7.4 Partial Fractions Solutions

1. $\displaystyle\int \frac{1}{x^2-4}\,dx$

Factor the denominator:

$$x^2 - 4 = (x-2)(x+2)$$

Partial fraction decomposition: Decompose the integrand into partial fractions:

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

Multiply both sides by $(x-2)(x+2)$ to clear the denominator:

$$1 = A(x+2) + B(x-2)$$

Solve for coefficients A and B:

- Let $x = 2$:

$$1 = A(2+2) + B(0) \implies 4A = 1 \implies A = \frac{1}{4}$$

- Let $x = -2$:

$$1 = A(0) + B(-2-2) \implies -4B = 1 \implies B = -\frac{1}{4}$$

Integrate: Rewrite the integral using the determined coefficients:

$$\int \left(\frac{1/4}{x-2} - \frac{1/4}{x+2} \right) dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

2. $\displaystyle\int \frac{1}{(x-1)(x+2)}\,dx$

Partial fraction decomposition:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Multiply by $(x-1)(x+2)$:

$$1 = A(x+2) + B(x-1)$$

Solve for coefficients:

- Let $x = 1$:

$$1 = A(1+2) \implies 3A = 1 \implies A = \frac{1}{3}$$

- Let $x = -2$:

$$1 = B(-2-1) \implies -3B = 1 \implies B = -\frac{1}{3}$$

Integrate:

$$\int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2} \right) dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

3. $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

Factor the denominator:

$$x^2 + 3x + 2 = (x+1)(x+2)$$

Partial fraction decomposition:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Multiply by $(x+1)(x+2)$:

$$1 = A(x+2) + B(x+1)$$

Solve for coefficients:

- Let $x = -1$:

$$1 = A(-1+2) \implies A = 1$$

- Let $x = -2$:

$$1 = B(-2+1) \implies -B = 1 \implies B = -1$$

Integrate:

$$\int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx$$
$$= \ln|x+1| - \ln|x+2| + C$$

4. $\displaystyle\int \frac{2x+3}{x^2+x-2}\,dx$

Factor the denominator:

$$x^2 + x - 2 = (x-1)(x+2)$$

Partial fraction decomposition:

$$\frac{2x+3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Multiply by $(x-1)(x+2)$:

$$2x + 3 = A(x+2) + B(x-1)$$

Solve for coefficients:

- Let $x = 1$:

$$2(1) + 3 = A(1+2) \implies 5 = 3A \implies A = \frac{5}{3}$$

- Let $x = -2$:

$$2(-2) + 3 = B(-2-1) \implies -1 = -3B \implies B = \frac{1}{3}$$

Integrate:

$$\int \left(\frac{5/3}{x-1} + \frac{1/3}{x+2} \right) dx$$
$$= \frac{5}{3}\ln|x-1| + \frac{1}{3}\ln|x+2| + C$$

5. $\displaystyle\int \frac{1}{x^3-x}\,dx$

Factor the denominator:

$$x^3 - x = x(x^2-1) = x(x-1)(x+1)$$

Partial fraction decomposition:

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Multiply by $x(x-1)(x+1)$:

$$1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Solve for coefficients:

- Let $x = 0$:

$$1 = A(-1)(1) \implies 1 = -A \implies A = -1$$

- Let $x = 1$:

$$1 = B(1)(2) \implies 1 = 2B \implies B = \frac{1}{2}$$

- Let $x = -1$:

$$1 = C(-1)(-2) \implies 1 = 2C \implies C = \frac{1}{2}$$

Integrate:

$$\int \left(-\frac{1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} \right) dx$$
$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

6. $\displaystyle\int \frac{x^2}{x^3-1}\,dx$

Method 1: Substitution

Let $u = x^3 - 1$. Then, $du = 3x^2\,dx$, which implies $x^2\,dx = \frac{1}{3}\,du$. Substituting into the integral:

$$\int \frac{1}{u} \cdot \frac{1}{3}\,du = \frac{1}{3}\ln|u| + C$$

Substituting back $u = x^3 - 1$:

$$= \frac{1}{3} \ln |x^3 - 1| + C$$

Method 2: Partial Fractions
Factor the denominator: $x^3 - 1 = (x-1)(x^2+x+1)$.

$$\frac{x^2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Multiply by $(x-1)(x^2+x+1)$:

$$x^2 = A(x^2+x+1) + (Bx+C)(x-1)$$

- Let $x = 1$: $1 = 3A \implies A = \frac{1}{3}$.
- Equate coefficients of x^2: $1 = A + B \implies 1 = \frac{1}{3} + B \implies B = \frac{2}{3}$.
- Equate constant terms: $0 = A - C \implies C = A = \frac{1}{3}$.

The integral becomes:

$$\frac{1}{3} \int \frac{1}{x-1}\, dx + \frac{1}{3} \int \frac{2x+1}{x^2+x+1}\, dx$$

Noting that the numerator $(2x+1)$ is the exact derivative of the denominator (x^2+x+1) in the second term:

$$= \frac{1}{3} \ln |x-1| + \frac{1}{3} \ln |x^2+x+1| + C$$

Using log properties $(\ln a + \ln b = \ln ab)$, this confirms the result from Method 1:

$$= \frac{1}{3} \ln |(x-1)(x^2+x+1)| + C = \frac{1}{3} \ln |x^3-1| + C$$

7. $\displaystyle\int \frac{1}{x(x^2+1)}\, dx$

Partial fraction decomposition:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiply by $x(x^2+1)$:

$$1 = A(x^2+1) + x(Bx+C)$$

Expanding terms:

$$1 = (A+B)x^2 + Cx + A$$

Solve for coefficients: Equating coefficients of like powers:

- Constant term: $A = 1$.
- x term: $C = 0$.
- x^2 term: $A + B = 0 \implies 1 + B = 0 \implies B = -1$.

Integrate:

$$\int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx$$

For the second term, let $u = x^2 + 1$, so $du = 2x\, dx$, or $\frac{1}{2} du = x\, dx$.

$$= \ln |x| - \frac{1}{2} \ln(x^2+1) + C$$

(Note: $|x^2+1|$ can be written as (x^2+1) since it is always positive.)

8. $\displaystyle\int \frac{x}{x^3+x^2}\, dx$

Simplify the integrand: Before applying partial fractions, reduce the fraction:

$$\frac{x}{x^3+x^2} = \frac{x}{x^2(x+1)} = \frac{1}{x(x+1)}$$

Partial fraction decomposition:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Multiply by $x(x+1)$:

$$1 = A(x+1) + Bx$$

Solve for coefficients:

- Let $x = 0$: $1 = A(1) \implies A = 1$.
- Let $x = -1$: $1 = B(-1) \implies B = -1$.

Integrate:

$$\int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$
$$= \ln|x| - \ln|x+1| + C$$

9. $\int \frac{1}{(x-1)^2(x+2)}\, dx$

Partial fraction decomposition: Include a term for each power of the repeated factor $(x-1)$:

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Multiply by $(x-1)^2(x+2)$:

$$1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Solve for coefficients:

- Let $x = 1$:
$$1 = B(1+2) \implies 3B = 1 \implies B = \frac{1}{3}$$
- Let $x = -2$:
$$1 = C(-2-1)^2 \implies 1 = C(-3)^2 \implies 9C = 1 \implies C = \frac{1}{9}$$
- Equate coefficients of x^2: Expansion gives $Ax^2 + Cx^2 = (A+C)x^2$. Since there is no x^2 term on the left side:
$$A + C = 0 \implies A = -C \implies A = -\frac{1}{9}$$

Integrate:

$$\int \left(-\frac{1/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{1/9}{x+2}\right) dx$$
$$= -\frac{1}{9}\ln|x-1| + \frac{1}{3}\int (x-1)^{-2}\, dx + \frac{1}{9}\ln|x+2|$$
$$= -\frac{1}{9}\ln|x-1| - \frac{1}{3(x-1)} + \frac{1}{9}\ln|x+2| + C$$

10. $\int \frac{x^4 + 2x^3 + 3x^2 + 4x + 5}{(x-1)(x^2+x+1)}\, dx$

Step 1: Long Division

The denominator expands to $x^3 - 1$. Since the degree of the numerator (4) is greater than the degree of the denominator (3), perform polynomial long division.

Dividing $x^4 + 2x^3 + 3x^2 + 4x + 5$ by $x^3 - 1$ yields:

$$\text{Quotient} = x + 2, \quad \text{Remainder} = 3x^2 + 5x + 7$$

Thus, the integral becomes:

$$\int (x+2)\, dx + \int \frac{3x^2 + 5x + 7}{(x-1)(x^2+x+1)}\, dx$$

The first part integrates to $\frac{1}{2}x^2 + 2x$. We focus now on the rational function.

Step 2: Partial Fraction Decomposition

$$\frac{3x^2 + 5x + 7}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}$$

Multiply by $(x-1)(x^2+x+1)$:

$$3x^2 + 5x + 7 = A(x^2 + x + 1) + (Bx + C)(x - 1)$$

Solve for coefficients:

- Let $x = 1$:
$$3(1) + 5(1) + 7 = A(1+1+1) \implies 15 = 3A \implies A = 5$$
- Equate coefficients of x^2:
$$3 = A + B \implies 3 = 5 + B \implies B = -2$$

- Equate constant terms:

$$7 = A - C \implies 7 = 5 - C \implies C = -2$$

Step 3: Integrate the Rational Part

$$\int \frac{5}{x-1}\,dx + \int \frac{-2x-2}{x^2+x+1}\,dx$$

The first term is $5\ln|x-1|$. For the second term, factor out -1 and split the numerator to match the derivative of the denominator $(2x+1)$:

$$-\int \frac{2x+2}{x^2+x+1}\,dx = -\int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{1}{x^2+x+1}\,dx$$

(a) $\int \frac{2x+1}{x^2+x+1}\,dx = \ln|x^2+x+1|$ (using u-substitution).

(b) $\int \frac{1}{x^2+x+1}\,dx$: Complete the square in the denominator.

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

Using the standard integral $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{u}{a})$, where $a = \frac{\sqrt{3}}{2}$:

$$\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Final Answer: Combining all parts:

$$\frac{1}{2}x^2 + 2x + 5\ln|x-1| - \ln(x^2+x+1) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$