7.3 Trigonometric Substitution

Trigonometric substitution is a way to evaluate integrals that involve square roots of quadratic expressions. By substituting a trigonometric function for the variable x, the integral can be transformed into a simpler form using the fundamental Pythagorean identities. This method is especially useful when dealing with integrals of the following forms:

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2}.$$

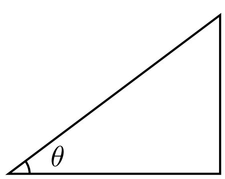
Common Substitutions

The table below summarizes the three standard trigonometric substitutions:

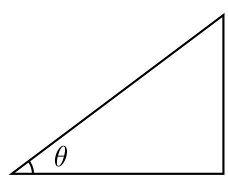
Expression	**Substitution**	**Identity**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Geometric Interpretation

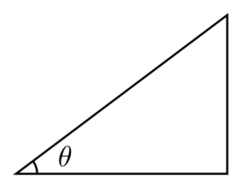
Each substitution corresponds to the sides of a right triangle:



Substitution: $x = a\sin\theta$



Substitution: $x = a\tan\theta$



Substitution: $x = a\sec\theta$

Steps for Using Trigonometric Substitution

1. Identify the form of the square root and the corresponding substitution from the table.
2. Replace x with the chosen trigonometric expression (e.g., $x = a\sin\theta$) and compute dx.
3. Simplify the square root using the trigonometric identity.
4. Rewrite the integral in terms of θ and solve.
5. If the integral is indefinite, return to the original variable x using the inverse trigonometric function or a reference triangle.

Example. Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$.

Example. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Example. Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$.

Remark. This example illustrates that even when trigonometric substitutions are possible, they may not always be the simplest approach. Direct substitution is often more efficient, so it is important to consider all options before proceeding.

Example. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$, where $a > 0$.

Example. Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$.

Example. Evaluate $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.