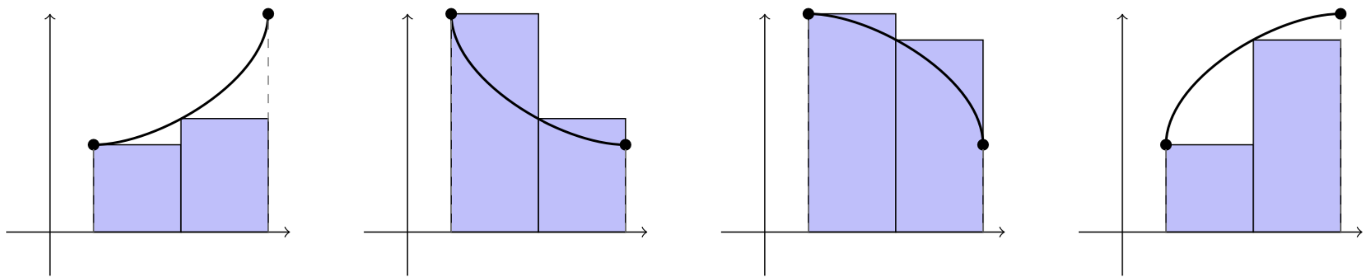
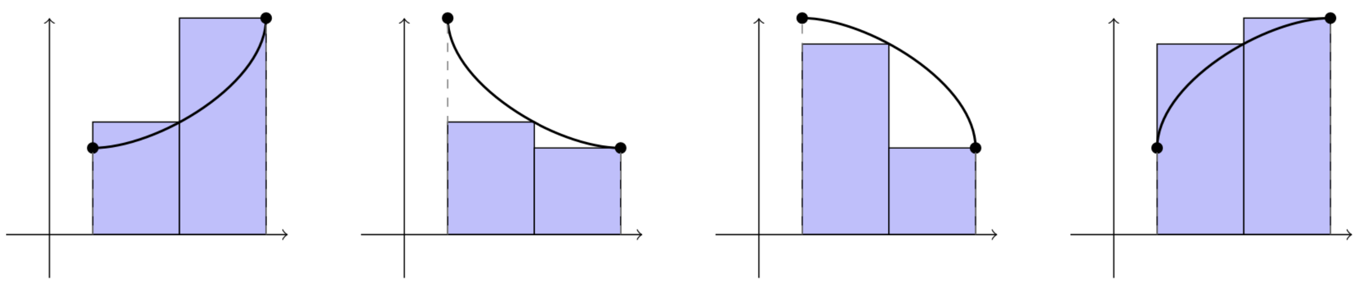
7.7 Approximate Integration

We approximate the integral $I = \int_a^b f(x)\,dx$ using four common methods:

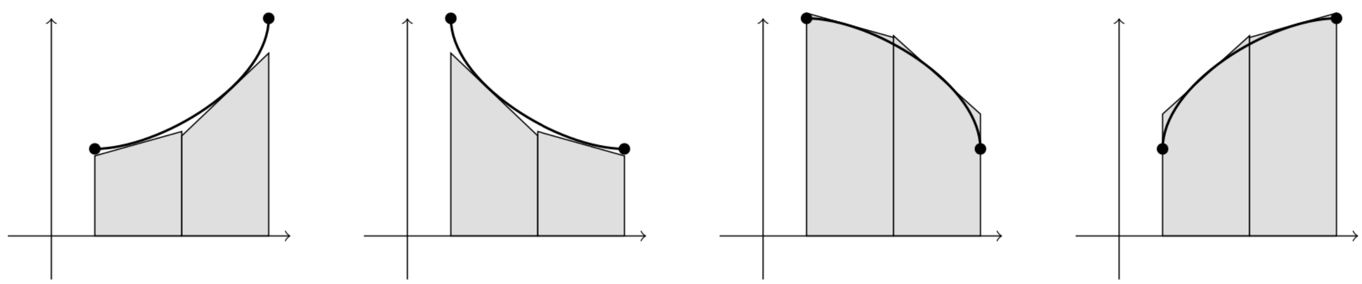
- Left-hand sum (L_n)



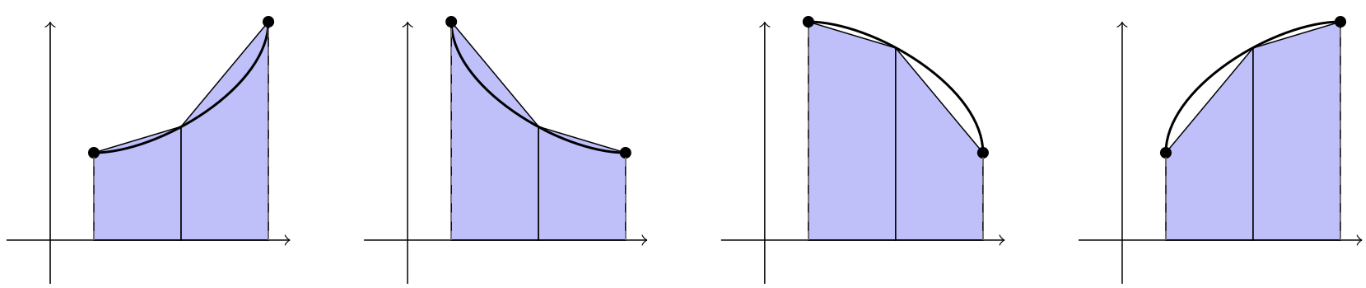
- Right-hand sum (R_n)



- Midpoint rule (M_n), (here, the tops of the rectangles have been “rotated”)



- Trapezoidal rule (T_n)



Relation of Each Method to the Integral

The behavior of each method depends on two key features of f:

1. **Increasing/Decreasing:** For an *increasing* function, the left-hand sum underestimates and the right-hand sum overestimates I. For a decreasing function, the roles are reversed.

2. **Concavity:**

   - If f is *concave up*, then the graph of f lies *below* the chord joining any two points on it. Consequently, the **trapezoidal rule** *overestimates* the true area for an increasing function. In contrast, the **midpoint rule** uses the value at the center of the interval; for a concave up function this value is lower than the average of the endpoints, so the midpoint rule *underestimates* the area.
   - If f is *concave down*, the graph lies *above* the chord. Thus, for an increasing function the trapezoidal rule *underestimates* I while the midpoint rule *overestimates* it.

Function Type	**Concave Up**	**Concave Down**
Increasing	L_n : Underestimate R_n : Overestimate M_n : Underestimate T_n : Overestimate	L_n : Underestimate R_n : Overestimate M_n : Overestimate T_n : Underestimate
Decreasing	L_n : Overestimate R_n : Underestimate M_n : Underestimate T_n : Overestimate	L_n : Overestimate R_n : Underestimate M_n : Overestimate T_n : Underestimate

Comparison of Integral Approximations

1. Increasing, Concave Up: Since the function is increasing, the left Riemann sum L_n underestimates the integral, while the right Riemann sum R_n overestimates it. Therefore:

$$L_n < I < R_n$$

Now, imagine constructing a rectangle whose height is determined using midpoints. The function is increasing, so the function value at the left endpoint is lower than at the midpoint. Since the function is also concave up, the secant line between the endpoints of the subinterval lies above the function's graph. This means that if we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would need to raise its top. In other words, the midpoint rectangle determines a trapezoid that lies below the curve, and the secant line determines a trapezoid that lies above the curve. Therefore:

$$M_n < I < T_n$$

Comparing L_n and M_n, the left endpoints are lower than the midpoints, so:

$$L_n < M_n.$$

Each trapezoid lies within the rectangle obtained from using right endpoints, so:

$$T_n < R_n.$$

Thus, the overall ordering is:

$$L_n < M_n < \int_a^b f(x)dx < T_n < R_n.$$

2. Increasing, Concave Down: Since the function is increasing, L_n underestimates the integral, while R_n overestimates it. Therefore:

$$L_n < I < R_n$$

Now, consider the midpoint rule. Since the function is concave down, the secant line between the endpoints lies below the function's graph. This means that if we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would then need to lower its top. In other words, the midpoint rectangle determines a trapezoid that lies above the curve, while the secant line determines a trapezoid that lies below the curve. Therefore:

$$T_n < I < M_n$$

Comparing L_n and T_n, each rectangle determined by the left endpoint lies within a trapezoid determined by the secant line, so:

$$L_n < T_n.$$

Since the midpoints are lower than the right endpoints:

$$M_n < R_n.$$

Thus, the overall ordering is:

$$L_n < T_n < \int_a^b f(x)dx < M_n < R_n.$$

3. Decreasing, Concave Up: Since the function is decreasing, L_n overestimates the integral, while R_n underestimates it. Therefore:

$$R_n < I < L_n$$

For the midpoint rule, since the function is concave up, the secant line between the endpoints lies above the function's graph. If we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would then need to raise its top. This means the midpoint rectangle determines a trapezoid that lies below the curve, while the secant line determines a trapezoid that lies above the curve. Therefore:

$$M_n < I < T_n$$

Comparing R_n and M_n, the right endpoints are lower than the midpoints, so:

$$R_n < M_n.$$

Each trapezoid lies within the rectangle obtained from using left endpoints, so:

$$T_n < L_n.$$

Thus, the overall ordering is:

$$R_n < M_n < \int_a^b f(x)dx < T_n < L_n.$$

4. Decreasing, Concave Down: Since the function is decreasing, L_n overestimates the integral, while R_n underestimates it. Therefore:

$$R_n < I < L_n$$

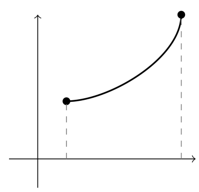
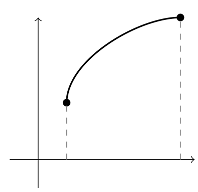
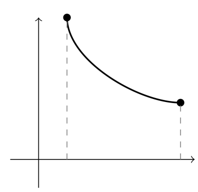
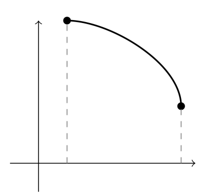
For the midpoint rule, since the function is concave down, the secant line between the endpoints lies below the function's graph. If we visualize "rotating" the top of the midpoint rectangle so that it is parallel to the secant line, we would then need to lower its top. This means the midpoint rectangle determines a trapezoid that lies above the curve, while the secant line determines a trapezoid that lies below the curve. Therefore:

$$T_n < I < M_n$$

Comparing R_n and T_n, each rectangle determined by the right endpoint lies within a trapezoid determined by the secant line, so $R_n < T_n$. Since the midpoints are lower than the left endpoints, we have $M_n < L_n$. Thus, the overall ordering is:

$$R_n < T_n < \int_a^b f(x)dx < M_n < L_n.$$

Summary of Orderings:

Case	**Graph**	**Order**
Increasing, Concave Up		$L_n < M_n < \int f(x)dx < T_n < R_n$
Increasing, Concave Down		$L_n < T_n < \int f(x)dx < M_n < R_n$
Decreasing, Concave Up		$R_n < M_n < \int f(x)dx < T_n < L_n$
Decreasing, Concave Down		$R_n < T_n < \int f(x)dx < M_n < L_n$