7.8 Improper Integrals

In defining a definite integral $\int_a^b f(x)\,dx$, we usually assume:

- The interval $[a, b]$ is finite.
- The function f does not have an infinite discontinuity on $[a, b]$.

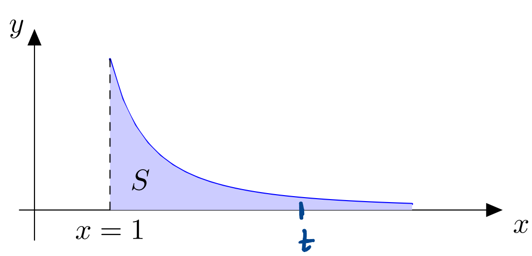
In this section, we extend the concept of definite integrals to cases where:

1. The interval of integration is infinite (e.g. from 1 to ∞).
2. The integrand has an infinite discontinuity within the interval of integration.

In either of these two cases, such an integral is called an *improper integral.*

Type 1: Infinite Intervals

Question. Consider the infinite region S that lies under the curve $y = \dfrac{1}{x^2}$, above the x-axis, and to the right of the line $x = 1$.



Is the area of S infinite?

We want to compute $\int_1^\infty \frac{1}{x^2}\,dx$

Let $A(t) = \int_1^t \frac{1}{x^2}\,dx$ be the area from 1 to t

Then $A(t) = \left[-\frac{1}{x}\right]_1^t = \left[-\frac{1}{t} - (-1)\right] = 1 - \frac{1}{t}$

As $t \to \infty$, $\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$

Therefore, the area is 1, which is finite!

Definition (Improper Integrals of Type 1)**.** Let f be a function defined on $[a, \infty)$ or $(-\infty, b]$. We define the improper integral in the following ways:

(a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number). If the limit exists, we say the integral *converges*; otherwise, it *diverges*.

(b) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

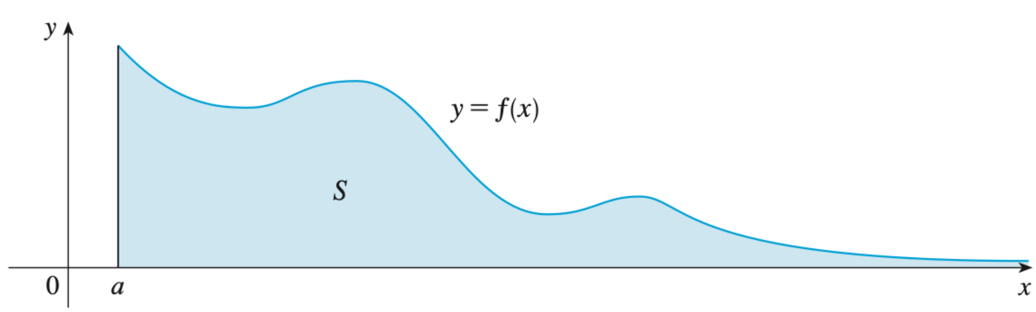
provided the limit exists (as a finite number).

(c) If $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are each convergent, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Any real number a can be used to split the interval.

Remark. Any of the improper integrals in Definition 1 can be interpreted as an area if f is a positive function. For example, in case (a), if $f(x) \geq 0$, the integral $\int_a^\infty f(x)\,dx$ (if convergent) can be viewed as the limit of the areas under f on $[a, t]$ as $t \to \infty$.



Example. Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

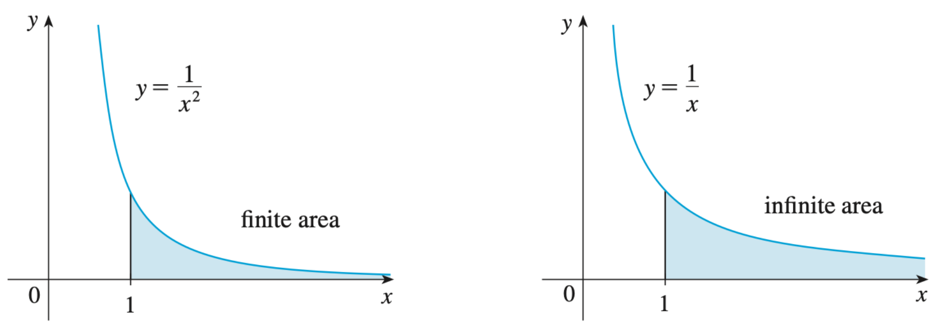
$$= \lim_{t\to\infty} \Big[\ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} \Big[\ln|t| - \ln|1|\Big]$$

$$= \lim_{t\to\infty} \ln t$$

$$= \infty \qquad \boxed{\text{Divergent}}$$

Remark. Compare the integrals $\int_1^\infty \frac{1}{x^2}\,dx$ and $\int_1^\infty \frac{1}{x}\,dx$.



The curves look similar, but the area under $\frac{1}{x^2}$ converges, while the area under $\frac{1}{x}$ diverges.

The values of $\frac{1}{x}$ don't decrease fast enough for the area to converge.

Example. Evaluate $\int_{-\infty}^{0} xe^x \, dx$.

$u = x$	$v = e^x$
$du = dx$	$dv = e^x dx$

We have $\int_{-\infty}^{0} xe^x dx = \lim_{t \to -\infty} \int_t^0 xe^x dx$

Hence,
$$\int_t^0 \overset{u \cdot dv}{xe^x dx} = \overset{u \cdot v}{\left[xe^x\right]_t^0} - \overset{v \cdot du}{\int_t^0 e^x dx}$$
$$= \left[0 \cdot e^0 - t \cdot e^t\right] - \left[e^x\right]_t^0$$
$$= -te^t - (e^0 - e^t)$$
$$= -te^t - 1 + e^t$$

Indeterminate of type $-\infty \cdot 0$

Now, $\lim_{t \to -\infty} \left[-te^t - 1 + e^t\right] = -1 - \lim_{t \to -\infty} te^t + \lim_{t \to -\infty} e^t$ $\quad \leftarrow 0$

Then
$$\lim_{t \to -\infty} te^t = \lim_{t \to -\infty} \frac{t}{e^{-t}} \quad \leftarrow \text{type } \frac{-\infty}{\infty}$$
$$= \lim_{t \to -\infty} \frac{1}{-e^{-t}} \quad \text{L'Hôpital}$$
$$= \lim_{t \to -\infty} \frac{-1}{e^{-t}} \qquad e^{-t} \to \infty$$
$$= 0$$

In total, we have $-1 - 0 + 0 = \boxed{-1}$

Example. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Example. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Type 2: Discontinuous Integrands

Sometimes the interval $[a, b]$ is finite, but the function f has a *vertical asymptote* or other kind of infinite discontinuity at some point in the interval. For instance, f might be continuous on $[a, b)$ and become unbounded as x approaches b.

Definition (Improper Integrals of Type 2)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

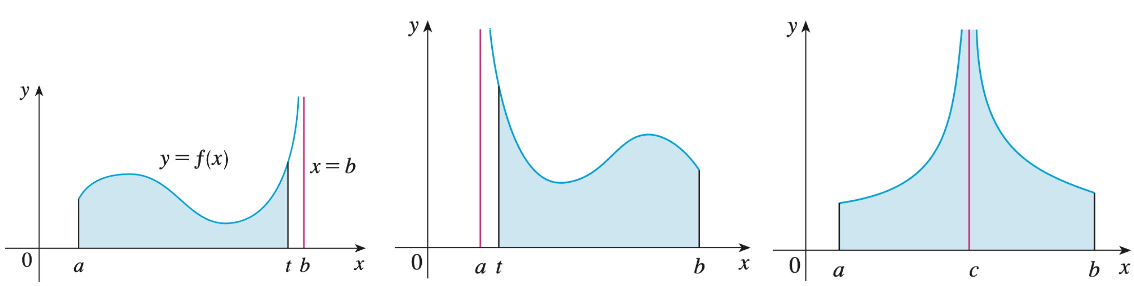
if this limit exists (as a finite number).

(c) If f has a discontinuity at some interior point c, where $a < c < b$, then we split:

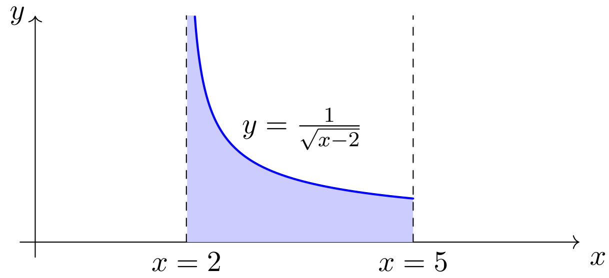
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \rightarrow$$

Need both to converge to be convergent!

provided both integrals on the right-hand side converge separately.



Example. Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.



This integral is improper because of $x=2$.

We can write this as $\displaystyle\lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$

Let $u = x-2$. Then $du = dx$.

As x goes from t to 5, u goes from $t-2$ to 3

$$\lim_{t\to 2^+} \int_{t-2}^{3} \frac{1}{\sqrt{u}}\,du = \lim_{t\to 2^+} \Big[2\sqrt{u}\Big]_{t-2}^{3}$$

$$= \lim_{t\to 2^+} \left[2\sqrt{3} - 2\sqrt{t-2}\right]$$

$$= 2\sqrt{3} - 0$$

$$= 2\sqrt{3}$$

Example. Determine whether $\int_0^{\pi/2} \sec x \, dx$ converges or diverges.

This is improper since as $x \to \frac{\pi}{2}^-$, $\cos x \to 0^+$ and $\sec x \to \infty$

We need to evaluate $\lim\limits_{t \to \frac{\pi}{2}^-} \int_0^t \sec x \, dx$

$$= \lim_{t \to \frac{\pi}{2}^-} \Big[\ln|\sec x + \tan x| \Big]_0^t$$

$$= \lim_{t \to \frac{\pi}{2}^-} \Big[\underbrace{\ln|\sec t + \tan t|}_{\to \infty} - \underbrace{\ln|\sec(0) + \tan(0)|}_{= 0} \Big]$$

$\boxed{\text{Diverges}}$

Example. Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$ if possible.

$\frac{1}{x-1}$ has a vertical asymptote at $x=1$. Split the integral at $x=1$

$$\int_0^3 \frac{dx}{x-1} = \overset{①}{\int_0^1 \frac{dx}{x-1}} + \overset{②}{\int_1^3 \frac{dx}{x-1}}$$

* We need both ① and ② to converge for the integral to converge.

$$① = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \Big[\ln|x-1|\Big]_0^t = \lim_{t\to 1^-} \Big[\ln|t-1| - \ln|-1|\Big] = \lim_{t\to 1^-} \ln|t-1|$$

As $t\to 1^-$, $|t-1| \to 0^+$. So $\displaystyle\lim_{t\to 1^-} \ln|t-1| = -\infty$

Since one of ① or ② diverges, the integral diverges.

Remark. If we ignore the vertical asymptote at $x = 1$, we might erroneously write:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big|_0^3 = \ln|2| - \ln|-1| = \ln 2,$$

Whenever you see $\int_a^b f(x)\,dx$, always check whether it is an improper integral.

Example. Evaluate $\int_0^1 \ln(x)\, dx$.

A Comparison Test for Improper Integrals

In many situations, finding the exact value of an improper integral is difficult or impossible, but we still want to know whether the integral converges or diverges. The following Comparison Test is very helpful in these cases. We state it for Type 1 (infinite intervals), but a similar result holds for Type 2 (discontinuous integrands).

Theorem (Comparison Test)**.** Suppose f and g are continuous functions on $[a, \infty)$ with

$$0 \leq g(x) \leq f(x) \quad \text{for all } x \geq a.$$

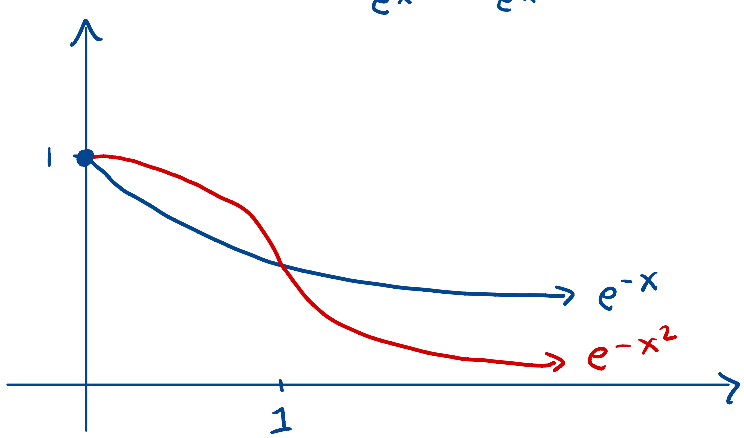
(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is also convergent

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is also divergent.

Example. Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ is convergent.

On the interval $[1, \infty)$, $0 \leq x \leq x^2$.

So $0 \leq e^x \leq e^{x^2}$. So $0 \leq \frac{1}{e^{x^2}} \leq \frac{1}{e^x}$. So $0 \leq e^{-x^2} \leq e^{-x}$.



Since $0 \leq e^{-x^2} \leq e^{-x}$ only holds on $[1, \infty)$, we split the integral:

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

On the interval $[0,1]$, e^{-x^2} is convergent.

By the Comparison Thm,

$$0 \le \int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx$$

Now $\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx$

$$= \lim_{t\to\infty} \left[-e^{-x}\right]_1^t$$

$$= \lim_{t\to\infty} \left[-e^{-t} + e^{-1}\right]$$

$-e^{-t} = \frac{-1}{e^t} \to 0$

$$= e^{-1}$$

Conclude: $\int_1^\infty e^{-x^2}\,dx$ converges by the comparison Thm,

and so $\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$ also converges.

Example. Show that $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.