9.3 Separable Differential Equations

Definition. A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. It can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

Question. How do we solve separable differential equations?

Solution.

1. Rewrite the equation by dividing both sides by $f(y)$:

   Let $h(y) = \frac{1}{f(y)}$, so the equation becomes

2. Multiply through by dx to separate variables:

3. Integrate both sides:

   This gives an implicit relationship between y and x.

4. **Why does this work?** Taking the derivative of both sides with respect to x,

   Applying the Chain Rule to the left-hand side,

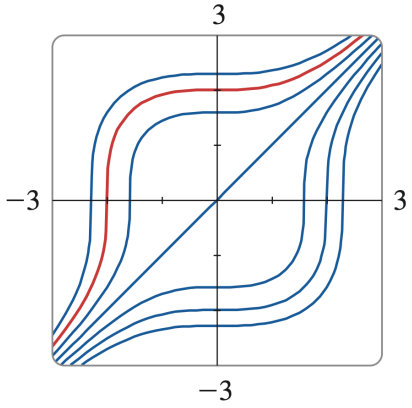
   so

   which matches the original separated form.

Example. Solve the differential equation

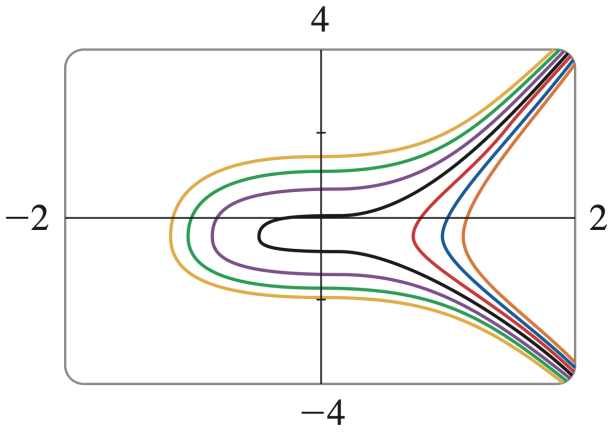
$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

Find the solution that satisfies the initial condition $y(0) = 2$.



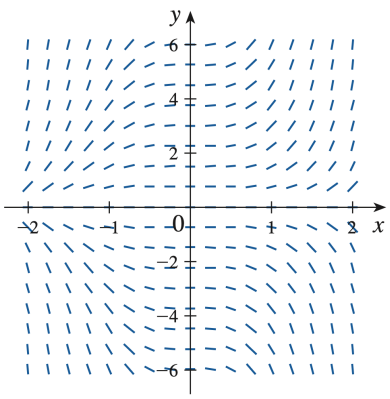
Example. Solve the differential equation

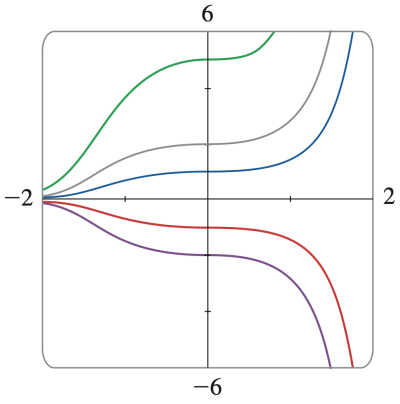
$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}.$$



Example. Solve the differential equation

$$y' = x^2 y.$$





Example. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine containing 0.03 kg/L of salt enters at 25 L/min, and the solution drains at the same rate. How much salt is in the tank after half an hour?