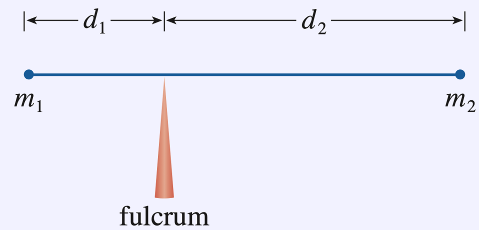
8.3 Center of Mass

Theorem (Law of the Lever)**.** Consider the situation illustrated below, where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.

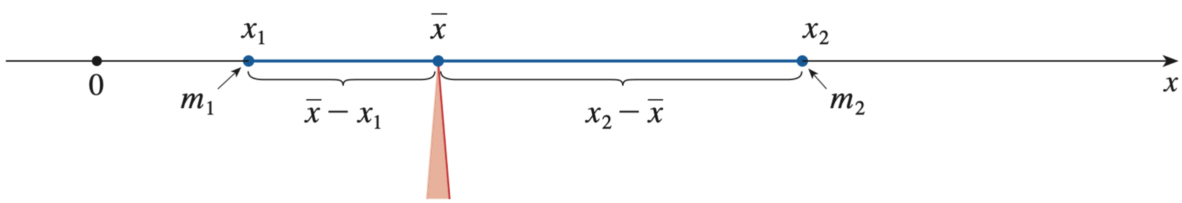


The rod will balance if

$$m_1 d_1 = m_2 d_2$$

This is called the Law of the Lever, a fact discovered by Archimedes.

Example. Suppose that the rod lies along the x-axis with m_1 at x_1 and m_2 at x_2. What is the center of mass $\overline{x}$?



$$m_1 d_1 = m_2 d_2$$

$$m_1(\overline{x} - x_1) = m_2(x_2 - \overline{x})$$

$$m_1\overline{x} - m_1 x_1 = m_2 x_2 - m_2\overline{x}$$

$$m_1\overline{x} + m_2\overline{x} = m_1 x_1 + m_2 x_2$$

$$\overline{x}(m_1 + m_2) = m_1 x_1 + m_2 x_2$$

$$\boxed{\overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

Question. Where is the center of mass $\overline{x}$ located in terms of moments?

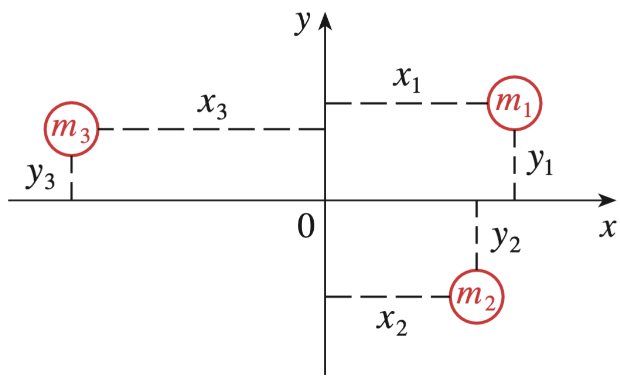
$m_1 x_1$ and $m_2 x_2$ are called the moments of the masses m_1 and m_2

$$\text{center of mass} = \frac{\text{sum of moments}}{\text{total mass}}$$

Question. Consider a system of n particles with masses $m_1, m_2, \ldots, m_n$, located at the points $x_1, x_2, \ldots, x_n$ on the x-axis. Where is the center of mass located?

$$\bar{x} = \frac{\text{sum of the moments}}{\text{total mass}} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{M}{m}$$

Example. Consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane. Where is the center of mass $\overline{x}$ located?

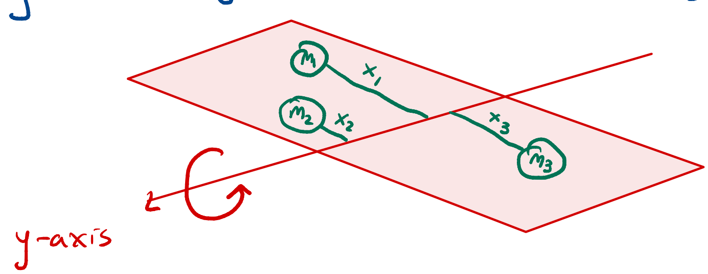


- The center of mass is located at some point $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\text{sum of the moments}}{\text{total mass}} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{M_y}{m}$$

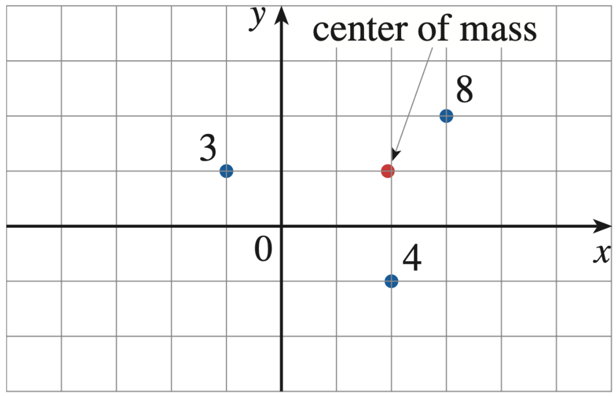
$$\bar{y} = \frac{\text{sum of the moments}}{\text{total mass}} = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m_1 + m_2 + \cdots + m_n} = \frac{M_x}{m}$$

- M_y is the total weighted distance to the y-axis. It measures the tendency of the system to rotate about the y-axis.



- M_x measures the tendency of the system to rotate about the x-axis.

Example. Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1), (2, -1)$, and $(3, 2)$, respectively.



$$M_y = 3(-1) + 4(2) + 8(3) = 29$$

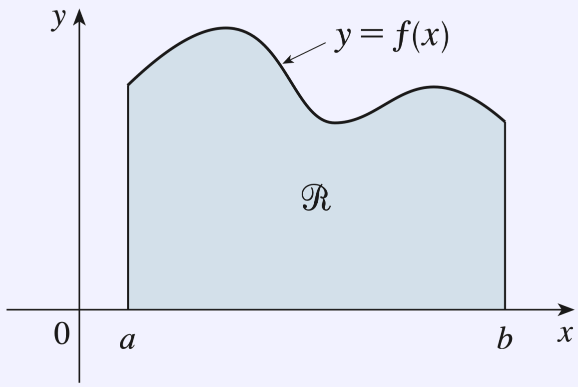
$$M_x = 3(1) + 4(-1) + 8(2) = 15$$

$$m = 3 + 4 + 8 = 15$$

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15} \qquad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

Center of mass is $(29/15, 1)$

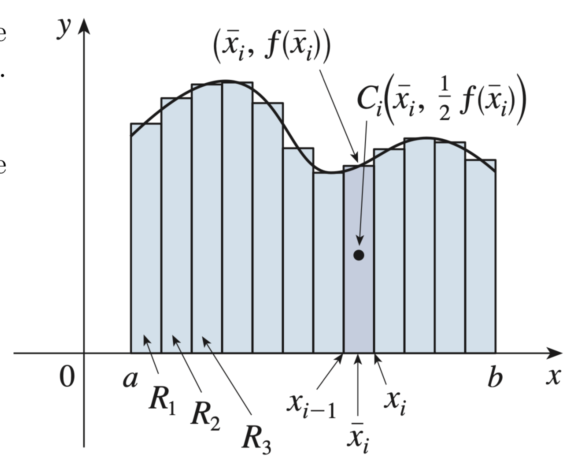
Theorem. Let $\mathscr{R}$ be a region in the plane occupied by a flat plate (lamina) with uniform density ρ. Suppose the region $\mathscr{R}$ is bounded by the curve $y = f(x)$, where $f(x)$ gives the height of the region at each point x along the horizontal axis. Let a and b be the horizontal bounds of the region $\mathscr{R}$, i.e., $\mathscr{R}$ lies between the vertical lines $x = a$ and $x = b$.



The center of mass of the plate, also called the centroid of R, with area A, is located at the point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{A}\int_a^b x\,f(x)\,dx \quad \text{and} \quad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

Note: If the plate has uniform density ρ, then its center of mass coincides with the centroid of R. If the density is not uniform, the center of mass is typically located at a different point.



- For the x-coordinate of the center of mass ($\overline{x}$), divide the region into n subintervals from $x = a$ to $x = b$. Each subinterval has width $\Delta x = \frac{b-a}{n}$.
- Choose sample points $\overline{x}_i$ in each subinterval to be the midpoint of the subinterval.
- For each strip at $\overline{x}_i$, the area is:

$$f(\overline{x}_i)\,\Delta x$$

- The mass of this strip is:

$$\text{mass} = \text{density} \cdot \text{area} = \rho \cdot f(\overline{x}_i)\,\Delta x$$

- The moment of this strip about the y-axis is:

$$\text{moment} = \text{distance} \cdot \text{mass} = \overline{x}_i \cdot \rho f(\overline{x}_i)\,\Delta x$$

- The total moment of all strips about the y-axis is:

$$M_y \approx \sum_{i=1}^{n} \overline{x}_i \, \rho \, f(\overline{x}_i) \Delta x$$

- As the number of subintervals n goes to infinity (and $\Delta x \to 0$), this sum becomes the integral:

$$M_y = \int_a^b x \, \rho \, f(x) \, dx$$

← Total weighted distance to the y-axis

- The x-coordinate of the center of mass is the weighted average of the moments:

$$\overline{x} = \frac{M_y}{m} = \frac{\int_a^b x \rho \, f(x) \, dx}{\rho \cdot A} = \frac{1}{A} \int_a^b x \, f(x) \, dx$$

- For the y-coordinate of the center of mass ($\overline{y}$), we need to calculate the moment of each strip with respect to the x-axis. This is the "weighted distance" of the strip to the x-axis. For a uniform strip, this is taken from the center of the strip. The distance of the center of the strip to the x-axis is:

↳ Things above and below the center line will average out

$$\frac{1}{2} f(\overline{x}_i)$$

- The moment about the x-axis for each strip is then:

$$\text{moment} = \text{distance} \cdot \text{mass} = \frac{1}{2} f(\overline{x}_i) \cdot \rho \, f(\overline{x}_i) \Delta x = \frac{1}{2} \rho \left[f(\overline{x}_i) \right]^2 \Delta x$$

- The total moment of all strips about the x-axis is:

$$M_x \approx \sum_{i=1}^{n} \frac{1}{2} \rho \left[f(\overline{x}_i) \right]^2 \Delta x$$

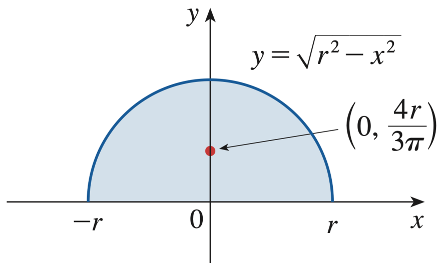
- Taking the limit as $n \to \infty$, this sum becomes the integral:

$$M_x = \int_a^b \frac{1}{2} \rho \left[f(x) \right]^2 dx$$

- The y-coordinate of the center of mass is:

$$\overline{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2} \rho \left[f(x) \right]^2 dx}{\rho \cdot A} = \frac{1}{A} \int_a^b \frac{1}{2} \left[f(x) \right]^2 dx$$

Example. Find the center of mass of a semicircular plate of radius r with uniform density.



- Place the semicircle so that $f(x) = \sqrt{r^2 - x^2}$ and $a = -r$, $b = r$

- By symmetry, $\bar{x} = 0$

- For $\bar{y}$,

$$\bar{y} = \frac{1}{A} \int_{-r}^{r} \frac{1}{2} [f(x)]^2 \, dx = \frac{1}{\frac{1}{2}\pi r^2} \cdot \frac{1}{2} \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

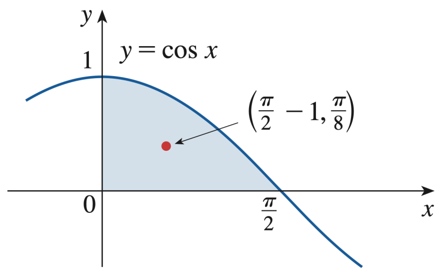
$$= \frac{1}{\pi r^2} \int_{-r}^{r} r^2 - x^2 \, dx$$

$$= \frac{1}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$= \frac{4r}{3\pi}$$

Final Answer: $\left(0, \frac{4r}{3\pi}\right)$

Example. Find the centroid of the region in the first quadrant bounded by the curves $y = \cos x$, $y = 0$, and $x = 0$.



$$A = \int_0^{\pi/2} \cos x \, dx = \left[\sin x\right]_0^{\pi/2} = 1$$

$$\bar{x} = \frac{1}{A}\int_0^{\pi/2} x\, f(x)\, dx = \int_0^{\pi/2} x \cos x \, dx$$

$$= \left[x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx \qquad \text{[Integration by Parts]}$$

$$= \frac{\pi}{2} - 1$$

$$\bar{y} = \frac{1}{A}\int_0^{\pi/2} \frac{1}{2}\left[f(x)\right]^2 dx = \frac{1}{2}\int_0^{\pi/2} \cos^2 x \, dx$$

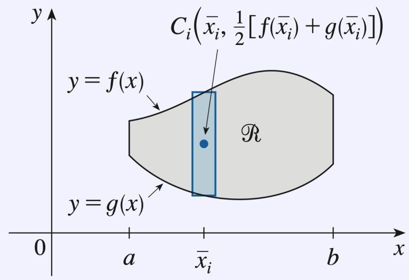
$$= \frac{1}{4}\int_0^{\pi/2} 1 + \cos 2x \, dx$$

$$= \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{\pi}{8}$$

Final Answer: $\left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$

Theorem. Suppose the region $\mathscr{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$.

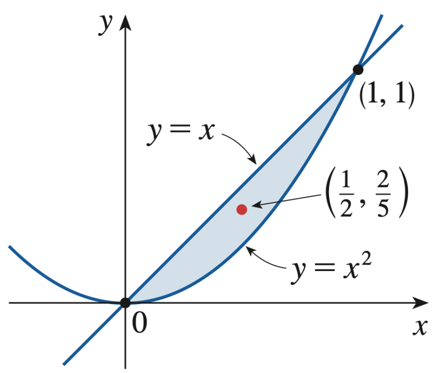


The centroid of $\mathscr{R}$ is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}\,dx$$

Example. Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.



$$A = \int_0^1 x - x^2\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0) = \frac{1}{6}$$

$$\overline{x} = \frac{1}{A}\int_0^1 x\left[x - x^2\right]dx = 6\int_0^1 x^2 - x^3\,dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left[f(x)^2 - g(x)^2\right] dx = \frac{1}{1/6}\cdot\frac{1}{2}\int_0^1 x^2 - x^4\, dx$$

$$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{2}{5}$$

Final Answer : $\left(\frac{1}{2}, \frac{2}{5}\right)$