7.4 Integration of Rational Functions by Partial Fractions

Question. What is the goal of a partial fraction decomposition?

Solution:
Write $\frac{P(x)}{Q(x)}$ as a sum of simpler fractions to make the function easier to integrate.

Question. What is a proper rational function? What is an improper rational function?

Solution:
Proper: $\deg P(x) < \deg Q(x)$. **Improper:** $\deg P(x) \geq \deg Q(x)$.

Question. Partial fractions only apply to proper fractions. What if $\frac{P(x)}{Q(x)}$ is improper?

Solution:
Apply polynomial long division to obtain:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $S(x)$ is a polynomial, and $\frac{R(x)}{Q(x)}$ is proper. Apply partial fraction decomposition to $\frac{R(x)}{Q(x)}$

Theorem (Partial Fraction Decomposition)**.** Any proper rational function $\frac{P(x)}{Q(x)}$ can be rewritten as a sum of simpler fractions, called partial fractions. The decomposition is built from the following components based on the factors of $Q(x)$:

1. **Distinct Linear Factors:** For each distinct linear term (a_1x+b_1) in $Q(x)$, the partial fraction decomposition includes a term of the form:

$$\frac{A_1}{a_1x+b_1}.$$

2. **Repeated Linear Factors:** For each repeated linear factor $(a_1x+b_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form:

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_m}{(a_1x+b_1)^m}.$$

3. **Irreducible Quadratic Factors:** For each irreducible quadratic term $(a_1x^2+b_1x+c_1)$ in $Q(x)$, where $b_1^2 - 4a_1c_1 < 0$, the partial fraction decomposition includes a term of the form:

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}.$$

4. **Repeated Irreducible Quadratic Factors:** For each repeated irreducible quadratic factor $(a_1x^2+b_1x+c_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form:

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2} + \cdots + \frac{A_mx+B_m}{(a_1x^2+b_1x+c_1)^m}.$$

Example. Evaluate the partial fraction decomposition of $\dfrac{3x^2+5x+2}{(x-1)(x+2)(x-3)}$.

Solution:

Case I: Distinct Linear Factors

Decomposition:

$$\frac{3x^2+5x+2}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}.$$

Example. Evaluate the partial fraction decomposition of $\dfrac{4x^2+7}{(x+1)^2(x-2)}$.

Solution:

Case II: Repeated Linear Factors

Decomposition:

$$\frac{4x^2+7}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}.$$

Example. Evaluate the partial fraction decomposition of $\dfrac{2x^2+3x+1}{(x^2+x+1)(x-1)}$.

Solution:

Case III: Irreducible Quadratic Factors

Decomposition:

$$\frac{2x^2+3x+1}{(x^2+x+1)(x-1)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}.$$

Example. Evaluate the partial fraction decomposition of $\dfrac{5x^3+2x^2+x+4}{(x^2+1)^2(x-2)}$.

Solution:

Case IV: Repeated Irreducible Quadratic Factors

Decomposition:

$$\frac{5x^3+2x^2+x+4}{(x^2+1)^2(x-2)} = \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x^2+1)^2} + \frac{C}{x-2}.$$

Procedure to Integrate Rational Functions by Partial Fractions

1. **Check if the rational function is proper:**
   - A rational function $\frac{P(x)}{Q(x)}$ is proper if $\deg(P) < \deg(Q)$.
   - If $\deg(P) \geq \deg(Q)$, perform polynomial long division to express it as:

   $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

   where $\deg(R) < \deg(Q)$.

2. **Factor the denominator $Q(x)$:**
   - Fully factor $Q(x)$ into linear and irreducible quadratic factors.
   - For example: $Q(x) = (x - a_1)(x - a_2)^2(x^2 + bx + c)$.

3. **Set up the partial fraction decomposition:**
   - Distinct Linear Factors
   - Repeated Linear Factors
   - Irreducible Quadratic Factors.
   - Repeated Irreducible Quadratic Factors.

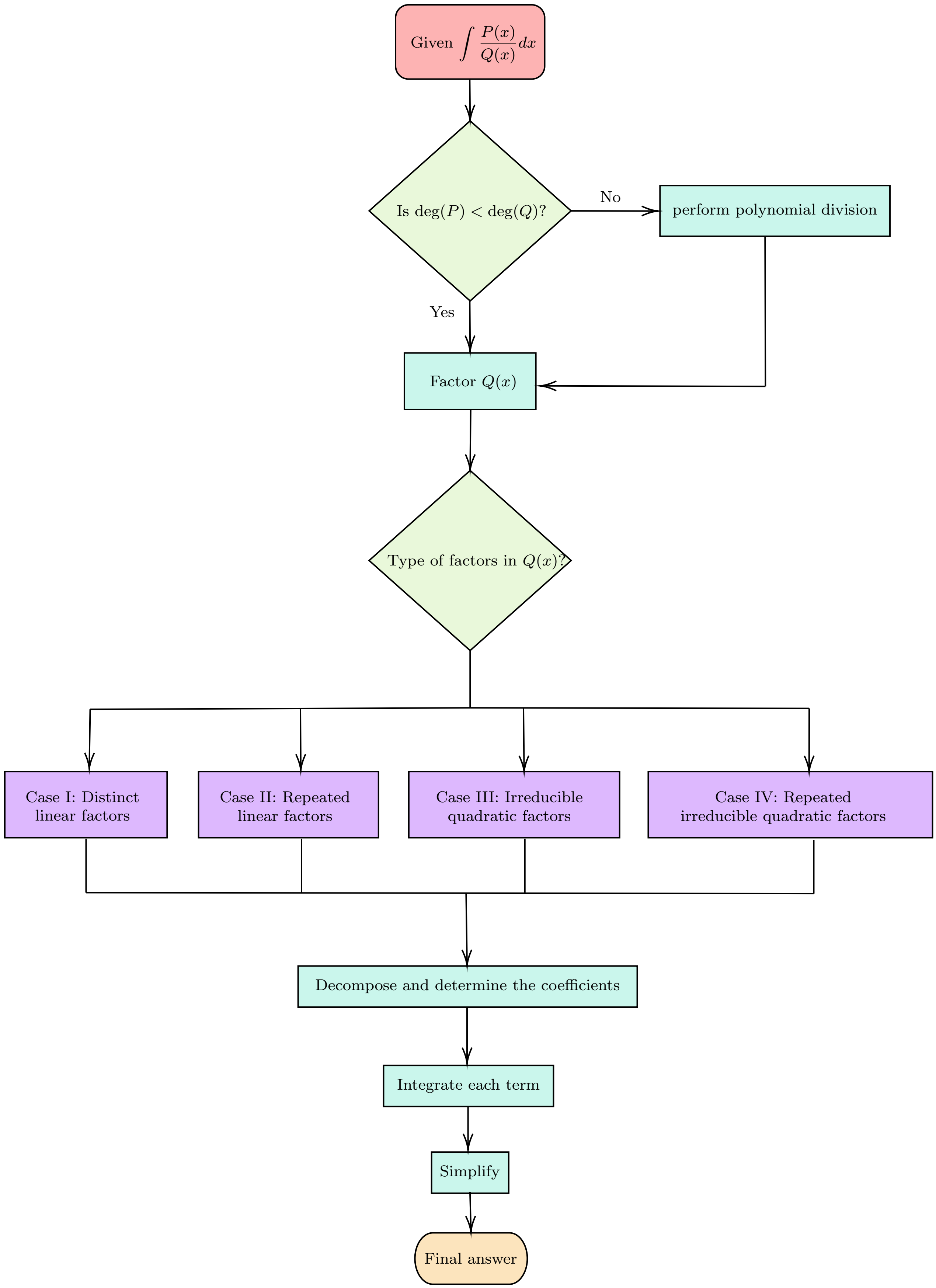
4. **Determine the coefficients:**
   - Multiply through by the denominator $Q(x)$ to eliminate fractions.
   - Expand and collect terms to form a polynomial equation.
   - Solve for the unknown coefficients by equating coefficients of corresponding powers of x.

5. **Integrate each term:**
   - Use basic integration formulas for linear and quadratic terms:

   $$\int \frac{1}{x-a}\,dx = \ln|x-a| + C, \quad \int \frac{Ax+B}{x^2+bx+c}\,dx = \text{(substitution or arctangent)}.$$

   - Combine the results.

6. **Simplify the final answer:**
   - Use logarithmic properties to combine terms where possible.
   - Ensure the answer is in its simplest form.


Given $\int \frac{P(x)}{Q(x)} dx$
Is $\deg(P) < \deg(Q)$?
No
perform polynomial division
Yes
Factor $Q(x)$
Type of factors in $Q(x)$?
Case I: Distinct linear factors
Case II: Repeated linear factors
Case III: Irreducible quadratic factors
Case IV: Repeated irreducible quadratic factors
Decompose and determine the coefficients
Integrate each term
Simplify
Final answer

Example. Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

Solution:

1. The degree of the numerator is less than the degree of the denominator, so we don't need to divide.

2. Factor the denominator:

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2).$$

3. The denominator has three distinct linear factors, so the partial fraction decomposition of the integrand is:

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

4. To determine the values of A, B, and C, we multiply both sides by the least common denominator, $x(2x-1)(x+2)$, obtaining:

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1).$$

Expanding the right-hand side, we get:

$$A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x) = (2A+B+2C)x^2 + (3A+2B-C)x - 2A.$$

Equating coefficients of corresponding terms on both sides gives the system of equations:

$$\begin{aligned} 2A+B+2C &= 1, \quad (\text{coefficient of } x^2) \\ 3A+2B-C &= 2, \quad (\text{coefficient of } x) \\ -2A &= -1. \quad (\text{constant term}) \end{aligned}$$

Solving the system gives $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$.

5. The integral becomes:

$$\begin{aligned} \int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \int \frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\,dx. \\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C. \end{aligned}$$

Example. Find $\int \frac{dx}{x^2 - 9}$.

Solution:

1. Proper
2. The denominator $x^2 - 9$ can be factored as:

$$x^2 - 9 = (x - 3)(x + 3).$$

3. Using the method of partial fractions, we write:

$$\frac{1}{x^2 - 9} = \frac{A}{x - 3} + \frac{B}{x + 3}.$$

4. To determine A and B, multiply through by the common denominator $(x - 3)(x + 3)$:

$$1 = A(x + 3) + B(x - 3).$$

Setting $x = 3$ gives $A = 1/6$. Setting $x = -3$ gives $B = -1/6$.
Thus, the partial fraction decomposition is:

$$\frac{1}{x^2 - 9} = \frac{1/6}{x - 3} - \frac{1/6}{x + 3}.$$

5. Hence:

$$\begin{aligned} \int \frac{dx}{x^2 - 9} &= \int \frac{1/6}{x - 3} - \frac{1/6}{x + 3}\, dx \\ &= \frac{1}{6} \ln|x - 3| - \frac{1}{6} \ln|x + 3| + C. \\ &= \frac{1}{6} \ln\left|\frac{x - 3}{x + 3}\right| + C. \end{aligned}$$

Example. Find $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Solution:

1. Improper. The first step is to perform polynomial long division.

$$
\begin{array}{r|l}
 & x+1 \\
\hline
x^3-x^2-x+1 & x^4+0x^3-2x^2+4x+1 \\
 & \underline{x^4-x^3-x^2+x} \\
 & \qquad x^3-x^2+3x+1 \\
 & \qquad \underline{x^3-x^2-x+1} \\
 & \qquad\qquad\quad 4x
\end{array}
$$

Thus, the quotient is:

$$x+1+\frac{4x}{x^3-x^2-x+1}.$$

2. Factor $Q(x)=x^3-x^2-x+1$. Since $Q(1)=0$, we know that $x-1$ is a factor. We get

$$Q(x)=(x-1)(x^2-1)=(x-1)(x-1)(x+1)=(x-1)^2(x+1).$$

3. The partial fraction decomposition for $\frac{4x}{Q(x)}$ is:

$$\frac{4x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}.$$

4. Multiplying through by the least common denominator $(x-1)^2(x+1)$ gives:

$$
\begin{aligned}
4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2 \\
&= Ax^2-A+Bx+B+Cx^2-2Cx+C \\
&= (A+C)x^2+(B-2C)x+(-A+B+C)
\end{aligned}
$$

Equating coefficients of x^2, x, and the constant term:

$$
\begin{aligned}
A+C&=0, &&\text{(coefficient of } x^2\text{)}\\
B-2C&=4, &&\text{(coefficient of } x\text{)}\\
-A+B+C&=0. &&\text{(constant term)}
\end{aligned}
$$

Solving this system, $A=1$, $B=2$, and $C=-1$. Thus, the partial fraction decomposition is:

$$\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}.$$

5. The integral becomes:

$$
\begin{aligned}
\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\,dx \\
&= \frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+C \\
&= \frac{x^2}{2}+x-\frac{2}{x-1}+\ln\left|\frac{x-1}{x+1}\right|+C.
\end{aligned}
$$

Example. Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$.

Solution:

1. **Check if the fraction is proper:** The degree of the numerator $2x^2 - x + 4$ is less than the degree of the denominator $x^3 + 4x$. Thus, the fraction is proper, and no division is needed.

2. **Factor the denominator:**
$$x^3 + 4x = x(x^2 + 4)$$
where $x^2 + 4$ is an irreducible quadratic factor ($b^2 - 4ac < 0$).

3. **Set up the partial fraction decomposition:**
$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

4. **Eliminate the denominators:** Multiply through by $x(x^2 + 4)$:

   Expand and collect terms:
$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x. \\ &= A(x^2) + 4A + Bx^2 + Cx \\ &= (A + B)x^2 + Cx + 4A. \end{aligned}$$

   Equate coefficients: Comparing coefficients of x^2, x, and the constant term:
$$\begin{aligned} A + B &= 2, \quad (\text{coefficient of } x^2) \\ C &= -1, \quad (\text{coefficient of } x) \\ 4A &= 4. \quad (\text{constant term}) \end{aligned}$$

   Solve for the constants:

   We get $A = 1$, $B = 1$, and $C = -1$.

5. **Rewrite the decomposition:**
$$\begin{aligned} \int \frac{2x^2 - x + 4}{x^3 + 4x} &= \int \frac{1}{x} + \frac{x - 1}{x^2 + 4}. \\ &= \int \frac{1}{x} + \frac{x}{x^2 + 4} + \frac{-1}{x^2 + 4}\, dx \\ &= \ln|x| + \frac{1}{2}\ln(x^2 + 4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C. \end{aligned}$$

   Key formula: $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a) + C$

Example. Evaluate $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx.$

Solution:

1. **Check if the fraction is proper:** The degree of the numerator $4x^2-3x+2$ is equal to the degree of the denominator $4x^2-4x+3$. Since the fraction is improper, we first perform polynomial long division.

2. **Perform polynomial long division:** Divide $4x^2-3x+2$ by $4x^2-4x+3$:

$$\frac{4x^2-3x+2}{4x^2-4x+3} = 1 + \frac{x-1}{4x^2-4x+3}.$$

3. **Attempt partial fraction decomposition:** The denominator $4x^2-4x+3$ is irreducible because its discriminant $(-4)^2-4(4)(3) = -32 < 0$.

   Partial fractions would decompose the integrand into $\frac{Ax+B}{4x^2-4x+3}$, but the numerator $x-1$ is already linear and matches this form. Therefore, applying partial fractions does not simplify the problem.

4. **Complete the square:** To simplify $4x^2-4x+3$, complete the square:

$$4x^2-4x+3 = (2x-1)^2+2.$$

5. **Substitute and solve:** Let $u = 2x-1$, so $du = 2\,dx$ or $dx = \frac{du}{2}$. Also, $x = \frac{1}{2}(u+1)$. The integral becomes:

$$\begin{aligned}
\int 1\,dx + \int \frac{x-1}{(2x-1)^2+2}\,dx &= \int 1\,dx + \int \frac{\frac{1}{2}(u+1)-1}{u^2+2}\cdot\frac{1}{2}\,du \\
&= \int 1\,dx + \frac{1}{4}\int \frac{u-1}{u^2+2}\,du. \\
&= \int 1\,dx + \frac{1}{4}\int \frac{u}{u^2+2}\,du - \frac{1}{4}\int \frac{1}{u^2+2}\,du. \\
&= x + \frac{1}{8}\ln|u^2+2| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \\
&= x + \frac{1}{8}\ln\big((2x-1)^2+2\big) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C.
\end{aligned}$$

Example. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Solution:

The given function can be decomposed into partial fractions by expressing it as a sum of terms, each corresponding to one of the factors in the denominator. The general form of the partial fraction decomposition is:

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3},$$

Example. Should we use partial fractions to solve $\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,dx$?

Solution:

We could, but substitution simplifies the integral quickly. Let:

$$u = x(x^2+3) = x^3 + 3x, \quad \text{so} \quad du = (3x^2+3)\,dx = 3(x^2+1)\,dx.$$

The integral becomes

$$\begin{aligned} &\int \frac{1}{u} \cdot \frac{1}{3}\,du \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln|x^3+3x| + C. \end{aligned}$$

Example. Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$.

Solution:

1. **Check if the rational function is proper:** Proper. no long division is needed.
2. **Factor the denominator:** Already a product of a linear factor x and a repeated irreducible quadratic factor $(x^2+1)^2$.
3. **Set up the partial fraction decomposition:** The partial fraction decomposition is:
$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$
4. **Eliminate the denominators:**
$$\begin{aligned} 1-x+2x^2-x^3 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

   Equate coefficients: Equate coefficients of x^4, x^3, x^2, x, and the constant term:
$$\begin{aligned} A+B &= 0, \quad (\text{coefficient of } x^4) \\ C &= -1, \quad (\text{coefficient of } x^3) \\ 2A+B+D &= 2, \quad (\text{coefficient of } x^2) \\ C+E &= -1, \quad (\text{coefficient of } x) \\ A &= 1. \quad (\text{constant term}) \end{aligned}$$

   Solve for the constants:
$$A=1, \quad B=-1, \quad C=-1, \quad D=1, \quad E=0.$$
5. **Substitute and solve:** Substitute the constants back into the partial fractions:
$$\begin{aligned} \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx &= \int \frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\,dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) - \arctan(x) - \frac{1}{2(x^2+1)} + C. \end{aligned}$$