7.4 Integration of Rational Functions by Partial Fractions

Question. What is the goal of a partial fraction decomposition?

Question. What is a proper rational function? What is an improper rational function?

Question. Partial fractions only apply to proper fractions. What if $\frac{P(x)}{Q(x)}$ is improper?

Theorem (Partial Fraction Decomposition)**.** Any proper rational function $\frac{P(x)}{Q(x)}$ can be rewritten as a sum of simpler fractions, called partial fractions. The decomposition is built from the following components based on the factors of $Q(x)$:

1. **Distinct Linear Factors:** For each distinct linear term (a_1x+b_1) in $Q(x)$, the partial fraction decomposition includes a term of the form:

$$\frac{A_1}{a_1x+b_1}.$$

2. **Repeated Linear Factors:** For each repeated linear factor $(a_1x+b_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form:

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_m}{(a_1x+b_1)^m}.$$

3. **Irreducible Quadratic Factors:** For each irreducible quadratic term $(a_1x^2+b_1x+c_1)$ in $Q(x)$, where $b_1^2-4a_1c_1<0$, the partial fraction decomposition includes a term of the form:

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}.$$

4. **Repeated Irreducible Quadratic Factors:** For each repeated irreducible quadratic factor $(a_1x^2+b_1x+c_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form:

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}+\frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2}+\cdots+\frac{A_mx+B_m}{(a_1x^2+b_1x+c_1)^m}.$$

Example. Evaluate the partial fraction decomposition of $\dfrac{3x^2+5x+2}{(x-1)(x+2)(x-3)}$.

Example. Evaluate the partial fraction decomposition of $\dfrac{4x^2+7}{(x+1)^2(x-2)}$.

Example. Evaluate the partial fraction decomposition of $\dfrac{2x^2+3x+1}{(x^2+x+1)(x-1)}$.

Example. Evaluate the partial fraction decomposition of $\dfrac{5x^3+2x^2+x+4}{(x^2+1)^2(x-2)}$.

Procedure to Integrate Rational Functions by Partial Fractions

1. **Check if the rational function is proper:**
   - A rational function $\frac{P(x)}{Q(x)}$ is proper if $\deg(P) < \deg(Q)$.
   - If $\deg(P) \geq \deg(Q)$, perform polynomial long division to express it as:

   $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

   where $\deg(R) < \deg(Q)$.

2. **Factor the denominator $Q(x)$:**
   - Fully factor $Q(x)$ into linear and irreducible quadratic factors.
   - For example: $Q(x) = (x - a_1)(x - a_2)^2(x^2 + bx + c)$.

3. **Set up the partial fraction decomposition:**
   - Distinct Linear Factors
   - Repeated Linear Factors
   - Irreducible Quadratic Factors.
   - Repeated Irreducible Quadratic Factors.

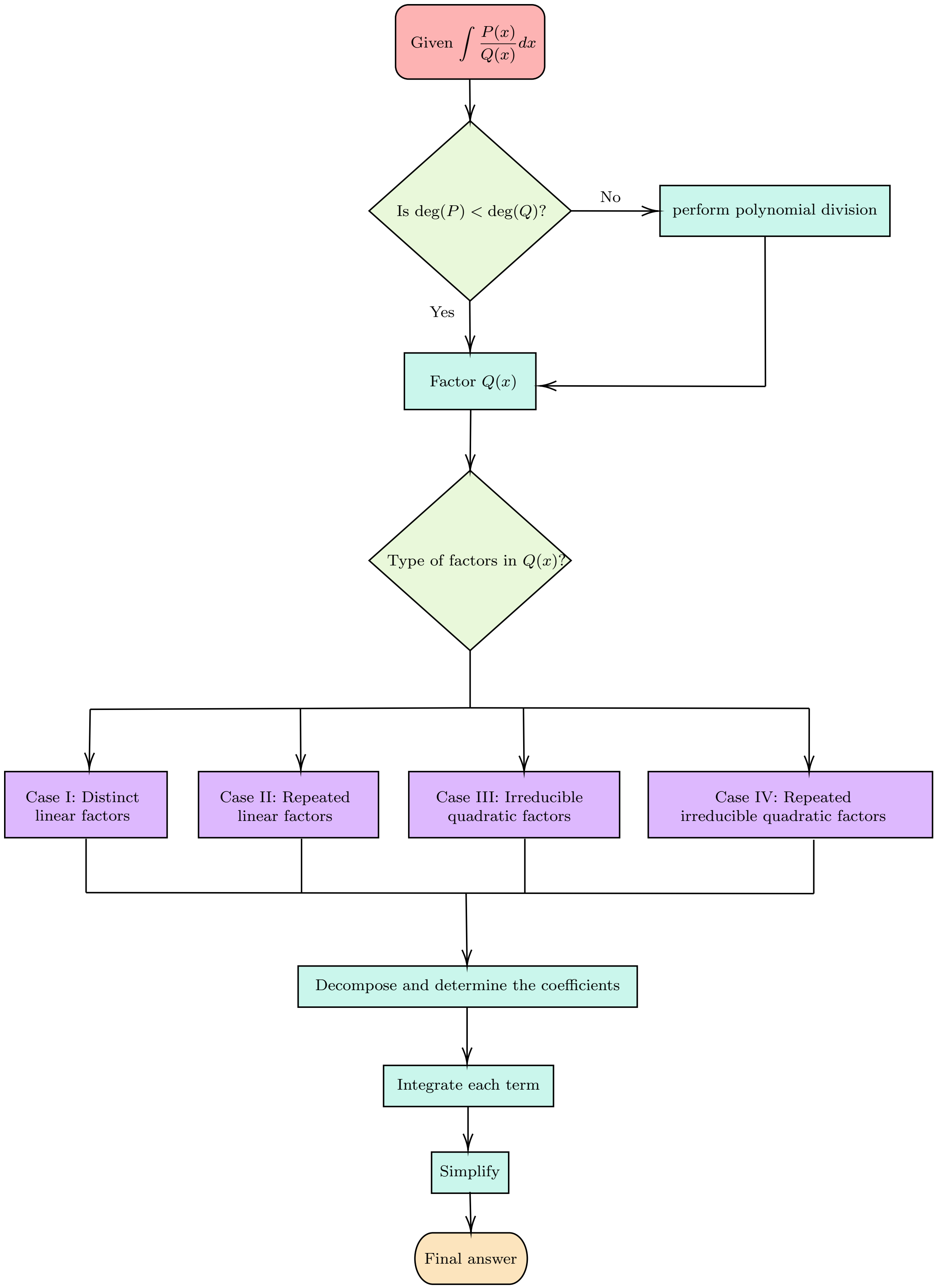
4. **Determine the coefficients:**
   - Multiply through by the denominator $Q(x)$ to eliminate fractions.
   - Expand and collect terms to form a polynomial equation.
   - Solve for the unknown coefficients by equating coefficients of corresponding powers of x.

5. **Integrate each term:**
   - Use basic integration formulas for linear and quadratic terms:

   $$\int \frac{1}{x-a}\,dx = \ln|x-a| + C, \quad \int \frac{Ax+B}{x^2+bx+c}\,dx = \text{(substitution or arctangent)}.$$

   - Combine the results.

6. **Simplify the final answer:**
   - Use logarithmic properties to combine terms where possible.
   - Ensure the answer is in its simplest form.



Given $\int \frac{P(x)}{Q(x)}dx$
Is $\deg(P) < \deg(Q)$?
No
perform polynomial division
Yes
Factor $Q(x)$
Type of factors in $Q(x)$?
Case I: Distinct linear factors
Case II: Repeated linear factors
Case III: Irreducible quadratic factors
Case IV: Repeated irreducible quadratic factors
Decompose and determine the coefficients
Integrate each term
Simplify
Final answer

Example. Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

Example. Find $\displaystyle\int \frac{dx}{x^2 - 9}$.

Example. Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$.

Example. Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$.

Example. Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx$.

Example. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Example. Should we use partial fractions to solve $\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,dx$?

Example. Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\, dx$.