MATH 2300 — Exam 1 Review Problems (Solutions)

§5.5: Suppose that

$$\int_0^5 f(x)\,dx = 10.$$

Evaluate

$$\int_0^{\sqrt{5}} 2x\,f(x^2)\,dx.$$

- Use the substitution

$$u = x^2 \quad \Longrightarrow \quad du = 2x\,dx.$$

- Change the limits: when $x = 0$, $u = 0$; when $x = \sqrt{5}$, $u = 5$.
- The integral becomes:

$$\begin{aligned}\int_0^{\sqrt{5}} 2x\,f(x^2)\,dx &= \int_0^5 f(u)\,du \\ &= 10.\end{aligned}$$

- $\boxed{10}$

§7.1: Evaluate the integral

$$\int_1^4 \sqrt{x}\,\ln x\,dx.$$

- **Choose parts:** Let

$$u = \ln x \quad \text{and} \quad dv = \sqrt{x}\,dx = x^{1/2}dx.$$

- **Differentiate and integrate:** Then,

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = \frac{2}{3}x^{3/2}.$$

- **Apply integration by parts:**

$$\begin{aligned}\int \sqrt{x}\,\ln x\,dx &= u\,v - \int v\,du \\ &= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{3/2}\cdot\frac{dx}{x} \\ &= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C.\end{aligned}$$

- **Evaluate from $x = 1$ to $x = 4$:** Since $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$ and $1^{3/2} = 1$, we have

$$\begin{aligned}I &= \left[\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2}\right]_1^4 \\ &= \left(\frac{2}{3}\cdot 8\ln 4 - \frac{4}{9}\cdot 8\right) - \left(\frac{2}{3}\cdot 1\ln 1 - \frac{4}{9}\cdot 1\right) \\ &= \left(\frac{16}{3}\ln 4 - \frac{32}{9}\right) - \left(0 - \frac{4}{9}\right) \\ &= \frac{16}{3}\ln 4 - \frac{32}{9} + \frac{4}{9} \\ &= \frac{16}{3}\ln 4 - \frac{28}{9}.\end{aligned}$$

- **Express the final answer:** Since $\ln 4 = 2\ln 2$, we may also write:

$$I = \frac{32}{3}\ln 2 - \frac{28}{9}.$$

- $\boxed{\frac{32}{3}\ln 2 - \frac{28}{9}}$

§7.1: Evaluate the integral

$$\int e^x \sin x \, dx.$$

- Let

$$I = \int e^x \sin x \, dx.$$

- Apply integration by parts. Choose:

$$u = \sin x \quad \text{and} \quad dv = e^x \, dx.$$

Then,

$$du = \cos x \, dx \quad \text{and} \quad v = e^x.$$

- By the integration by parts formula,

$$I = u\,v - \int v\,du = e^x \sin x - \int e^x \cos x \, dx.$$

- Let

$$J = \int e^x \cos x \, dx.$$

Apply integration by parts again with:

$$u = \cos x \quad \text{and} \quad dv = e^x \, dx.$$

Then,

$$du = -\sin x \, dx \quad \text{and} \quad v = e^x.$$

- Using integration by parts for J:

$$\begin{aligned} J &= u\,v - \int v\,du \\ &= e^x \cos x - \int e^x (-\sin x) \, dx \\ &= e^x \cos x + \int e^x \sin x \, dx \\ &= e^x \cos x + I. \end{aligned}$$

- Substitute the expression for J back into the equation for I:

$$\begin{aligned} I &= e^x \sin x - J \\ &= e^x \sin x - \Big(e^x \cos x + I\Big). \end{aligned}$$

- Solve for I:

$$\begin{aligned} I &= e^x \sin x - e^x \cos x - I, \\ 2I &= e^x \sin x - e^x \cos x, \\ I &= \frac{e^x}{2}\Big(\sin x - \cos x\Big) + C. \end{aligned}$$

- $\boxed{\frac{e^x}{2}(\sin x - \cos x) + C}$

§7.2: Compute

$$\int \sec^4(x)\tan^3(x)\,dx.$$

- **Substitute:** Let

$$u = \tan x \quad \Longrightarrow \quad du = \sec^2 x\,dx.$$

- **Express the integrand in terms of u:**
  - Write one factor of $\sec^2 x$ in the integrand as:

$$\sec^4 x = \sec^2 x \cdot \sec^2 x = \sec^2 x\,(1+\tan^2 x) = \sec^2 x\,(1+u^2).$$

  - Thus, the integrand becomes:

$$\sec^4(x)\tan^3(x)\,dx = \tan^3 x\,(1+\tan^2 x)\,\sec^2 x\,dx.$$

- **Rewrite in terms of u:** Replace $\tan x$ by u and $\sec^2 x\,dx$ by du:

$$\int \sec^4(x)\tan^3(x)\,dx = \int u^3(1+u^2)\,du.$$

- **Expand and integrate:**

$$\begin{aligned}\int u^3(1+u^2)\,du &= \int \left(u^3+u^5\right)du\\ &= \int u^3\,du + \int u^5\,du\\ &= \frac{u^4}{4}+\frac{u^6}{6}+C.\end{aligned}$$

- **Substitute back:** Replace u by $\tan x$ to obtain the final answer:

$$\boxed{\frac{\tan^4 x}{4}+\frac{\tan^6 x}{6}+C}.$$

§7.3: Evaluate

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

- Use the substitution $x = 2\tan\theta$ so that:

$$dx = 2\sec^2\theta\,d\theta, \quad x^2 = 4\tan^2\theta, \quad \sqrt{x^2+4} = 2\sec\theta.$$

- Substitute into the integral:

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta\,d\theta}{(4\tan^2\theta)(2\sec\theta)}\\ &= \frac{1}{4}\int \frac{\sec\theta\,d\theta}{\tan^2\theta}.\end{aligned}$$

- Rewrite:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta} = \frac{\cos\theta}{\sin^2\theta}.$$

- Thus, the integral becomes:

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

- Let $u = \sin\theta$ so that $du = \cos\theta\,d\theta$. Then:

$$\frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\left(-\frac{1}{u}\right) + C.$$

- Return to x: since $x = 2\tan\theta$, the triangle gives

$$\sin\theta = \frac{x}{\sqrt{x^2+4}}.$$

- Therefore, the antiderivative is:

$$-\frac{1}{4\sin\theta} = -\frac{\sqrt{x^2+4}}{4x} + C.$$

- $\boxed{-\frac{\sqrt{x^2+4}}{4x} + C}$

§7.4: Find
$$\int \frac{1}{x^2-4x+3}\,dx.$$

- Factor the denominator:
$$x^2-4x+3=(x-1)(x-3).$$
- Write the partial fraction decomposition:
$$\frac{1}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}.$$
- Multiply both sides by $(x-1)(x-3)$:
$$1=A(x-3)+B(x-1).$$
- Let $x=3$:
$$1=A(0)+B(2) \quad\Longrightarrow\quad B=\frac{1}{2}.$$
- Let $x=1$:
$$1=A(-2)+B(0) \quad\Longrightarrow\quad A=-\frac{1}{2}.$$
- Hence,
$$\int \frac{dx}{x^2-4x+3}=-\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x-3|+C=\frac{1}{2}\ln\left|\frac{x-3}{x-1}\right|+C.$$
- $\boxed{\frac{1}{2}\ln\left|\frac{x-3}{x-1}\right|+C}$

§7.4: Set up the partial fraction decomposition for

$$\frac{4x^2+7x+3}{(5x+3)^3(x^2+1)(x+1)^2}.$$

You do not need to solve for the constants; simply express the decomposition in the proper form.

- Recognize that the denominator consists of three distinct factors:
  - A linear factor raised to the third power: $(5x+3)^3$.
  - An irreducible quadratic factor: x^2+1.
  - A repeated linear factor: $(x+1)^2$.
- For the linear factor $(5x+3)^3$, the corresponding partial fractions are:

$$\frac{A}{5x+3}+\frac{B}{(5x+3)^2}+\frac{C}{(5x+3)^3}.$$

- For the irreducible quadratic factor x^2+1, the corresponding partial fraction is:

$$\frac{Dx+E}{x^2+1}.$$

- For the repeated linear factor $(x+1)^2$, the corresponding partial fractions are:

$$\frac{F}{x+1}+\frac{G}{(x+1)^2}.$$

- Therefore, the complete partial fraction decomposition is:

$$\boxed{\frac{4x^2+7x+3}{(5x+3)^3(x^2+1)(x+1)^2}=\frac{A}{5x+3}+\frac{B}{(5x+3)^2}+\frac{C}{(5x+3)^3}+\frac{Dx+E}{x^2+1}+\frac{F}{x+1}+\frac{G}{(x+1)^2}}.$$

- (Note: To determine the constants A, B, C, D, E, F, and G, you would multiply through by the common denominator and equate coefficients of like powers of x. However, for this problem you are only required to set up the decomposition.)

§7.7: Suppose f is continuous, decreasing, and concave down on $[2, 8]$. Let L_4, R_4, T_4, M_4 denote the left-hand, right-hand, trapezoidal, and midpoint approximations to

$$\int_2^8 f(x)\,dx,$$

using 4 equal subintervals. Order these approximations from smallest to largest.

- For a decreasing function:
  - The left-hand sum L_4 is an *overestimate*.
  - The right-hand sum R_4 is an *underestimate*.
- With concavity (concave down), the midpoint approximation M_4 overestimates, and the trapezoidal rule T_4 underestimates.
- Therefore, the order is:

$$R_4 < T_4 < I < M_4 < L_4.$$

- $\boxed{R_4 < T_4 < I < M_4 < L_4}$

§7.8: Given that

$$\int_3^\infty \frac{1}{x^2+4}\,dx$$

converges, use the Comparison Theorem to determine the convergence of

$$\int_3^\infty \frac{\sin^2 x}{x^2+4}\,dx.$$

- Note that for all x:

$$0 \le \sin^2 x \le 1.$$

- Hence,

$$0 \le \frac{\sin^2 x}{x^2+4} \le \frac{1}{x^2+4}.$$

- Since $\int_3^\infty \frac{1}{x^2+4}\,dx$ converges, the Comparison Theorem implies that $\int_3^\infty \frac{\sin^2 x}{x^2+4}\,dx$ also converges.

- $\boxed{\text{The integral converges.}}$

§7.8: Determine whether the following improper integral converges or diverges. If it converges, compute its value:

$$\int_2^4 \frac{1}{\sqrt{4-x}}\,dx.$$

- The integrand becomes unbounded as $x \to 4^-$. Write the integral as a limit:

$$\int_2^4 \frac{dx}{\sqrt{4-x}} = \lim_{t\to 4^-} \int_2^t \frac{dx}{\sqrt{4-x}}.$$

- Substitute $u = 4-x$ so that $du = -dx$. Change the limits: when $x = 2$, $u = 2$; when $x = t$, $u = 4-t$.

- Then,

$$\begin{aligned}\int_2^t \frac{dx}{\sqrt{4-x}} &= \int_{u=2}^{4-t} \frac{-du}{\sqrt{u}} \\ &= \int_{u=4-t}^{2} \frac{du}{\sqrt{u}}.\end{aligned}$$

- Compute the antiderivative:

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}.$$

- Thus,

$$\int_{4-t}^{2} \frac{du}{\sqrt{u}} = 2\sqrt{2} - 2\sqrt{4-t}.$$

- Taking the limit as $t \to 4^-$ (so that $\sqrt{4-t} \to 0$):

$$\lim_{t\to 4^-}\left(2\sqrt{2} - 2\sqrt{4-t}\right) = 2\sqrt{2}.$$

- $\boxed{2\sqrt{2}}$

§7.8: Determine whether the following improper integral converges or diverges. If it converges, compute its value:

$$\int_0^{\pi/2} \tan x \, dx.$$

- Rewrite $\tan x$ as $\tan x = \dfrac{\sin x}{\cos x}$..
- Let $u = \cos x$. So $du = -\sin x \, dx$.
- When $x = 0$, $u = \cos 0 = 1$. When $x = \pi/2$: $u = \cos(\pi/2) = 0$.
- The integral becomes:

$$\begin{aligned} \int_{x=0}^{x=\pi/2} \tan x \, dx &= \int_{u=1}^{u=0} \left(-\frac{1}{u}\right) du \\ &= \int_{u=1}^{u=0} -\frac{du}{u} \\ &= \int_{u=0}^{u=1} \frac{du}{u}. \end{aligned}$$

- This is an improper integral at the lower limit. Write it as a limit:

$$\int_0^1 \frac{du}{u} = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{du}{u} = \lim_{\epsilon \to 0^+} \Big[\ln |u|\Big]_\epsilon^1.$$

  Since $\ln 1 = 0$, this becomes:

$$\lim_{\epsilon \to 0^+} \Big(0 - \ln \epsilon\Big) = \lim_{\epsilon \to 0^+} \big(-\ln \epsilon\big).$$

  As $\epsilon \to 0^+$, $\ln \epsilon \to -\infty$, so

$$-\ln \epsilon \to +\infty.$$

- Since the limit diverges to $+\infty$, the original integral does not converge.
- $\boxed{\text{The integral diverges.}}$