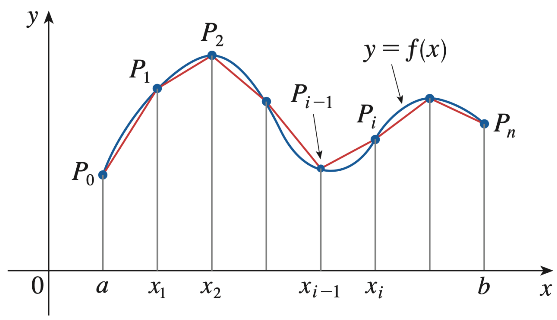
8.1 / 10.2 Arc Length

Theorem (Arc Length Formula)**.** Let $y = f(x)$ be a curve defined on the interval $[a, b]$, and suppose that $f'(x)$ is continuous on $[a, b]$. The arc length L of the curve is given by:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

1. **Polygonal Approximation:**

   - Divide the interval $[a, b]$ into n subintervals of equal width Δx.
   - For each i, let $y_i = f(x_i)$ and consider the points $P_i = (x_i, y_i)$ on the curve.
   - Approximate the curve by a polygonal path connecting these points.



2. **Length of a Single Segment:**

   - The length of a segment connecting two consecutive points P_{i-1} and P_i is:

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

   - By the Mean Value Theorem, $\Delta y_i = f'(x_i^*)\Delta x$ for some x_i^* in $[x_{i-1}, x_i]$.
   - Substitute this into the segment length:

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \Delta x \sqrt{1 + (f'(x_i^*))^2}$$

3. **Total Length of the Polygonal Path:**

   - Sum the lengths of all segments:

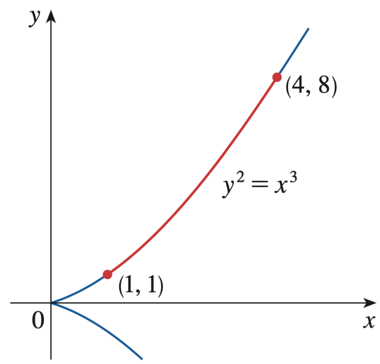
$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \Delta x \sqrt{1 + (f'(x_i^*))^2}$$

4. **Take the Limit as $n \to \infty$:**

   - As $n \to \infty$, $\Delta x \to 0$, and the sum becomes a definite integral:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



The top half of the curve is given by $y = x^{3/2}$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

u-sub:

$u = 1 + \frac{9}{4}x$

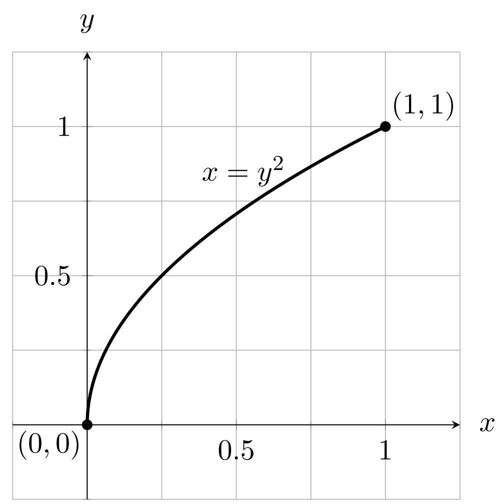
$du = \frac{9}{4}dx$

$$\approx 7.633$$

Theorem. If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example. Find the length of the arc of the parabola $x = y^2$ from $(0, 0)$ to $(1, 1)$.



$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1 + (2y)^2}\, dy$$

$$= \int_0^1 \sqrt{1 + 4y^2}\, dy$$

> u-sub: $u = 2y$
> $du = 2\,dy$

$$= \frac{1}{2} \int_0^2 \sqrt{1 + u^2}\, du$$

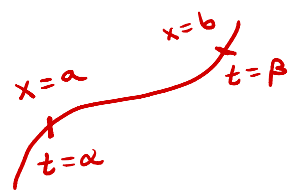
> trig-sub: $u = \tan\theta$
> $du = \sec^2\theta\, d\theta$

$$= \frac{1}{2} \int_0^{\arctan(2)} \sec^3\theta\, d\theta$$

$$\approx 1.478$$

Theorem. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where $f'(t)$ and $g'(t)$ are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



Suppose the curve can be written as $y = h(x)$ for $a \le x \le b$, where $a = f(\alpha)$ and $b = f(\beta)$.

[If C is not a function $y = h(x)$, break into pieces where it is]

Use the arc length formula $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ and change variables to t. When $x = a$, $t = \alpha$. When $x = b$, $t = \beta$.

Also, $dx = f'(t) \cdot dt = \frac{dx}{dt} \cdot dt$.

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\ \frac{dx}{dt} \cdot dt = \int_{\alpha}^{\beta} \sqrt{\frac{(dx/dt)^2}{(dx/dt)^2} + \frac{(dy/dt)^2}{(dx/dt)^2}} \cdot \frac{dx}{dt}\, dt$$

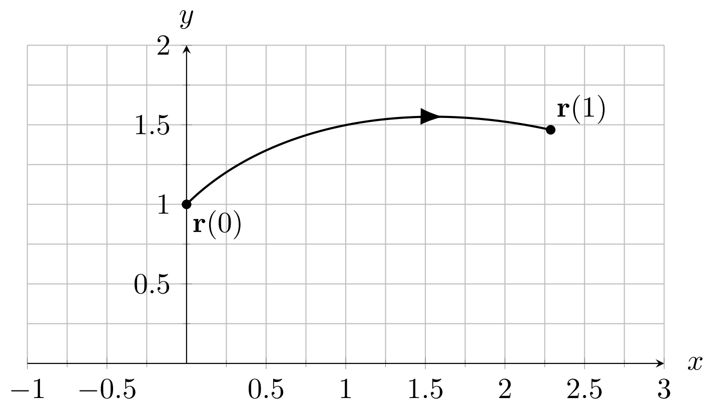
$$= \int_{\alpha}^{\beta} \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{dx/dt} \cdot \frac{dx}{dt}\, dt$$

$$= \int_{\alpha}^{\beta} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$

Example. Find the arc length of the parametric curve

$$\mathbf{r}(t) = \big(e^t \sin t,\ e^t \cos t\big), \qquad 0 \le t \le 1.$$

$x = e^t \sin t$
$y = e^t \cos t$



$\sqrt{2}\,(e-1)$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = e^t \cos t + e^t \sin t = e^t(\cos t + \sin t)$$

$$\frac{dy}{dt} = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t)$$

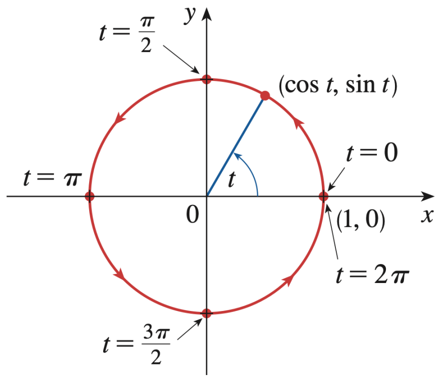
$$L = \int_0^1 \sqrt{e^{2t}(\cos t + \sin t)^2 + e^{2t}(\cos t - \sin t)^2}\, dt$$

$$= \int_0^1 \sqrt{e^{2t}(\cos^2 t + 2\sin t \cos t + \sin^2 t) + e^{2t}(\cos^2 t - 2\sin t \cos t + \sin^2 t)}\, dt$$

$$= \int_0^1 \sqrt{e^{2t}(1 + 2\sin t \cos t) + e^{2t}(1 - 2\sin t \cos t)}\, dt$$

$$= \int_0^1 \sqrt{2e^{2t}}\, dt = \int_0^1 \sqrt{2}\, e^t\, dt = \sqrt{2}\,\big[e^t\big]_0^1 = \sqrt{2}\,(e-1)$$

Example. Find the length of the unit circle described by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.



$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -\sin t \quad \text{and} \quad \frac{dy}{dt} = \cos t$$

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} 1\, dt$$

$$= \Big[t \Big]_0^{2\pi}$$

$$= 2\pi$$