## 9.1 Differential Equations

**Definition.** A **differential equation** is an equation involving an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that appears in the equation.

**Example.**

1. $y' = y$ If $\boxed{y=e^x}$ then $y'=e^x$ so $y'=y$ ✓

   The simplest exponential growth equation. The more $y$ you have, the faster it grows—like compound interest or unrestricted population growth.

2. $y'' = 0$ If $\boxed{y=mx+b}$ then $y''=0$ ✓

   The second derivative being zero means the slope is constant. So $y$ must be a linear function. This represents motion at constant velocity.

3. $y'' = 1$ If $\boxed{y=\frac{1}{2}x^2+bx+c}$ then $y''=1$ ✓

   Constant acceleration—like an object falling under gravity (ignoring air resistance). The solution will be a quadratic function of $x$.

4. $y'' + y = 0$ If $\boxed{y=\sin x}$ then $y''=-\sin x$ so $y''+y=0$ ✓

   This describes simple harmonic motion—systems like a mass on a spring or a pendulum (for small angles). The solution is sinusoidal: $y = \sin x$ or $\cos x$.

5. $y'' + 4y = 0$ If $\boxed{y=\sin 2x}$ then $y''=-4\sin 2x$ so $y''+4y=0$ ✓

   Another harmonic oscillator, but with a faster frequency. Appears when the spring is stiffer or the object oscillates faster.

6. $y' = xy$ If $\boxed{y=e^{\frac{1}{2}x^2}}$ then $y'=xe^{\frac{1}{2}x^2}=xy$ ✓

   This first-order equation says the rate of change of $y$ depends on both $x$ and $y$. It shows up in problems where growth depends on position, like certain population or physics models.

**Definition.** To **solve** a differential equation means to find all functions that satisfy the equation.

**Example.** Consider the differential equation $y' = x^3$. The general solution is $y = \frac{x^4}{4} + C$, where $C$ is an arbitrary constant.

**Example.** Determine whether the function $y = x + \frac{1}{x}$ is a solution of the given differential equation.

(a) $xy' + y = 2x$

(b) $xy'' + 2y' = 0$

We first compute $y' = 1 - \frac{1}{x^2}$ and $y'' = \frac{2}{x^3}$

(a) $xy' + y = x\left(1 - \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right)$

$= x - \frac{1}{x} + x + \frac{1}{x}$

$= 2x$ ✓

(b) $xy'' + 2y' = x\left(\frac{2}{x^3}\right) + 2\left(1 - \frac{1}{x^2}\right)$

$= \frac{2}{x^2} + 2 - \frac{2}{x^2}$

$= 2$ ✗

**Example.** Determine whether the given function is a solution of the differential equation:

$$y = e^{2x}, \qquad y' - 2y = 0$$

1. $y' = 2e^{2x}$

2. So, $y' - 2y = 2e^{2x} - 2 \cdot e^{2x} = 0$

3. Hence $y = e^{2x}$ is a solution to $y' - 2y$ ✓

**Example.** Show that every member of the family of functions

$$y = \frac{1+ce^t}{1-ce^t}$$

is a solution of the differential equation

$$y' = \frac{1}{2}(y^2 - 1).$$

Quotient rule

Expand

① $$y' = \frac{(1-ce^t)(ce^t) - (1+ce^t)(-ce^t)}{(1-ce^t)^2} = \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1-ce^t)^2}$$

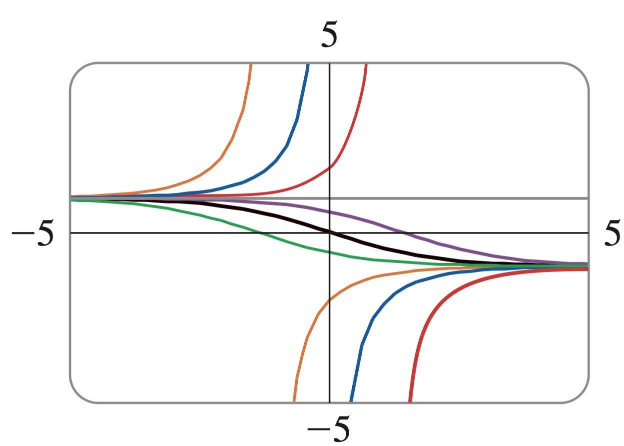
Simplify $$= \frac{2ce^t}{(1-ce^t)^2}$$

Common denominator

② $$\frac{1}{2}(y^2-1) = \frac{1}{2}\left[\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1\right] = \frac{1}{2}\left[\frac{(1+ce^t)^2}{(1-ce^t)^2} - \frac{(1-ce^t)^2}{(1-ce^t)^2}\right]$$

combine $$= \frac{1}{2} \cdot \frac{4ce^t}{(1-ce^t)^2} = \frac{2ce^t}{(1-ce^t)^2}$$

Hence, for every value of $c$, $y' = \frac{1}{2}(y^2-1)$ ✓



In practice, we often care less about finding the whole family of solutions (the *general solution*) and more about finding the specific one that meets a given condition like $y(t_0) = y_0$. This condition is called an **initial condition**, and solving the differential equation with this condition is called an **initial-value problem**.

Visually, an initial condition helps us pick out the one curve from the family of solutions that passes through the point $(t_0, y_0)$.

**Example.** Find a solution of the differential equation

$$y' = \frac{1}{2}(y^2 - 1)$$

that satisfies the initial condition $y(0) = 2$.

From the previous example, we know that

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution for all values of $c$. Substituting $t = 0$ and $y = 2$,

$$2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1+c}{1-c} \Rightarrow 2(1-c) = 1 + c \Rightarrow 2 - 2c = 1 + c$$
$$\Rightarrow 1 = 3c$$
$$\Rightarrow c = 1/3$$

The solution to the initial value problem is $y = \dfrac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \dfrac{3 + e^t}{3 - e^t}$