7.8 Improper Integrals

A definite integral is called **improper** if the interval is unbounded ($a = -\infty$ or $b = \infty$) or if the integrand becomes infinite at an endpoint (or at a point inside the interval).

Strategy: rewrite the integral using limits.

- **Infinite interval:** $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx.$
- **Infinite discontinuity:** if $f(x) \to \infty$ as $x \to a^+$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$. If f blows up at $c \in (a,b)$, then *split the integral*:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

  and convert *each* piece to a limit.

Convergence: the improper integral **converges** if the relevant limit(s) exist and are finite; otherwise it **diverges**. If you split at c, both limits must converge.

Notes

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx.$
2. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx.$
3. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx.$
4. $\displaystyle\int_{-\infty}^0 xe^x\,dx.$
5. $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$
6. $\displaystyle\int_0^3 \frac{dx}{x-1}.$

WebAssign

1. $\displaystyle\int_8^9 \frac{1}{9x-1}\,dx.$
2. $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx.$
3. $\displaystyle\int_{-\infty}^\infty \frac{\sin(x)}{1+2x^2}\,dx.$
4. $\displaystyle\int_1^3 \ln(x-1)\,dx.$
5. $\displaystyle\int_0^\infty e^{-6x}\,dx.$
6. $\displaystyle\int_{-\infty}^0 \frac{x}{(x^2+2)^4}\,dx.$
7. $\displaystyle\int_{-\infty}^\infty \frac{x^5}{x^6+1}\,dx.$
8. $\displaystyle\int_e^\infty \frac{11}{x(\ln(x))^3}\,dx.$
9. $\displaystyle\int_0^1 \frac{5}{x^5}\,dx.$
10. $\displaystyle\int_3^{11} \frac{1}{\sqrt{11-x}}\,dx.$
11. $\displaystyle\int_2^\infty \frac{1}{x-\ln(x)}\,dx.$
12. $\displaystyle\int_1^\infty \frac{8+\cos(x)}{\sqrt{x^4+x^2}}\,dx.$

Practice

1. $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$
2. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
3. $\displaystyle\int_0^\infty e^{-x}\,dx$
4. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$
6. $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$
7. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$
8. $\displaystyle\int_0^1 \ln(x)\,dx$
9. $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$
10. $\displaystyle\int_0^2 \frac{dx}{(2-x)^{1/3}}$
11. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$
12. $\displaystyle\int_0^{\pi/2} \tan(x)\,dx$
13. $\displaystyle\int_1^\infty \frac{1}{x\,(\ln x)^2}\,dx$
14. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$