9.3 Separable Differential Equations (Solutions)

Multiple Choice Practice

1. **(C)** $\dfrac{dy}{dx} = xy$
   Separable: $\frac{1}{y}dy = x\,dx$

2. **(C)** $\dfrac{dy}{dx} = x + y$
   Not separable — cannot write as a product $g(x)h(y)$

3. **(C)** $y = Ce^{x^3/3}$
   From $\frac{dy}{dx} = x^2 y \Rightarrow \frac{1}{y}dy = x^2 dx \Rightarrow \ln|y| = \frac{1}{3}x^3 + C$

4. **(A)** $y = Cx^3$
   From $\frac{dy}{dx} = \frac{3y}{x} \Rightarrow \frac{1}{y}dy = \frac{3}{x}dx \Rightarrow \ln|y| = 3\ln|x| + C$

5. **(A)** $y = \dfrac{1}{2-x}$
   From $\frac{dy}{dx} = y^2 \Rightarrow -\frac{1}{y} = x + C$, use $y(0) = 2 \Rightarrow C = -\frac{1}{2}$

6. **(B)** $y = 3e^{x - x^3/3}$
   Separable: $\frac{1}{y}dy = (1 - x^2)dx \Rightarrow \ln|y| = x - \frac{1}{3}x^3 + C$, then apply initial condition

7. **(B)** $y = Cx^2$
   From $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{1}{y}dy = \frac{2}{x}dx \Rightarrow \ln|y| = 2\ln|x| + C$

8. **(A)** $y^2 = \dfrac{2x^3}{3} + C$
   From $\frac{dy}{dx} = \frac{x^2}{y} \Rightarrow y\,dy = x^2 dx \Rightarrow \frac{1}{2}y^2 = \frac{2}{3}x^3 + C$

Free Response Practice

1. Solve $y' = xy$

Note: $y = 0$ is a contant solution. If $y \neq 0$, then

$$\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{1}{y}\,dy &= x\,dx \\
\int \frac{1}{y}\,dy &= \int x\,dx \\
\ln|y| &= \frac{1}{2}x^2 + C \\
|y| &= e^{\frac{1}{2}x^2 + C} = e^C \cdot e^{x^2/2} \\
y &= \pm e^C \cdot e^{x^2/2} \\
y &= Ae^{x^2/2}
\end{aligned}$$

where A is any constant, including 0.

2. Solve $y' = \frac{2y}{x}$

Note: $y = 0$ is a contant solution. If $y \neq 0$, then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2y}{x} \\
\frac{1}{y}\,dy &= \frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx \\
\ln|y| &= 2\ln|x| + C \\
|y| &= e^C \cdot |x|^2 \\
y &= \pm e^C \cdot x^2 \\
y &= Ax^2
\end{aligned}$$

where A is any constant, including 0.

3. Solve $y' = x(1 + y^2)$

$$\begin{aligned}
\frac{dy}{dx} &= x(1 + y^2) \\
\frac{1}{1+y^2}\,dy &= x\,dx \\
\int \frac{1}{1+y^2}\,dy &= \int x\,dx \\
\arctan y &= \frac{1}{2}x^2 + C \\
y &= \tan\left(\frac{1}{2}x^2 + C\right)
\end{aligned}$$

4. Solve $y' = y^2$, with $y(0) = 1$

$$\begin{aligned}
\frac{dy}{dx} &= y^2 \\
\frac{1}{y^2}\,dy &= dx \\
\int y^{-2}\,dy &= \int dx \\
-\frac{1}{y} &= x + C \Rightarrow y = \frac{-1}{x+C} \\
y(0) = 1 \Rightarrow \frac{-1}{C} &= 1 \Rightarrow C = -1 \\
&\boxed{y = \frac{-1}{x-1}}
\end{aligned}$$

5. Solve $y' = (1-x^2)y$, with $y(0) = 2$

$$\begin{aligned}
\frac{dy}{dx} &= (1-x^2)y \\
\frac{1}{y}\,dy &= (1-x^2)\,dx \\
\int \frac{1}{y}\,dy &= \int (1-x^2)\,dx \\
\ln|y| &= x - \frac{1}{3}x^3 + C \\
|y| &= e^C \cdot e^{x-x^3/3} \\
y &= \pm e^C \cdot e^{x-x^3/3} = Ae^{x-x^3/3}
\end{aligned}$$

$$y(0) = 2 \Rightarrow A = 2$$

$$\boxed{y = 2e^{x-x^3/3}}$$

6. Solve $y' = x^2(1+y)$, with $y(1) = 0$

$$\begin{aligned}
\frac{dy}{dx} &= x^2(1+y) \\
\frac{1}{1+y}\,dy &= x^2\,dx \\
\int \frac{1}{1+y}\,dy &= \int x^2\,dx \\
\ln|1+y| &= \frac{1}{3}x^3 + C \\
|1+y| &= e^C \cdot e^{x^3/3} \\
1+y &= \pm e^C \cdot e^{x^3/3} = Ae^{x^3/3} \\
y &= Ae^{x^3/3} - 1 \\
y(1) = 0 &\Rightarrow Ae^{1/3} = 1 \Rightarrow A = e^{-1/3}
\end{aligned}$$

$$\boxed{y = e^{(x^3-1)/3} - 1}$$

7. Solve $y' = \frac{y}{x}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln|y| &= \ln|x| + C \\
|y| &= e^C|x| \\
y &= \pm e^C x = Ax
\end{aligned}$$

$$\boxed{y = Ax}$$

8. Solve $y' = \frac{x}{y}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x}{y} \\
y\,dy &= x\,dx \\
\int y\,dy &= \int x\,dx \\
\frac{1}{2}y^2 &= \frac{1}{2}x^2 + C
\end{aligned}$$

$$\boxed{y^2 = x^2 + C}$$

9. Solve $y' = \frac{2y}{x+1}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2y}{x+1} \\
\frac{1}{y}\,dy &= \frac{2}{x+1}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x+1}\,dx \\
\ln|y| &= 2\ln|x+1| + C \\
|y| &= e^C|x+1|^2 \\
y &= \pm e^C(x+1)^2 = A(x+1)^2
\end{aligned}$$

$$\boxed{y = A(x+1)^2}$$