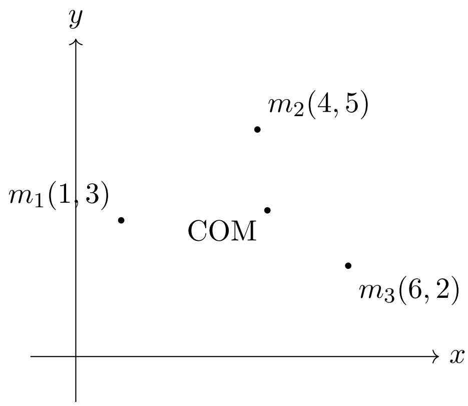
8.3 Center of Mass (Solutions)

1. **Three Point Masses.**



- **Total mass:**

$$M = 2 + 3 + 4 = 9\,\text{kg}.$$

- **Compute the moments:**

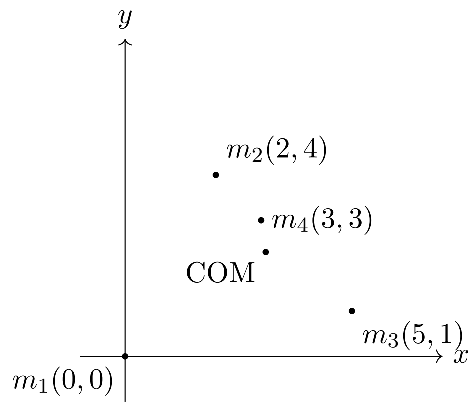
$$\begin{aligned} M_x &= 2(1) + 3(4) + 4(6) \\ &= 2 + 12 + 24 = 38 \end{aligned}$$

$$\begin{aligned} M_y &= 2(3) + 3(5) + 4(2) \\ &= 6 + 15 + 8 = 29. \end{aligned}$$

- **Center of Mass:**

$$\bar{x} = \frac{38}{9},$$
$$\bar{y} = \frac{29}{9}.$$

2. **Four Point Masses.**



- **Total mass:**

$$M = 1 + 2 + 3 + 4 = 10\,\text{kg}.$$

- **Compute the moments:**

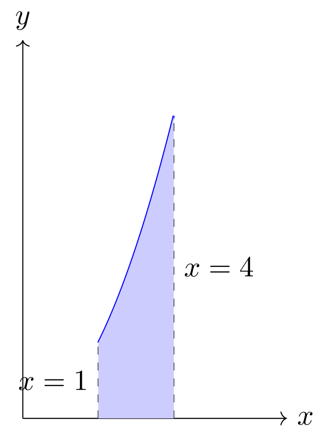
$$\begin{aligned} M_x &= 1(0) + 2(2) + 3(5) + 4(3) \\ &= 0 + 4 + 15 + 12 = 31 \end{aligned}$$

$$\begin{aligned} M_y &= 1(0) + 2(4) + 3(1) + 4(3) \\ &= 0 + 8 + 3 + 12 = 23. \end{aligned}$$

- **Center of Mass:**

$$\bar{x} = \frac{31}{10} = 3.1,$$
$$\bar{y} = \frac{23}{10} = 2.3.$$

3. **Region Bounded by $y = x^2$, $y = 0$, $x = 1$, and $x = 2$.**



- **Area:**

$$A = \int_1^2 (x^2 - 0)\,dx = \int_1^2 x^2\,dx$$
$$= \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3 - 1^3}{3} = \frac{8-1}{3} = \frac{7}{3}.$$

- **x-coordinate of the centroid:**

$$\bar{x} = \frac{1}{A}\int_1^2 x\,(x^2)\,dx = \frac{1}{A}\int_1^2 x^3\,dx$$
$$= \frac{1}{7/3}\left[\frac{x^4}{4}\right]_1^2 = \frac{3}{7}\left(\frac{16-1}{4}\right)$$
$$= \frac{3}{7}\cdot\frac{15}{4} = \frac{45}{28}.$$

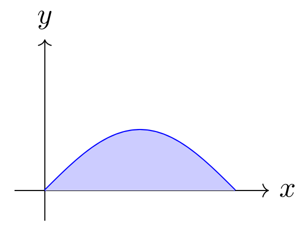
- **y-coordinate of the centroid:**

$$\bar{y} = \frac{1}{A}\int_1^2 \frac{(x^2)^2}{2}\,dx = \frac{1}{A}\int_1^2 \frac{x^4}{2}\,dx$$
$$= \frac{1}{7/3}\cdot\frac{1}{2}\left[\frac{x^5}{5}\right]_1^2 = \frac{3}{7}\cdot\frac{1}{2}\left(\frac{32-1}{5}\right)$$
$$= \frac{3}{7}\cdot\frac{31}{10} = \frac{93}{70}.$$

Thus, the center of mass is

$$\left(\frac{45}{28}, \frac{93}{70}\right).$$

4. **Region Bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.**



- **Area:**

$$A = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi$$
$$= [-\cos\pi] - [-\cos 0] = (1) - (-1) = 2.$$

- **x-coordinate of the centroid:**

$$\bar{x} = \frac{1}{A}\int_0^\pi x\sin x\,dx.$$

Using integration by parts (with $u = x$, $dv = \sin x\,dx$):

$$\int_0^\pi x\sin x\,dx = [-x\cos x]_0^\pi + \int_0^\pi \cos x\,dx$$
$$= [-\pi\cos\pi + 0] + [\sin x]_0^\pi$$
$$= -\pi(-1) + (0 - 0) = \pi.$$

Thus,

$$\bar{x} = \frac{\pi}{2}.$$

- **y-coordinate of the centroid:**

$$\bar{y} = \frac{1}{A}\int_0^\pi \frac{[\sin x]^2}{2}\,dx = \frac{1}{2A}\int_0^\pi \sin^2 x\,dx.$$

Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$\int_0^\pi \sin^2 x\,dx = \int_0^\pi \frac{1-\cos 2x}{2}\,dx$$
$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$
$$= \frac{1}{2}(\pi - 0) = \frac{\pi}{2}.$$

Therefore,

$$\bar{y} = \frac{1}{2\cdot 2}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

Thus, the center of mass is

$$\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

5. **Uniform Semicircular Lamina.**

A uniform semicircular lamina of radius R (with the flat side along the x-axis) is symmetric about the y-axis. Therefore,

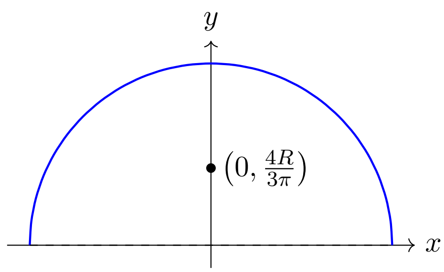
$$\bar{x} = 0.$$

It is a well-known result that the y-coordinate of the centroid is given by:

$$\bar{y} = \frac{4R}{3\pi}.$$

Thus, the center of mass is

$$\left(0, \frac{4R}{3\pi}\right).$$



6. **Uniform Triangular Lamina.**

Consider the triangle with vertices $(0,0)$, $(a,0)$, and (a,a). The centroid (center of mass) of a triangle is the average of its vertices:

$$\bar{x} = \frac{0+a+a}{3} = \frac{2a}{3},$$
$$\bar{y} = \frac{0+0+a}{3} = \frac{a}{3}.$$

Thus, the center of mass is

$$\left(\frac{2a}{3}, \frac{a}{3}\right).$$