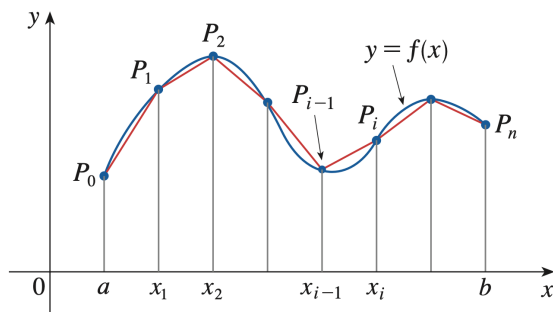


8.1 / 10.2 Arc Length

Theorem (Arc Length Formula). Let $y = f(x)$ be a curve defined on the interval $[a, b]$, and suppose that $f'(x)$ is continuous on $[a, b]$. The arc length L of the curve is given by:

1. Polygonal Approximation:

- Divide the interval $[a, b]$ into n subintervals of equal width Δx .
- For each i , let $y_i = f(x_i)$ and consider the points $P_i = (x_i, y_i)$ on the curve.
- Approximate the curve by a polygonal path connecting these points.



2. Length of a Single Segment:

- The length of a segment connecting two consecutive points P_{i-1} and P_i is:
 - By the Mean Value Theorem, $\Delta y_i = f'(x_i^*)\Delta x$ for some x_i^* in $[x_{i-1}, x_i]$.
 - Substitute this into the segment length:

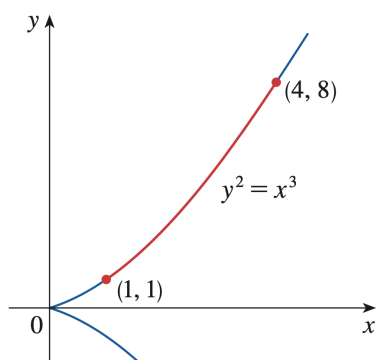
3. Total Length of the Polygonal Path:

- Sum the lengths of all segments:

4. Take the Limit as $n \rightarrow \infty$:

- As $n \rightarrow \infty$, $\Delta x \rightarrow 0$, and the sum becomes a definite integral:

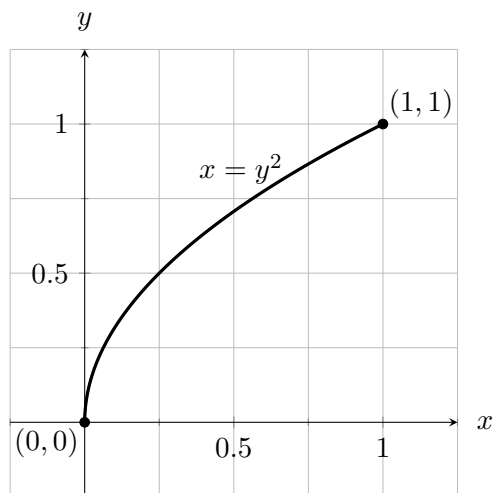
Example. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



Theorem. If a curve has the equation $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y , we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

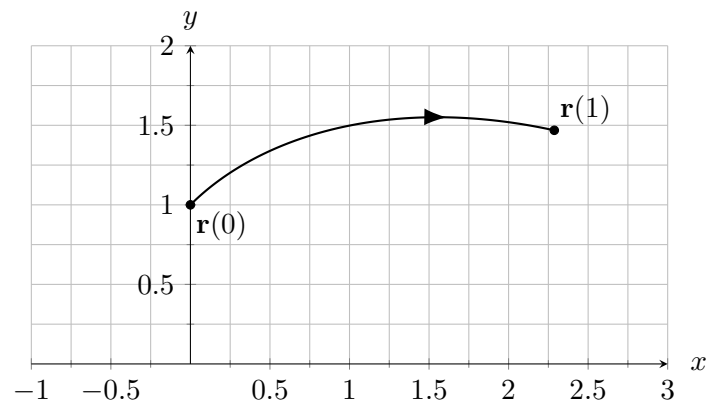
Example. Find the length of the arc of the parabola $x = y^2$ from $(0, 0)$ to $(1, 1)$.



Theorem. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where $f'(t)$ and $g'(t)$ are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β , then the length of C is:

Example. Find the arc length of the parametric curve

$$\mathbf{r}(t) = (e^t \sin t, e^t \cos t), \quad 0 \leq t \leq 1.$$



Example. Find the length of the unit circle described by $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

