8.1, 10.2 Arc Length (Solutions)

1. The arc length formula is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

For $f(x) = \ln(\cos x)$:

- Compute the derivative:

$$f'(x) = \frac{d}{dx}\ln(\cos x) = -\tan x.$$

- Compute $1 + (f'(x))^2$:

$$1 + \tan^2 x = \sec^2 x.$$

- Compute the arc length integral:

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx \\ &= \int_0^{\frac{\pi}{4}} \sec x\, dx. \end{aligned}$$

- Use the integral result:

$$\int \sec x\, dx = \ln|\sec x + \tan x|.$$

- Evaluate from 0 to $\frac{\pi}{4}$:

$$\begin{aligned} L &= \ln|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}| - \ln|\sec 0 + \tan 0| \\ &= \ln(\sqrt{2} + 1) - \ln 1 \\ &= \ln(\sqrt{2} + 1). \end{aligned}$$

2. To find the arc length of the function $f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$ over the interval $[0, 2]$, we use the standard arc length formula:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

First, we determine the derivative of the function:

$$f'(x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2}.$$

Next, we simplify the expression under the square root, $1 + (f'(x))^2$:

$$1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4 + e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4}.$$

Recognizing that the numerator is a perfect square, $(e^x + e^{-x})^2$, the expression simplifies to:

$$1 + (f'(x))^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2.$$

Substituting this back into the integral yields:

$$L = \int_0^2 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx = \int_0^2 \frac{e^x + e^{-x}}{2}\, dx.$$

We can split the integral for easier evaluation:

$$L = \frac{1}{2}\int_0^2 e^x\, dx + \frac{1}{2}\int_0^2 e^{-x}\, dx.$$

Evaluating the antiderivatives from 0 to 2:

$$\begin{aligned} L &= \frac{1}{2}\left[e^x\right]_0^2 + \frac{1}{2}\left[-e^{-x}\right]_0^2 \\ &= \frac{1}{2}(e^2 - 1) + \frac{1}{2}(-e^{-2} - (-1)) \\ &= \frac{e^2 - 1 - e^{-2} + 1}{2} \\ &= \frac{e^2 - e^{-2}}{2}. \end{aligned}$$

3. Consider the curve defined by $x = g(y) = \frac{1}{3}y^{3/2} - y^{1/2}$ on the interval $[1, 4]$. The arc length is given by:

$$L = \int_a^b \sqrt{1 + (g'(y))^2}\, dy.$$

We begin by computing the derivative with respect to y:

$$g'(y) = \frac{d}{dy}\left(\frac{1}{3}y^{3/2} - y^{1/2}\right) = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}.$$

Next, we simplify the integrand $1 + (g'(y))^2$:

$$\begin{aligned} 1 + \left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2 &= 1 + \frac{1}{4}\left(y - 2 + y^{-1}\right) \\ &= \frac{4 + y - 2 + y^{-1}}{4} \\ &= \frac{y + 2 + y^{-1}}{4}. \end{aligned}$$

This expression forms a perfect square:

$$\frac{(y^{1/2} + y^{-1/2})^2}{4} = \left(\frac{y^{1/2} + y^{-1/2}}{2}\right)^2.$$

The arc length integral therefore becomes:

$$L = \int_1^4 \frac{y^{1/2} + y^{-1/2}}{2}\, dy = \frac{1}{2}\int_1^4 y^{1/2}\, dy + \frac{1}{2}\int_1^4 y^{-1/2}\, dy.$$

Computing the antiderivatives:

$$\int y^{1/2}\, dy = \frac{2}{3}y^{3/2} \quad \text{and} \quad \int y^{-1/2}\, dy = 2y^{1/2}.$$

Evaluating from 1 to 4:

$$\begin{aligned} L &= \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_1^4 + \frac{1}{2}\left[2y^{1/2}\right]_1^4 \\ &= \frac{1}{2}\left(\frac{2}{3}(8) - \frac{2}{3}(1)\right) + \frac{1}{2}\left(2(2) - 2(1)\right) \\ &= \frac{1}{2}\left(\frac{14}{3}\right) + \frac{1}{2}(2) \\ &= \frac{7}{3} + 1 = \frac{10}{3}. \end{aligned}$$

4. We wish to find the arc length of the curve $x = \frac{2}{3}y^{3/2}$ over the interval $[0, 4]$. First, compute the derivative $g'(y)$:

$$g'(y) = \frac{d}{dy}\left(\frac{2}{3}y^{3/2}\right) = y^{1/2}.$$

Substituting this into the arc length formula:

$$L = \int_0^4 \sqrt{1 + (g'(y))^2}\, dy = \int_0^4 \sqrt{1 + y}\, dy.$$

To evaluate this integral, we use the substitution $u = 1 + y$, which implies $du = dy$. The limits of integration change as follows:

- When $y = 0$, $u = 1$.
- When $y = 4$, $u = 5$.

The integral becomes:

$$L = \int_1^5 u^{1/2}\, du.$$

Evaluating the antiderivative:

$$L = \left[\frac{2}{3}u^{3/2}\right]_1^5 = \frac{2}{3}\left(5^{3/2} - 1^{3/2}\right) = \frac{2}{3}\left(5\sqrt{5} - 1\right).$$

Thus, the arc length is:

$$L = \frac{10\sqrt{5} - 2}{3}.$$

5. Consider the parametric curve given by $x = t^2$ and $y = t^3$ for $t \in [0, 1]$. The derivatives with respect to t are:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2.$$

The arc length integral is:

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt.$$

We can factor t^2 out from under the radical:

$$L = \int_0^1 \sqrt{t^2(4 + 9t^2)}\, dt = \int_0^1 t\sqrt{4 + 9t^2}\, dt.$$

We use the substitution $u = 4 + 9t^2$, so $du = 18t\, dt$, or $t\, dt = \frac{1}{18} du$. The limits change from $t \in [0, 1]$ to $u \in [4, 13]$.

$$L = \frac{1}{18} \int_4^{13} u^{1/2}\, du.$$

Evaluating the integral:

$$L = \frac{1}{18} \left[\frac{2}{3} u^{3/2}\right]_4^{13} = \frac{1}{27} \left(13^{3/2} - 4^{3/2}\right) = \frac{1}{27} \left(13\sqrt{13} - 8\right).$$

6. We determine the arc length of one arch of the cycloid defined by $x = t - \sin t$ and $y = 1 - \cos t$ for $t \in [0, 2\pi]$.

First, we compute the derivatives:

$$\frac{dx}{dt} = 1 - \cos t, \quad \frac{dy}{dt} = \sin t.$$

The integrand for the arc length is:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(1 - \cos t)^2 + \sin^2 t}.$$

Expanding and simplifying the terms inside the square root:

$$\begin{aligned}(1 - \cos t)^2 + \sin^2 t &= 1 - 2\cos t + \cos^2 t + \sin^2 t \\ &= 1 - 2\cos t + 1 \\ &= 2 - 2\cos t.\end{aligned}$$

Thus, the integral becomes:

$$L = \int_0^{2\pi} \sqrt{2(1 - \cos t)}\, dt.$$

Using the half-angle identity $1 - \cos t = 2\sin^2 \frac{t}{2}$, we substitute:

$$L = \int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt = \int_0^{2\pi} 2\left|\sin \frac{t}{2}\right| dt.$$

Since $\sin \frac{t}{2} \geq 0$ for $t \in [0, 2\pi]$, we can remove the absolute value bars:

$$L = 2\int_0^{2\pi} \sin \frac{t}{2}\, dt.$$

Let $u = \frac{t}{2}$, so $du = \frac{1}{2}dt \implies dt = 2du$. The limits become 0 to π:

$$L = 2\int_0^{\pi} \sin u\,(2du) = 4\int_0^{\pi} \sin u\, du.$$

Finally, evaluating the definite integral:

$$L = 4\left[-\cos u\right]_0^{\pi} = 4\left(-\cos \pi - (-\cos 0)\right) = 4(1 + 1) = 8.$$