**
   $$\frac{1}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}$$

   (c) **Multiply both sides by** $(x-2)(x+2)$:
   $$1=A(x+2)+B(x-2)$$

   (d) **Solve for A and B:**
   Substitute $x=2$:
   $$1=A(2+2)+B(2-2)\Rightarrow 1=4A\Rightarrow A=\frac{1}{4}$$
   Substitute $x=-2$:
   $$1=A(-2+2)+B(-2-2)\Rightarrow 1=-4B\Rightarrow B=-\frac{1}{4}$$

   (e) **Rewrite the integral:**
   $$\int\left(\frac{1/4}{x-2}-\frac{1/4}{x+2}\right)dx$$

   (f) **Integrate each term:**
   $$\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+C$$

2. $\displaystyle\int \frac{1}{(x-1)(x+2)}\,dx$

   (a) **Set up the partial fraction decomposition:**
   $$\frac{1}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$$

   (b) **Multiply both sides by** $(x-1)(x+2)$ to clear fractions:
   $$1=A(x+2)+B(x-1)$$

   (c) **Solve for A and B:**
   Substitute $x=1$:
   $$1=A(1+2)+B(1-1)\Rightarrow 1=3A\Rightarrow A=\frac{1}{3}$$
   Substitute $x=-2$:
   $$1=A(-2+2)+B(-2-1)\Rightarrow 1=-3B\Rightarrow B=-\frac{1}{3}$$

   (d) **Rewrite the integral:**
   $$\int\left(\frac{1/3}{x-1}-\frac{1/3}{x+2}\right)dx$$

   (e) **Integrate each term:**
   $$\frac{1}{3}\ln|x-1|-\frac{1}{3}\ln|x+2|+C$$

3. $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

   (a) **Factor the denominator:**
   $$x^2+3x+2=(x+1)(x+2)$$

   (b) **Set up the partial fraction decomposition:**
   $$\frac{1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

   (c) **Multiply both sides by** $(x+1)(x+2)$:
   $$1=A(x+2)+B(x+1)$$

   (d) **Solve for A and B:**
   Substitute $x=-1$:
   $$1=A(-1+2)+B(-1+1)\Rightarrow 1=A(1)\Rightarrow A=1$$
   Substitute $x=-2$:
   $$1=A(-2+2)+B(-2+1)\Rightarrow 1=B(-1)\Rightarrow B=-1$$

   (e) **Rewrite the integral:**
   $$\int\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx$$

   (f) **Integrate each term:**
   $$\ln|x+1|-\ln|x+2|+C$$

4. $\displaystyle\int \frac{2x+3}{x^2+x-2}\,dx$

   (a) **Factor the denominator:**
   $$x^2+x-2=(x-1)(x+2)$$

   (b) **Set up the partial fraction decomposition:**
   $$\frac{2x+3}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$$

   (c) **Multiply both sides by** $(x-1)(x+2)$:
   $$2x+3=A(x+2)+B(x-1)$$

   (d) **Solve for A and B:**
   Substitute $x=1$:
   $$2(1)+3=A(1+2)+B(1-1)$$
   $$\Rightarrow 2+3=3A\Rightarrow A=\frac{5}{3}$$
   Substitute $x=-2$:
   $$2(-2)+3=A(-2+2)+B(-2-1)$$
   $$\Rightarrow -4+3=-3B\Rightarrow B=\frac{1}{3}$$

   (e) **Rewrite the integral:**
   $$\int\left(\frac{5/3}{x-1}+\frac{1/3}{x+2}\right)dx$$

   (f) **Integrate each term:**
   $$\frac{5}{3}\ln|x-1|+\frac{1}{3}\ln|x+2|+C$$

5. $\displaystyle\int \frac{1}{x^3 - x}\,dx$

(a) **Factor the denominator:**

$$x^3 - x = x(x-1)(x+1)$$

(b) **Set up the partial fraction decomposition:**

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

(c) **Multiply both sides by** $x(x-1)(x+1)$ to clear fractions:

$$1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

(d) **Expand and solve for A, B, and C:** Expanding each term:

$$1 = A(x^2-1) + B(x^2+x) + C(x^2-x)$$

Grouping like terms:

$$1 = (A+B+C)x^2 + (B-C)x - A$$

Equating coefficients:

$$\begin{aligned} A+B+C &= 0 \\ B-C &= 0 \\ -A &= 1 \end{aligned}$$

Solving:

$$\begin{aligned} A &= -1 \\ B = C &= \frac{1}{2} \end{aligned}$$

(e) **Rewrite the integral:**

$$\int \left(\frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} \right) dx$$

(f) **Integrate each term:**

$$-\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

6. $\displaystyle\int \frac{x^2}{x^3 - 1}\,dx$

(a) **Use u-substitution:** Let $u = x^3 - 1$, so that $du = 3x^2 dx$.

(b) **Rewrite the integral:**

$$\int \frac{x^2}{x^3-1}\,dx = \int \frac{x^2}{u}\,dx$$

Since $du = 3x^2 dx$, we solve for dx:

$$dx = \frac{du}{3x^2}$$

Substituting:

$$\begin{gathered} \int \frac{x^2}{u} \cdot \frac{du}{3x^2} \\ \int \frac{1}{3} \cdot \frac{du}{u} \end{gathered}$$

(c) **Integrate:**

$$\frac{1}{3}\ln|u| + C$$

(d) **Substitute back** $u = x^3 - 1$:

$$\frac{1}{3}\ln|x^3 - 1| + C$$

(e) **Final Answer:**

$$\frac{1}{3}\ln|x^3 - 1| + C$$

Alternatively,

(a) **Factor the denominator:**

$$x^3 - 1 = (x-1)(x^2+x+1)$$

(b) **Set up the partial fraction decomposition:**

$$\frac{x^2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

(c) **Multiply both sides by** $(x-1)(x^2+x+1)$:

$$x^2 = A(x^2+x+1) + (Bx+C)(x-1)$$

(d) **Expand and solve for A, B, and C:** Expanding:

$$x^2 = A(x^2+x+1) + Bx^2 - Bx + Cx - C$$

Grouping like terms:

$$x^2 = (A+B)x^2 + (A-B+C)x + (A-C)$$

Equating coefficients:

$$\begin{aligned} A+B &= 1 \\ A-B+C &= 0 \\ A-C &= 0 \end{aligned}$$

Solving:

$$A = \frac{1}{3}, \quad B = \frac{2}{3}, \quad C = \frac{1}{3}$$

(e) **Rewrite the integral:**

$$\int \left(\frac{1/3}{x-1} + \frac{(2x+1)/3}{x^2+x+1} \right) dx$$

(f) **Integrate each term:**

$$\frac{1}{3}\ln|x-1| + \frac{1}{3}\int \frac{2x+1}{x^2+x+1}dx$$

Using substitution $u = x^2+x+1$, so $du = (2x+1)dx$:

$$\frac{1}{3}\ln|x-1| + \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|x-1| + \frac{1}{3}\ln|x^2+x+1| + C$$

(g) **Final Answer:**

$$\frac{1}{3}\ln\big|(x-1)(x^2+x+1)\big| + C$$

7. $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$

(a) **Set up the partial fraction decomposition:**

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

(b) **Multiply both sides by** $x(x^2+1)$ to clear fractions:

$$1 = A(x^2+1) + (Bx+C)x$$

(c) **Expand and solve for A, B, and C:** Expanding:

$$1 = Ax^2 + A + Bx^2 + Cx$$

Grouping like terms:

$$1 = (A+B)x^2 + Cx + A$$

Equating coefficients:

$$\begin{aligned} A+B &= 0 \\ C &= 0 \\ A &= 1 \end{aligned}$$

Solving:

$$A = 1, \quad B = -1, \quad C = 0$$

(d) **Rewrite the integral:**

$$\int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$

(e) **Integrate each term:**

$$\ln|x| - \frac{1}{2}\ln|x^2+1| + C$$

8. $\displaystyle\int \frac{x}{x^3+x^2}\,dx$

(a) **Factor the denominator:**

$$x^3 + x^2 = x^2(x+1)$$

(b) **Set up the partial fraction decomposition:**

$$\frac{x}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

(c) **Multiply both sides by** $x^2(x+1)$ to clear fractions:

$$x = Ax(x+1) + B(x+1) + Cx^2$$

(d) **Expand and solve for A, B, and C:** Expanding:

$$x = Ax^2 + Ax + Bx + B + Cx^2$$

Grouping like terms:

$$x = (A+C)x^2 + (A+B)x + B$$

Equating coefficients:

$$\begin{aligned} A+C &= 0 \\ A+B &= 1 \\ B &= 0 \end{aligned}$$

Solving:

$$B = 0, \quad A = 1, \quad C = -1$$

(e) **Rewrite the integral:**

$$\int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

(f) **Integrate each term:**

$$\ln|x| - \ln|x+1| + C$$

9. $\displaystyle\int \frac{1}{(x-1)^2(x+2)}\,dx$

(a) **Set up the partial fraction decomposition:**

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

(b) **Multiply both sides by** $(x-1)^2(x+2)$ to clear fractions:

$$1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

(c) **Expand the terms:**

$$\begin{aligned} A(x^2+x-2) &= Ax^2 + Ax - 2A \\ B(x+2) &= Bx + 2B \\ C(x^2-2x+1) &= Cx^2 - 2Cx + C \end{aligned}$$

Adding them together:

$$1 = (A+C)x^2 + (A-2C+B)x + (-2A+2B+C)$$

(d) **Solve for A, B, and C:** Equating coefficients:

$$\begin{aligned} A+C &= 0 \quad \text{(coefficient of } x^2\text{)} \\ A-2C+B &= 0 \quad \text{(coefficient of } x\text{)} \\ -2A+2B+C &= 1 \quad \text{(constant term)} \end{aligned}$$

Solving step-by-step:

$$C = -A \quad \text{(from first equation)}$$

Substitute into the second equation:

$$\begin{aligned} A - 2(-A) + B &= 0 \\ A + 2A + B &= 0 \\ 3A + B &= 0 \quad \Rightarrow B = -3A \end{aligned}$$

Substitute into the third equation:

$$\begin{aligned} -2A + 2(-3A) + (-A) &= 1 \\ -2A - 6A - A &= 1 \\ -9A &= 1 \\ A &= -\frac{1}{9} \end{aligned}$$

Using $B = -3A$:

$$B = -3 \times -\frac{1}{9} = \frac{1}{3}$$

Using $C = -A$:

$$C = -(-\frac{1}{9}) = \frac{1}{9}$$

Verified by substituting back into the equations.

(e) **Rewrite the integral:**

$$\int \left(\frac{-1/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{1/9}{x+2}\right) dx$$

(f) **Integrate each term:**

$$-\frac{1}{9}\ln|x-1| - \frac{1}{3}\cdot\frac{1}{x-1} + \frac{1}{9}\ln|x+2| + C$$

10. $\displaystyle\int \frac{x^4+2x^3+3x^2+4x+5}{(x-1)(x^2+x+1)}\,dx$

(a) **Perform Long Division** The numerator $x^4+2x^3+3x^2+4x+5$ has a higher degree than the denominator $x^3-1=(x-1)(x^2+x+1)$, so we perform polynomial long division:

$$\begin{array}{r||l}
 & x+2 \\
\hline
x^3-1 & x^4+2x^3+3x^2+4x+5 \\
 & x^4-x \\
\hline
 & 2x^3+3x^2+5x+5 \\
 & -(2x^3-2) \\
\hline
 & 3x^2+5x+7
\end{array}$$

This gives the quotient $x+2$ and remainder $3x^2+5x+7$, so we rewrite the integral:

$$\int\left(x+2+\frac{3x^2+5x+7}{(x-1)(x^2+x+1)}\right)dx.$$

(b) **Split the Integral**

$$\int(x+2)dx+\int\frac{3x^2+5x+7}{(x-1)(x^2+x+1)}dx.$$

(c) **Evaluate the First Integral**

$$\int x\,dx+\int 2\,dx=\frac{x^2}{2}+2x.$$

(d) **Partial Fraction Decomposition** Express the fraction as:

$$\frac{3x^2+5x+7}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.$$

Multiply both sides by $(x-1)(x^2+x+1)$ and expand:

$$3x^2+5x+7=A(x^2+x+1)+(Bx+C)(x-1).$$

Expanding and grouping like terms:

$$(A+B)x^2+(A-B+C)x+(A-C)=3x^2+5x+7.$$

Solving for A, B, C:

$$\begin{aligned}A+B&=3,\\ A-B+C&=5,\\ A-C&=7.\end{aligned}$$

Solving,

$$A=5,\quad B=-2,\quad C=-2.$$

(e) **Rewrite the Integral**

$$\int\left(\frac{5}{x-1}+\frac{-2x-2}{x^2+x+1}\right)dx.$$

(f) **Integrate Each Term**

- First term:

$$\int\frac{5}{x-1}dx=5\ln|x-1|.$$

- Second term:

$$\int\frac{-2x-2}{x^2+x+1}\,dx.$$

We split this integral into two parts:

$$-\int\frac{2x+1}{x^2+x+1}\,dx-\int\frac{1}{x^2+x+1}\,dx.$$

The first part uses the substitution:

$$u=x^2+x+1,\quad\text{so}\quad du=(2x+1)dx.$$

and so the first integral simplifies as:

$$-2\ln|x^2+x+1|.$$

We now evaluate the second integral:

$$\int\frac{1}{x^2+x+1}\,dx.$$

Completing the square:

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

This is now in the standard form:

$$\int\frac{1}{a^2+u^2}\,du=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right).$$

Here, $a^2=\frac{3}{4}$ so $a=\frac{\sqrt{3}}{2}$, and $u=x+\frac{1}{2}$. Using the formula:

$$\int\frac{1}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}\,dx=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

Since our integral had a coefficient of -1, we obtain:

$$-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

(g) **Final Answer**

$$\frac{x^2}{2}+2x+5\log|x-1|-\log|x^2+x+1|$$
$$-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C.$$

7.8 Improper Integrals Solutions

1. $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx$
   - **Antiderivative:**
   $$\int \frac{dx}{1+x^2} = \arctan x.$$
   - **Evaluate the Limit:**
   $$\lim_{b\to\infty}\Big[\arctan b - \arctan 0\Big] = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$
   - **Answer:** $\frac{\pi}{2}$.

2. $\displaystyle\int_0^{1} \frac{1}{\sqrt{x}}\,dx$
   - **Rewrite:** $\frac{1}{\sqrt{x}} = x^{-1/2}$.
   - **Integrate:**
   $$\int x^{-1/2}\,dx = 2x^{1/2} + C.$$
   - **Evaluate from 0 to 1:**
   $$2(1)^{1/2} - 2(0)^{1/2} = 2.$$
   - **Answer:** 2.

3. $\displaystyle\int_0^{\infty} e^{-x}\,dx$
   - **Antiderivative:**
   $$\int e^{-x}\,dx = -e^{-x} + C.$$
   - **Evaluate:**
   $$\lim_{b\to\infty}\Big[-e^{-b} + e^{0}\Big] = (0+1) = 1.$$
   - **Answer:** 1.

4. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$
   - **Antiderivative:**
   $$\int x^{-2}\,dx = -x^{-1} + C.$$
   - **Evaluate:**
   $$\lim_{b\to\infty}\left[-\frac{1}{b} + \frac{1}{1}\right] = 0 + 1 = 1.$$
   - **Answer:** 1.

5. $\displaystyle\int_0^{1} \frac{1}{\sqrt{1-x}}\,dx$
   - **Substitution:** Let $u = 1-x$ so that $du = -dx$.
   - **Change Limits:** When $x=0$, $u=1$; when $x=1$, $u=0$.
   - **Rewrite the Integral:**
   $$\int_0^1 \frac{dx}{\sqrt{1-x}} = -\int_1^0 u^{-1/2}\,du = \int_0^1 u^{-1/2}\,du.$$
   - **Integrate:**
   $$\int u^{-1/2}\,du = 2u^{1/2} \implies 2(1^{1/2} - 0^{1/2}) = 2.$$
   - **Answer:** 2.

6. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$
   - **Antiderivative:**
   $$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$$
   - **Evaluate:**
   $$\arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$
   - **Answer:** π.

7. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$
   - **Integration by Parts:** Let
   $$u = \ln x \implies du = \frac{dx}{x},$$
   $$dv = \frac{dx}{x^2} \implies v = -\frac{1}{x}.$$
   - **Apply Formula:**
   $$\int \frac{\ln x}{x^2}\,dx = u\,v - \int v\,du = -\frac{\ln x}{x} - \int\left(-\frac{1}{x}\right)\frac{dx}{x} = -\frac{\ln x}{x} + \int\frac{dx}{x^2}.$$
   - **Integrate the Remaining Term:**
   $$\int \frac{dx}{x^2} = -\frac{1}{x} + C.$$
   - **Evaluate from 1 to ∞:**
   $$\lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b}\right] - \left[-\frac{\ln 1}{1} - \frac{1}{1}\right] = \Big(0-0\Big) - \Big(0-(-1)\Big) = 1.$$
   - **Answer:** 1.

8. $\displaystyle\int_0^{1} \ln(x)\,dx$
   - **Antiderivative:**
   $$\int \ln x\,dx = x\ln x - x + C.$$
   - **Evaluate:** At $x=1$, $1\cdot\ln 1 - 1 = -1$. At $x=0$, note that $\lim_{x\to 0^+} x\ln x = 0$. Thus,
   $$(-1) - (0) = -1.$$
   - **Answer:** -1.

9. $\displaystyle\int_0^{\infty} \frac{x}{(1+x^2)^2}\,dx$
   - **Substitution:** Let $u = 1+x^2$ so that $du = 2x\,dx$, hence $x\,dx = \frac{1}{2}du$.
   - **Change Limits:** When $x=0$, $u=1$; as $x\to\infty$, $u\to\infty$.

- **Rewrite the Integral:**

$$\int_0^\infty \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{u=1}^\infty u^{-2}\,du.$$

- **Integrate:**

$$\frac{1}{2}\int_1^\infty u^{-2}\,du = \frac{1}{2}\left[-\frac{1}{u}\right]_1^\infty = \frac{1}{2}(0-(-1)) = \frac{1}{2}.$$

- **Answer:** $\frac{1}{2}$.

10. $\displaystyle\int_0^2 \frac{dx}{(2-x)^{1/3}}$

- **Substitution:** Let $u = 2-x$ so that $du = -dx$.
- **Change Limits:** When $x=0$, $u=2$; when $x=2$, $u=0$.
- **Rewrite the Integral:**

$$\int_0^2 \frac{dx}{(2-x)^{1/3}} = -\int_{u=2}^0 u^{-1/3}\,du = \int_0^2 u^{-1/3}\,du.$$

- **Integrate:**

$$\int u^{-1/3}\,du = \frac{3}{2}\,u^{2/3} + C,$$

$$\text{so}\quad \int_0^2 u^{-1/3}\,du = \frac{3}{2}\left[2^{2/3}-0\right] = \frac{3}{2}\,2^{2/3}.$$

- **Answer:** $\frac{3}{2}\,2^{2/3}$.

11. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

- **Substitution:** Let $x = \sin\theta$, so that $dx = \cos\theta\,d\theta$ and $\sqrt{1-x^2} = \cos\theta$.
- **Change Limits:** When $x=0$, $\theta=0$; when $x=1$, $\theta=\frac{\pi}{2}$.
- **Rewrite the Integral:**

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\,d\theta = \int_0^{\pi/2} \sin^2\theta\,d\theta.$$

- **Use the Identity:** $\sin^2\theta = \frac{1}{2}\Big(1-\cos 2\theta\Big)$:

$$\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2}\Big(1-\cos 2\theta\Big)\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

- **Answer:** $\frac{\pi}{4}$.

12. $\displaystyle\int_0^{\pi/2} \tan x\,dx$

- **Antiderivative:**

$$\int \tan x\,dx = -\ln|\cos x| + C.$$

- **Evaluate the Limit:** At $x=\pi/2$, $\cos x \to 0$ so that $-\ln|\cos x| \to \infty$. Thus,

$$\lim_{x\to\pi/2^-}(-\ln|\cos x|) - (-\ln|\cos 0|) = \infty - 0.$$

- **Answer:** The integral diverges.

13. $\displaystyle\int_1^\infty \frac{1}{x\,(\ln x)^2}\,dx$

- **Substitution:** Let $u = \ln x$, so that $du = \frac{dx}{x}$.
- **Change Limits:** When $x=1$, $u=0$; as $x\to\infty$, $u\to\infty$.
- **Rewrite the Integral:**

$$\int_1^\infty \frac{dx}{x\,(\ln x)^2} = \int_0^\infty \frac{du}{u^2}.$$

- **Examine the Integral:**

$$\int_\varepsilon^A \frac{du}{u^2} = \left[-\frac{1}{u}\right]_\varepsilon^A = -\frac{1}{A} + \frac{1}{\varepsilon}.$$

As $\varepsilon \to 0^+$, $\frac{1}{\varepsilon} \to \infty$.

- **Answer:** The integral diverges.

14. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$

- **Substitution:** Let $x = t^2$ so that $dx = 2t\,dt$ and $\sqrt{x} = t$.
- **Change Limits:** When $x=0$, $t=0$; as $x\to\infty$, $t\to\infty$.
- **Rewrite the Integral:**

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^\infty \frac{2t\,dt}{t\,(1+t^2)} = 2\int_0^\infty \frac{dt}{1+t^2}.$$

- **Integrate:**

$$2\int_0^\infty \frac{dt}{1+t^2} = 2\left[\arctan t\right]_0^\infty = 2\left(\frac{\pi}{2}-0\right) = \pi.$$

- **Answer:** π.