9.3 Separable Differential Equations

Definition. A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. It can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

Question. How do we solve separable differential equations?

Solution.

1. Rewrite the equation by dividing both sides by $f(y)$:

$$\frac{1}{f(y)} \cdot \frac{dy}{dx} = g(x)$$

   Let $h(y) = \frac{1}{f(y)}$, so the equation becomes

$$h(y) \frac{dy}{dx} = g(x)$$

2. Multiply through by dx to separate variables:

$$h(y)\, dy = g(x)\, dx$$

3. Integrate both sides:

$$\int h(y)\, dy = \int g(x)\, dx$$

   This gives an implicit relationship between y and x.

4. **Why does this work?** Taking the derivative of both sides with respect to x,

$$\frac{d}{dx}\left(\int h(y)\, dy \right) = \frac{d}{dx}\left(\int g(x)\, dx \right)$$

   Applying the Chain Rule to the left-hand side,

$$\frac{d}{dy}\left(\int h(y)\, dy \right) \cdot \frac{dy}{dx} = g(x)$$

   so

$$h(y) \cdot \frac{dy}{dx} = g(x)$$

   which matches the original separated form.

This justifies moving the dx to the other side in Step 2

Example. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

Find the solution that satisfies the initial condition $y(0) = 2$.

Separate variables: $y^2\,dy = x^2\,dx$

Integrate both sides: $\int y^2\,dy = \int x^2\,dx$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

Solve for y: $y^3 = x^3 + 3C$

$$y = \sqrt[3]{x^3 + 3C}$$

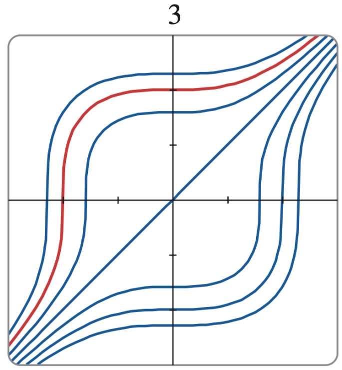
$$y = \sqrt[3]{x^3 + K}$$

Replace $3C$ by a new constant K

Apply the initial condition:

$$2 = \sqrt[3]{0^3 + K} \Rightarrow 2 = \sqrt[3]{K} \Rightarrow K = 8$$

Final Answer $\boxed{y = \sqrt[3]{x^3 + 8}}$



Example. Solve the differential equation

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}.$$

Separate Variables:

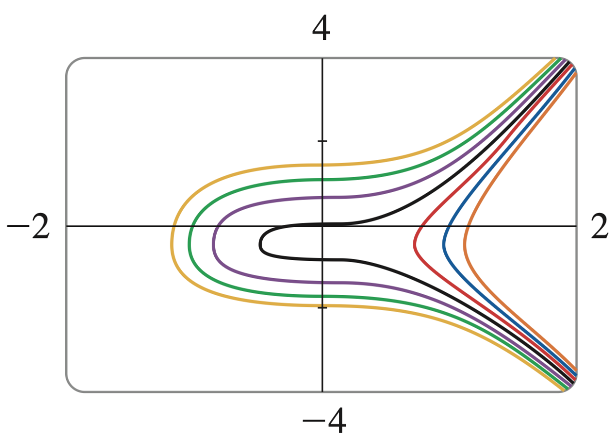
$$(2y + \cos y)\, dy = 6x^2\, dx$$

Integrate both sides:

$$\int 2y + \cos y \; dy = \int 6x^2\, dx$$

$$y^2 + \sin y = 2x^3 + C$$

Note: It is not possible to solve for y as a function of x, so we leave the solution as an implicit relationship.



Example. Solve the differential equation

$$y' = x^2 y.$$

Rewrite the equation: $\frac{dy}{dx} = x^2 y$

Note: $y = 0$ is a constant solution. If $y \neq 0$, we can separate variables.

$$\frac{dy}{y} = x^2\,dx \Rightarrow \int \frac{1}{y}\,dy = \int x^2\,dx \Rightarrow \ln|y| = \frac{x^3}{3} + C$$

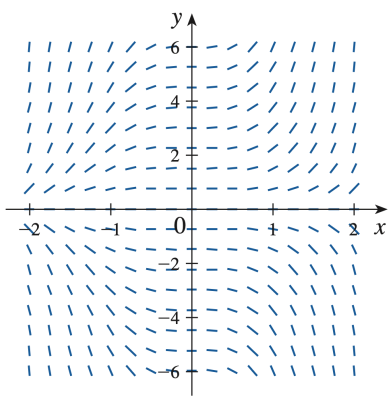
Solve for y:

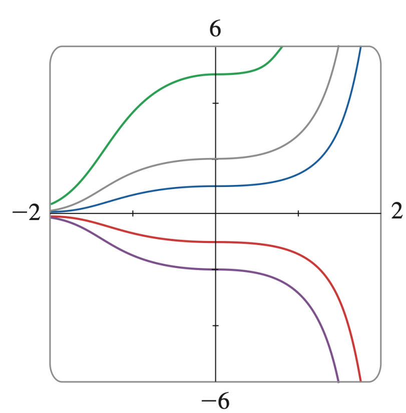
$$|y| = e^{x^3/3 + C} = e^C \cdot e^{x^3/3}$$

$$\Rightarrow y = \pm e^C \cdot e^{x^3/3}$$ Let $A = \pm e^C$ and absorb into one constant

$$\Rightarrow y = A \cdot e^{x^3/3}$$

A can be any real number, including 0.





Bonus

Example. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine containing 0.03 kg/L of salt enters at 25 L/min, and the solution drains at the same rate. How much salt is in the tank after half an hour?

Let $y(t)$ be the amount of salt (in kg) at time t (in minutes)

We know $y(0) = 20$ and we want $y(30)$

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

Rate in

$0.03 \text{ kg/L} \cdot 25 \text{ L/min}$

$= 0.75 \text{ kg/min}$

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200}$$

Rate out

$y(t)/5000 \text{ kg/L} \cdot 25 \text{ L/min}$

$= y(t)/200 \text{ kg/min}$

$$\frac{dy}{dt} = \frac{150 - y(t)}{200}$$

Solve: $\displaystyle\int \frac{1}{150-y}\, dy = \int \frac{1}{200}\, dt$

Apply the initial condition $y(0) = 20$

$$y(t) = 150 - 130e^{-t/200}$$

Compute $y(30) = 38.1$ kg