9.1 Differential Equations

Definition. A **differential equation** is an equation involving an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that appears in the equation.

Example.

1. $y' = y$

The simplest exponential growth equation. The more y you have, the faster it grows—like compound interest or unrestricted population growth.

2. $y'' = 0$

The second derivative being zero means the slope is constant. So y must be a linear function. This represents motion at constant velocity.

3. $y'' = 1$

Constant acceleration—like an object falling under gravity (ignoring air resistance). The solution will be a quadratic function of x.

4. $y'' + y = 0$

This describes simple harmonic motion—systems like a mass on a spring or a pendulum (for small angles). The solution is sinusoidal: $y = \sin x$ or $\cos x$.

5. $y'' + 4y = 0$

Another harmonic oscillator, but with a faster frequency. Appears when the spring is stiffer or the object oscillates faster.

6. $y' = xy$

This first-order equation says the rate of change of y depends on both x and y. It shows up in problems where growth depends on position, like certain population or physics models.

Definition. To **solve** a differential equation means to find all functions that satisfy the equation.

Example. Consider the differential equation $y' = x^3$. The general solution is $y = \frac{x^4}{4} + C$, where C is an arbitrary constant.

Example. Determine whether the function $y = x + \frac{1}{x}$ is a solution of the given differential equation.

(a) $xy' + y = 2x$

(b) $xy'' + 2y' = 0$

Example. Determine whether the given function is a solution of the differential equation:

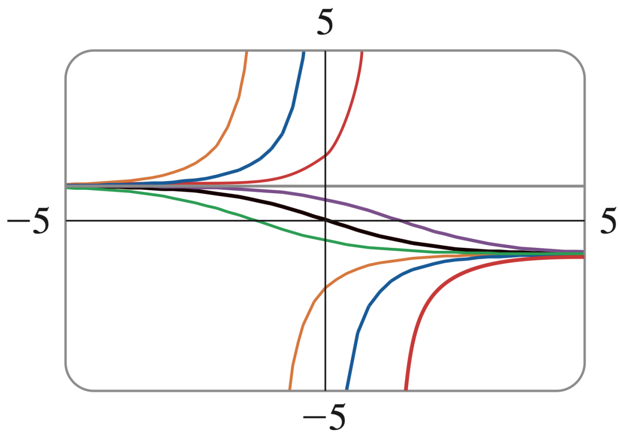
$$y = e^{2x}, \qquad y' - 2y = 0$$

Example. Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation

$$y' = \frac{1}{2}(y^2 - 1).$$



In practice, we often care less about finding the whole family of solutions (the *general solution*) and more about finding the specific one that meets a given condition like $y(t_0) = y_0$. This condition is called an **initial condition**, and solving the differential equation with this condition is called an **initial-value problem**.

Visually, an initial condition helps us pick out the one curve from the family of solutions that passes through the point (t_0, y_0).

Example. Find a solution of the differential equation

$$y' = \frac{1}{2}(y^2 - 1)$$

that satisfies the initial condition $y(0) = 2$.