7.8 Improper Integrals Solutions

1. $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx$

   This is an integral over an infinite interval. We evaluate the limit:

$$\lim_{b\to\infty} \int_0^{b} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \Big[\arctan x\Big]_0^b$$

$$= \lim_{b\to\infty} (\arctan b - \arctan 0) = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

2. $\displaystyle\int_0^{1} \frac{1}{\sqrt{x}}\,dx$

   The integrand has an infinite discontinuity at $x = 0$.

$$\lim_{a\to 0^+} \int_a^{1} x^{-1/2}\,dx = \lim_{a\to 0^+} \Big[2x^{1/2}\Big]_a^1$$

$$= \lim_{a\to 0^+} (2(1) - 2\sqrt{a}) = 2 - 0 = \boxed{2}$$

3. $\displaystyle\int_0^{\infty} e^{-x}\,dx$

   Evaluate as a limit to infinity:

$$\lim_{b\to\infty} \int_0^{b} e^{-x}\,dx = \lim_{b\to\infty} \Big[-e^{-x}\Big]_0^b$$

$$= \lim_{b\to\infty} (-e^{-b} - (-e^{0})) = 0 + 1 = \boxed{1}$$

4. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

$$\lim_{b\to\infty} \int_1^{b} x^{-2}\,dx = \lim_{b\to\infty} \Big[-\frac{1}{x}\Big]_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} - (-1)\right) = 0 + 1 = \boxed{1}$$

5. $\displaystyle\int_0^{1} \frac{1}{\sqrt{1-x}}\,dx$

   The integrand is discontinuous at $x = 1$. Let $u = 1 - x$, so $du = -dx$.

   - Limits: As $x \to 0, u \to 1$. As $x \to 1^-, u \to 0^+$.

$$\int_0^{1} (1-x)^{-1/2}\,dx = \int_0^{1} u^{-1/2}\,du$$

   The new integral is improper at $u = 0$:

$$\lim_{a\to 0^+} \Big[2u^{1/2}\Big]_a^1 = 2(1) - 0 = \boxed{2}$$

6. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

   Discontinuous at $x = -1$ and $x = 1$. Technically, we split the integral at 0.

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

   Evaluating the limits at the endpoints:

$$\lim_{x\to 1^-} \arcsin x - \lim_{x\to -1^+} \arcsin x$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

7. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

Rewrite as an improper integral:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx.$$

Use integration by parts with

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx, \qquad dv = x^{-2}\,dx \quad \Rightarrow \quad v = -\frac{1}{x}.$$

Then

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2}\,dx &= [uv]_1^b - \int_1^b v\,du \\ &= \left[-\frac{\ln x}{x}\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\,dx\right) \\ &= \left[-\frac{\ln x}{x}\right]_1^b + \int_1^b \frac{1}{x^2}\,dx \\ &= \left[-\frac{\ln x}{x}\right]_1^b + \left[-\frac{1}{x}\right]_1^b.\end{aligned}$$

Now take the limit. Since $\frac{\ln b}{b} \to 0$ as $b\to\infty$ and $\ln 1 = 0$,

$$\lim_{b\to\infty}\left[-\frac{\ln x}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - 0\right) = 0,$$

and

$$\lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b} - (-1)\right) = 1.$$

Therefore,

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

8. $\int_0^1 \ln(x)\,dx$

Discontinuous at $x = 0$. $\int \ln x\,dx = x\ln x - x$.

$$\Big[x\ln x - x\Big]_1 - \lim_{a\to 0^+}(a\ln a - a)$$

We use L'Hôpital's Rule for the limit: $\lim_{a\to 0^+} \frac{\ln a}{1/a} = 0$.

$$= (0-1) - (0-0) = \boxed{-1}$$

9. $\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$

Let $u = 1 + x^2$, $du = 2x\,dx$.

- Limits: $x = 0 \to u = 1$, $x \to \infty \to u \to \infty$.

$$\begin{aligned}\frac{1}{2}\lim_{b\to\infty}\int_1^b u^{-2}\,du &= \frac{1}{2}\lim_{b\to\infty}\left[-\frac{1}{u}\right]_1^b \\ &= \frac{1}{2}(0-(-1)) = \boxed{\frac{1}{2}}\end{aligned}$$

10. $\int_0^2 \frac{dx}{(2-x)^{1/3}}$

Discontinuous at $x = 2$. Let $u = 2 - x$.

- Limits: $x = 0 \to u = 2$, $x \to 2^- \to u \to 0^+$.

$$\int_0^2 u^{-1/3}\,du = \lim_{a\to 0^+}\left[\frac{3}{2}u^{2/3}\right]_a^2$$
$$= \frac{3}{2}(2^{2/3}) - 0 = \boxed{3\cdot 2^{-1/3}}$$

11. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

Discontinuous at $x = 1$. Trig substitution $x = \sin\theta$ removes the singularity.

- Limits: $x = 0 \to \theta = 0$, $x = 1 \to \theta = \pi/2$.

$$\int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \int_0^{\pi/2} \sin^2\theta\,d\theta$$

The new integral is proper. Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$:

$$\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

12. $\displaystyle\int_0^{\pi/2} \tan x\,dx$

Discontinuous at $x = \pi/2$. Antiderivative is $-\ln|\cos x|$.

$$\lim_{b\to\pi/2^-}\Big[-\ln(\cos x)\Big]_0^b$$

As $b \to \pi/2^-$, $\cos b \to 0^+$, so $\ln(\cos b) \to -\infty$.

$$-(-\infty) - (-\ln 1) = \infty. \quad \boxed{\text{Diverges}}$$

13. $\displaystyle\int_1^{\infty} \frac{1}{x\,(\ln x)^2}\,dx$

Let $u = \ln x$, $du = dx/x$.

- Limits: $x = 1 \to u = 0$, $x \to \infty \to u \to \infty$.

This transforms into $\int_0^\infty u^{-2}\,du$. This integral must be split because it is improper at both 0 and ∞.

$$\int_0^1 u^{-2}du + \int_1^\infty u^{-2}du$$

Looking at the first part: $\lim_{a\to 0^+}[-1/u]_a^1 = -1 - (-\infty) = \infty$. Since one part diverges, the whole integral diverges.

$$\boxed{\text{Diverges}}$$

14. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}\,(1+x)}$

Improper at 0 and ∞. Let $x = t^2$, $dx = 2t\,dt$.

- Limits: $x = 0 \to t = 0$, $x \to \infty \to t \to \infty$.

$$\int_0^\infty \frac{2t\,dt}{t(1+t^2)} = 2\int_0^\infty \frac{1}{1+t^2}\,dt$$

The substitution removed the singularity at 0. We only evaluate the limit at ∞:

$$2\lim_{b\to\infty}\Big[\arctan t\Big]_0^b = 2\left(\frac{\pi}{2} - 0\right) = \boxed{\pi}$$