8.3 Center of Mass

Discrete Systems

Theorem (Law of the Lever)**.** If two masses m_1 and m_2 are placed on opposite sides of a fulcrum at distances d_1 and d_2, respectively, then the rod will balance provided that

$$m_1 d_1 = m_2 d_2.$$

In the case where m_1 is located at x_1, m_2 at x_2, and the center of mass is at $\bar{x}$, the balancing condition can be written as

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}),$$

which implies

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Definition (Center of Mass for a System of Particles)**.** The **center of mass** of a system of particles is the point

$$(\bar{x}, \bar{y}).$$

For particles on a line, the center of mass is

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.$$

For particles in the plane with masses $m_1, m_2, \ldots, m_n$ located at

$$(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n),$$

the center of mass is

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}, \qquad \bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n}.$$

Moments

Definition (Moment)**.** The *moment* of a mass m about a point (or an axis) is the product of the mass and its distance from that point (or axis). For a system of particles, the moment about the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i.$$

In the plane, we define:

$$M_y = \sum_{i=1}^{n} m_i x_i \quad \text{(moment about the } y\text{–axis)},$$

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \text{(moment about the } x\text{–axis)}.$$

Regions Bounded by Curves

Region Above the x-Axis: Suppose a lamina has **uniform density** and occupies the region bounded above by $y = f(x)$, below by the x-axis, and from $x = a$ to $x = b$. Then

$$A = \int_a^b f(x)\,dx,$$

$$M_y = \int_a^b x\,f(x)\,dx, \qquad M_x = \int_a^b \frac{[f(x)]^2}{2}\,dx.$$

Hence the center of mass (centroid) is

$$\bar{x} = \frac{1}{A}\int_a^b x\,f(x)\,dx, \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{[f(x)]^2}{2}\,dx.$$

Region Between Two Curves: Suppose a lamina has **uniform density** and occupies the region bounded above by $y = f(x)$, below by $y = g(x)$, and from $x = a$ to $x = b$, where $f(x) \geq g(x)$ on $a \leq x \leq b$. Then

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx,$$

$$M_y = \int_a^b x\big(f(x) - g(x)\big)\,dx, \qquad M_x = \int_a^b \frac{[f(x)]^2 - [g(x)]^2}{2}\,dx.$$

Hence the center of mass (centroid) is

$$\bar{x} = \frac{1}{A}\int_a^b x\big(f(x) - g(x)\big)\,dx, \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{[f(x)]^2 - [g(x)]^2}{2}\,dx.$$

Symmetry Principle

Theorem (Symmetry Principle)**.** If a region D is symmetric with respect to a line ℓ, then the centroid of D lies on ℓ.

Center of Mass Problems

1. A system consists of three point masses:
   - $m_1 = 2$ kg at $(1, 3)$
   - $m_2 = 3$ kg at $(4, 5)$
   - $m_3 = 4$ kg at $(6, 2)$

   Compute the center of mass of the system.

2. A system consists of four point masses:
   - $m_1 = 1$ kg at $(0, 0)$
   - $m_2 = 2$ kg at $(2, 4)$
   - $m_3 = 3$ kg at $(5, 1)$
   - $m_4 = 4$ kg at $(3, 3)$

   Compute the center of mass of the system.

3. Compute the center of mass of the region bounded by

$$y = x^2, \quad y = 0, \quad x = 1, \quad \text{and} \quad x = 2.$$

4. Compute the center of mass of the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = \pi,$$

5. Use symmetry to find the center of mass of a uniform semicircular lamina of radius R.

6. Use symmetry to determine the center of mass of a uniform triangular lamina with vertices at $(0, 0)$, $(a, 0)$, and (a, a).