8.1, 10.2 Arc Length

Theorem (Arc Length Formula for $y = f(x)$)**.** If f is differentiable on $[a, b]$ and f' is continuous, then the arc length L of the curve $y = f(x)$ is given by

$$L = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Theorem (Arc Length for $x = g(y)$)**.** If a curve is given by $x = g(y)$ for $y \in [c, d]$ with g differentiable and g' continuous, then the arc length L is

$$L = \int_c^d \sqrt{1 + \big(g'(y)\big)^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Theorem (Arc Length for Parametric Equations)**.** Let a curve C be defined by the parametric equations

$$x = f(t), \quad y = g(t), \quad t \in [\alpha, \beta],$$

where f and g are differentiable and their derivatives are continuous. Then the arc length L of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Arc Length Problems

1. Find the arc length of the curve $f(x) = \ln(\cos x)$ over the interval $0 \le x \le \frac{\pi}{4}$.[1]
2. Find the arc length of the curve $f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$ over the interval $0 \le x \le 2$.
3. Find the arc length of the curve $x = \frac{1}{3}y^{3/2} - y^{1/2}$ over the interval $1 \le y \le 4$.
4. Find the arc length of the curve $x = \frac{2}{3}y^{3/2}$ over the interval $0 \le y \le 4$.
5. Find the arc length of the curve defined by

$$x = t^2, \quad y = t^3, \quad 0 \le t \le 1.$$

6. Find the arc length of the cycloid given by

$$x = t - \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi.$$

[1] These problems are hard to come up with. That is, how can we ensure that $\sqrt{1 + [f'(x)]^2}$ can be integrated? For a discussion of this, see `https://math.colorado.edu/~chda1090/arclengthprobs.pdf`