7.3 Trigonometric Substitution (Solutions)

1. $\displaystyle\int \frac{x}{\sqrt{9+3x^2}}\,dx$

   (a) *Choose a substitution.* Let $u = 9+3x^2$. Then $du = 6x\,dx$, so $x\,dx = \frac{1}{6}\,du$.

   (b) *Rewrite and integrate.*

$$\begin{aligned}\int \frac{x}{\sqrt{9+3x^2}}\,dx &= \int \frac{1}{\sqrt{u}}\left(\frac{1}{6}\,du\right)\\ &= \frac{1}{6}\int u^{-1/2}\,du\\ &= \frac{1}{6}\cdot 2u^{1/2}+C\\ &= \frac{1}{3}\sqrt{u}+C.\end{aligned}$$

   (c) *Substitute back.*

$$\boxed{\int \frac{x}{\sqrt{9+3x^2}}\,dx = \frac{1}{3}\sqrt{9+3x^2}+C.}$$

2. $\displaystyle\int \frac{x}{(9+2x^2)^{3/2}}\,dx$

   (a) *Choose a substitution.* Let $u = 9+2x^2$. Then $du = 4x\,dx$, so $x\,dx = \frac{1}{4}\,du$.

   (b) *Rewrite and integrate.*

$$\begin{aligned}\int \frac{x}{(9+2x^2)^{3/2}}\,dx &= \int u^{-3/2}\left(\frac{1}{4}\,du\right)\\ &= \frac{1}{4}\int u^{-3/2}\,du\\ &= \frac{1}{4}\left(\frac{u^{-1/2}}{-1/2}\right)+C\\ &= -\frac{1}{2}u^{-1/2}+C.\end{aligned}$$

   (c) *Substitute back.*

$$\boxed{\int \frac{x}{(9+2x^2)^{3/2}}\,dx = -\frac{1}{2\sqrt{9+2x^2}}+C.}$$

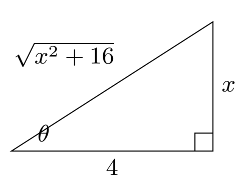
3. $\displaystyle\int \frac{x^3}{\sqrt{16+x^2}}\,dx$

   Quick note. This integral can also be done efficiently by u-sub with $u = 16+x^2$ and by rewriting x^3 as $x^2\cdot x$.

   (a) *Identify the form.*

$$\sqrt{16+x^2} = \sqrt{x^2+4^2} \quad\Rightarrow\quad x = 4\tan\theta.$$

   (b) *Draw the triangle (for* $\tan\theta = \frac{x}{4}$*).*



   (c) *Substitute.* Let $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\,d\theta$.

   (d) *Simplify the radical.*

$$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = 4\sec\theta.$$

   (e) *Rewrite and integrate.*

$$\begin{aligned}\int \frac{x^3}{\sqrt{16+x^2}}\,dx &= \int \frac{(4\tan\theta)^3}{4\sec\theta}\,(4\sec^2\theta)\,d\theta\\ &= 64\int \tan^3\theta\,\sec\theta\,d\theta\\ &= 64\int (\sec^2\theta-1)\,\sec\theta\tan\theta\,d\theta.\end{aligned}$$

Let $u = \sec\theta$. Then $du = \sec\theta\tan\theta\,d\theta$, and

$$64\int(\sec^2\theta - 1)\,\sec\theta\tan\theta\,d\theta = 64\int(u^2-1)\,du$$
$$= 64\left(\frac{u^3}{3} - u\right) + C$$
$$= \frac{64}{3}\sec^3\theta - 64\sec\theta + C.$$

(f) *Back-substitute.* From the triangle, $\sec\theta = \dfrac{\sqrt{x^2+16}}{4}$. Thus

$$\frac{64}{3}\sec^3\theta - 64\sec\theta = \frac{64}{3}\left(\frac{\sqrt{x^2+16}}{4}\right)^3 - 64\left(\frac{\sqrt{x^2+16}}{4}\right)$$
$$= \frac{1}{3}(x^2+16)^{3/2} - 16\sqrt{x^2+16}.$$

(g) *Final answer.*

$$\boxed{\int\frac{x^3}{\sqrt{16+x^2}}\,dx = \frac{1}{3}(x^2+16)^{3/2} - 16\sqrt{x^2+16} + C.}$$

4. $\displaystyle\int\sqrt{5-2x^2}\,dx$

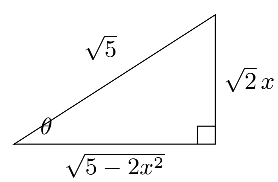
(a) *Identify the form.*

$$\sqrt{5-2x^2} = \sqrt{5}\sqrt{1-\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2}\ \text{ with } u = \frac{\sqrt{2}\,x}{\sqrt{5}},\ \text{so use } u = \sin\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{5}{2}}\sin\theta, \qquad dx = \sqrt{\frac{5}{2}}\cos\theta\,d\theta.$$

(c) *Reference triangle (for $\sin\theta = \dfrac{\sqrt{2}\,x}{\sqrt{5}}$).*



(d) *Rewrite the integral.*

$$\sqrt{5-2x^2} = \sqrt{5-5\sin^2\theta} = \sqrt{5}\,\cos\theta,$$

so

$$\int\sqrt{5-2x^2}\,dx = \int(\sqrt{5}\,\cos\theta)\left(\sqrt{\frac{5}{2}}\cos\theta\right)d\theta$$
$$= \frac{5}{\sqrt{2}}\int\cos^2\theta\,d\theta.$$

(e) *Integrate.*

$$\int\cos^2\theta\,d\theta = \int\frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C,$$

so

$$\int\sqrt{5-2x^2}\,dx = \frac{5}{2\sqrt{2}}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C.$$

(f) *Back-substitute.* From the reference triangle,

$$\sin\theta = \frac{\sqrt{2}\,x}{\sqrt{5}}, \qquad \cos\theta = \frac{\sqrt{5-2x^2}}{\sqrt{5}},$$

so

$$\theta = \arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right).$$

Also,

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)\left(\frac{\sqrt{5-2x^2}}{\sqrt{5}}\right) \\ &= \frac{2\sqrt{2}\,x\sqrt{5-2x^2}}{5}.\end{aligned}$$

(g) *Final answer.*

$$\boxed{\int \sqrt{5-2x^2}\,dx = \frac{x}{2}\sqrt{5-2x^2} + \frac{5}{2\sqrt{2}}\arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right) + C.}$$

5. $\displaystyle\int \frac{dx}{\sqrt{5-2x^2}}$

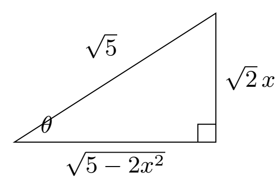
(a) *Identify the form.*

$$\sqrt{5-2x^2} = \sqrt{5}\sqrt{1-\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2} \text{ with } u = \frac{\sqrt{2}\,x}{\sqrt{5}}, \text{ so use } u = \sin\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{5}{2}}\sin\theta, \qquad dx = \sqrt{\frac{5}{2}}\cos\theta\,d\theta.$$

(c) *Reference triangle (for* $\sin\theta = \dfrac{\sqrt{2}\,x}{\sqrt{5}}$*).*



(d) *Rewrite the integral.*

$$\sqrt{5-2x^2} = \sqrt{5}\,\cos\theta,$$

so

$$\int \frac{dx}{\sqrt{5-2x^2}} = \int \frac{\sqrt{\frac{5}{2}}\cos\theta}{\sqrt{5}\,\cos\theta}\,d\theta = \frac{1}{\sqrt{2}}\int d\theta.$$

(e) *Integrate.*

$$\frac{1}{\sqrt{2}}\int d\theta = \frac{\theta}{\sqrt{2}} + C.$$

(f) *Back-substitute.*

$$\theta = \arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right).$$

(g) *Final answer.*

$$\boxed{\int \frac{dx}{\sqrt{5-2x^2}} = \frac{1}{\sqrt{2}}\arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right) + C.}$$

6. $\displaystyle\int \frac{dx}{\sqrt{16+x^2}}$

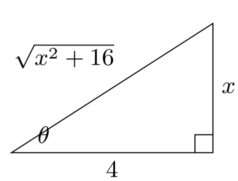
(a) *Identify the form.*

$$\sqrt{16+x^2} = \sqrt{x^2+4^2} \quad\Rightarrow\quad \sqrt{x^2+a^2} \text{ with } a = 4, \text{ so use } x = 4\tan\theta.$$

(b) *Trig substitution.*

$$x = 4\tan\theta, \qquad dx = 4\sec^2\theta\,d\theta.$$

(c) *Reference triangle (for* $\tan\theta = \dfrac{x}{4}$*).*



(d) *Rewrite the integral.*

$$\sqrt{16+x^2} = 4\sec\theta,$$

so

$$\int \frac{dx}{\sqrt{16+x^2}} = \int \frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int \sec\theta\,d\theta.$$

(e) *Integrate.*

$$\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C.$$

(f) *Back-substitute.* From the triangle,

$$\sec\theta = \frac{\sqrt{x^2+16}}{4}, \qquad \tan\theta = \frac{x}{4},$$

so

$$\ln|\sec\theta+\tan\theta| = \ln\left|\frac{\sqrt{x^2+16}+x}{4}\right|.$$

(g) *Final answer.*

$$\boxed{\int \frac{dx}{\sqrt{16+x^2}} = \ln\left|x+\sqrt{x^2+16}\right| + C.}$$

7. $\displaystyle\int \frac{dx}{\sqrt{8x^2-11}}$

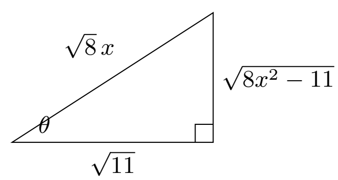
(a) *Identify the form.*

$$\sqrt{8x^2-11} = \sqrt{11}\sqrt{\left(\frac{\sqrt{8}\,x}{\sqrt{11}}\right)^2 - 1} \quad\Rightarrow\quad \sqrt{u^2-a^2} \text{ with } u = \frac{\sqrt{8}\,x}{\sqrt{11}}, \text{ so use } u = \sec\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{11}{8}}\sec\theta, \qquad dx = \sqrt{\frac{11}{8}}\sec\theta\tan\theta\,d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \dfrac{\sqrt{8}\,x}{\sqrt{11}}$*).*



(d) *Rewrite the integral.*

$$\sqrt{8x^2-11} = \sqrt{11}\tan\theta,$$

so

$$\int \frac{dx}{\sqrt{8x^2-11}} = \int \frac{\sqrt{\frac{11}{8}}\sec\theta\tan\theta}{\sqrt{11}\tan\theta}\,d\theta = \frac{1}{\sqrt{8}}\int \sec\theta\,d\theta.$$

(e) *Integrate.*

$$\frac{1}{\sqrt{8}}\int \sec\theta\,d\theta = \frac{1}{\sqrt{8}}\ln|\sec\theta+\tan\theta| + C.$$

(f) *Back-substitute.* From the triangle,

$$\sec\theta = \frac{\sqrt{8}\,x}{\sqrt{11}}, \qquad \tan\theta = \frac{\sqrt{8x^2-11}}{\sqrt{11}},$$

so

$$\ln|\sec\theta+\tan\theta| = \ln\left|\frac{\sqrt{8}\,x+\sqrt{8x^2-11}}{\sqrt{11}}\right|.$$

(g) *Final answer.*

$$\boxed{\int \frac{dx}{\sqrt{8x^2-11}} = \frac{1}{2\sqrt{2}}\ln\left|\sqrt{8}\,x+\sqrt{8x^2-11}\right| + C.}$$

8. $\displaystyle\int \frac{1}{x\sqrt{x^2-16}}\,dx$

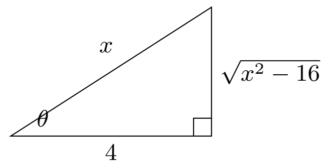
(a) *Identify the form.*

$$\sqrt{x^2-16} = \sqrt{x^2-4^2} \quad\Rightarrow\quad \sqrt{x^2-a^2} \text{ with } a = 4, \text{ so use } x = 4\sec\theta.$$

(b) *Trig substitution.*

$$x = 4\sec\theta, \qquad dx = 4\sec\theta\tan\theta\,d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \dfrac{x}{4}$*).*



(d) *Rewrite the integral.*

$$\sqrt{x^2-16} = \sqrt{16\sec^2\theta - 16} = 4\tan\theta,$$

so

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-16}}\,dx &= \int \frac{1}{(4\sec\theta)(4\tan\theta)}\,(4\sec\theta\tan\theta)\,d\theta \\ &= \frac{1}{4}\int d\theta.\end{aligned}$$

(e) *Integrate.*

$$\frac{1}{4}\int d\theta = \frac{\theta}{4} + C.$$

(f) *Back-substitute (solve for θ using* $\tan\theta$*).* From the triangle,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2-16}}{4},$$

so

$$\theta = \arctan\!\left(\frac{\sqrt{x^2-16}}{4}\right).$$

(g) *Final answer.*

$$\boxed{\int \frac{1}{x\sqrt{x^2-16}}\,dx = \frac{1}{4}\arctan\!\left(\frac{\sqrt{x^2-16}}{4}\right) + C.}$$

(h) *Note.* Our book defines $\sec^{-1}$ to return $\theta \in (0,\pi/2)\cup(\pi,3\pi/2)$. With this convention, $\theta = \sec^{-1}\left(\frac{x}{4}\right)$ automatically handles both branches: for $x > 4$ it returns an acute angle, and for $x < -4$ it returns a quadrant-III angle. We can also solve using tangent: the triangle gives the acute reference angle $\alpha = \arctan\left(\frac{\sqrt{x^2-16}}{4}\right) \in (0,\pi/2)$; on the $x < -4$ branch the needed angle is $\theta = \pi + \alpha$ (since $\tan(\pi+\alpha) = \tan\alpha$). But the antiderivative after substitution is $\frac{1}{4}\theta + C$, so

$$\frac{1}{4}\theta + C = \frac{1}{4}(\pi+\alpha) + C = \frac{1}{4}\alpha + \left(C + \frac{\pi}{4}\right),$$

and $C + \frac{\pi}{4}$ is just a new constant. Thus using arctan produces the same antiderivative (up to $+C$).

9. $\displaystyle\int \frac{x^2}{\sqrt{10-3x^2}}\,dx$

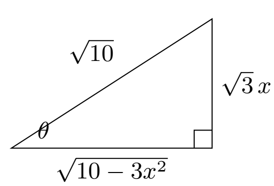
(a) *Identify the form.*

$$\sqrt{10-3x^2} = \sqrt{10}\sqrt{1-\left(\frac{\sqrt{3}\,x}{\sqrt{10}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2}\ \text{ with } u = \frac{\sqrt{3}\,x}{\sqrt{10}},\ \text{so use } u = \sin\theta.$$

(b) *Trig substitution.*

$$x = \sqrt{\frac{10}{3}}\sin\theta, \qquad dx = \sqrt{\frac{10}{3}}\cos\theta\,d\theta.$$

(c) *Reference triangle (for* $\sin\theta = \dfrac{\sqrt{3}\,x}{\sqrt{10}}$*).*



(d) *Rewrite the integral.*

$$x^2 = \frac{10}{3}\sin^2\theta, \qquad \sqrt{10-3x^2} = \sqrt{10(1-\sin^2\theta)} = \sqrt{10}\cos\theta,$$

so

$$\int \frac{x^2}{\sqrt{10-3x^2}}\,dx = \int \frac{\frac{10}{3}\sin^2\theta}{\sqrt{10}\cos\theta}\left(\sqrt{\frac{10}{3}}\cos\theta\right)d\theta$$
$$= \frac{10}{3\sqrt{3}}\int \sin^2\theta\,d\theta.$$

(e) *Integrate.*

$$\int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C,$$

so

$$\int \frac{x^2}{\sqrt{10-3x^2}}\,dx = \frac{5}{3\sqrt{3}}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C.$$

(f) *Back-substitute.*

$$\theta = \arcsin\left(\frac{\sqrt{3}\,x}{\sqrt{10}}\right), \qquad \sin 2\theta = 2\sin\theta\cos\theta = \frac{\sqrt{3}\,x\sqrt{10-3x^2}}{5}.$$

(g) *Final answer.*

$$\boxed{\int \frac{x^2}{\sqrt{10-3x^2}}\,dx = \frac{5}{3\sqrt{3}}\arcsin\left(\frac{\sqrt{3}\,x}{\sqrt{10}}\right) - \frac{x}{6}\sqrt{10-3x^2} + C.}$$

10. $\displaystyle\int \frac{\sqrt{9x^2-16}}{x^4}\,dx$

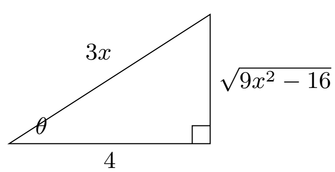
(a) *Identify the form.*

$$\sqrt{9x^2-16} = \sqrt{(3x)^2-4^2} \quad\Rightarrow\quad \sqrt{u^2-a^2} \text{ with } u = 3x,\ a = 4, \text{ so use } u = 4\sec\theta.$$

(b) *Trig substitution.*

$$3x = 4\sec\theta \quad\Rightarrow\quad x = \frac{4}{3}\sec\theta, \qquad dx = \frac{4}{3}\sec\theta\tan\theta\,d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \dfrac{3x}{4}$*).*



(d) *Rewrite the integral.* From $(3x)^2 - 4^2 = (4\sec\theta)^2 - 4^2$, we get

$$\sqrt{9x^2-16} = 4\tan\theta, \qquad x^4 = \left(\frac{4}{3}\right)^4\sec^4\theta = \frac{256}{81}\sec^4\theta.$$

Therefore,

$$\int \frac{\sqrt{9x^2-16}}{x^4}\,dx = \int \frac{4\tan\theta}{\frac{256}{81}\sec^4\theta}\left(\frac{4}{3}\sec\theta\tan\theta\right)d\theta$$
$$= \frac{27}{16}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{27}{16}\int \sin^2\theta\cos\theta\,d\theta.$$

(e) *Integrate.* Let $u = \sin\theta$, $du = \cos\theta\,d\theta$.

$$\frac{27}{16}\int \sin^2\theta\cos\theta\,d\theta = \frac{27}{16}\int u^2\,du = \frac{9}{16}\sin^3\theta + C.$$

(f) *Back-substitute.* From the triangle,

$$\sin\theta = \frac{\sqrt{9x^2-16}}{3x} \quad\Rightarrow\quad \sin^3\theta = \frac{(9x^2-16)^{3/2}}{27x^3}.$$

Hence

$$\frac{9}{16}\sin^3\theta = \frac{9}{16}\cdot\frac{(9x^2-16)^{3/2}}{27x^3} = \frac{1}{48}\,\frac{(9x^2-16)^{3/2}}{x^3}.$$

(g) *Final answer.*

$$\boxed{\int \frac{\sqrt{9x^2-16}}{x^4}\,dx = \frac{1}{48}\,\frac{(9x^2-16)^{3/2}}{x^3} + C.}$$

11. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$

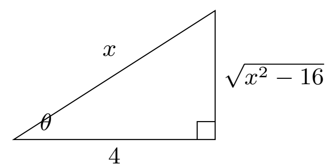
(a) *Identify the form.*

$$\sqrt{x^2-16} = \sqrt{x^2-4^2} \quad\Rightarrow\quad \sqrt{x^2-a^2} \text{ with } a=4, \text{ so use } x = 4\sec\theta.$$

(b) *Trig substitution.*

$$x = 4\sec\theta, \qquad dx = 4\sec\theta\tan\theta\,d\theta.$$

(c) *Reference triangle (for* $\sec\theta = \frac{x}{4}$*).*



(d) *Rewrite the integral.*

$$x^2 = 16\sec^2\theta, \qquad \sqrt{x^2-16} = 4\tan\theta,$$

so

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-16}}\,dx &= \int \frac{1}{(16\sec^2\theta)(4\tan\theta)}\,(4\sec\theta\tan\theta)\,d\theta \\ &= \frac{1}{16}\int \cos\theta\,d\theta.\end{aligned}$$

(e) *Integrate.*

$$\frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C.$$

(f) *Back-substitute.* From the triangle,

$$\sin\theta = \frac{\sqrt{x^2-16}}{x}.$$

(g) *Final answer.*

$$\boxed{\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \frac{\sqrt{x^2-16}}{16x} + C.}$$

12. $\displaystyle\int_0^{\sqrt{5/2}} \sqrt{5-2x^2}\,dx$

(a) *Identify the form.*

$$\sqrt{5-2x^2} = \sqrt{5}\sqrt{1-\left(\frac{\sqrt{2}\,x}{\sqrt{5}}\right)^2} \quad\Rightarrow\quad \sqrt{a^2-u^2} \text{ with } u = \frac{\sqrt{2}\,x}{\sqrt{5}}, \text{ so use } u = \sin\theta.$$

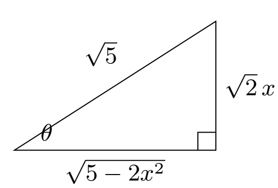
(b) *Trig substitution.*

$$x = \sqrt{\frac{5}{2}}\sin\theta, \qquad dx = \sqrt{\frac{5}{2}}\cos\theta\,d\theta.$$

Change bounds:

$$x = 0 \Rightarrow \theta_1 = 0, \qquad x = \sqrt{\frac{5}{2}} \Rightarrow \theta_2 = \frac{\pi}{2}.$$

(c) *Reference triangle (for* $\sin\theta = \frac{\sqrt{2}\,x}{\sqrt{5}}$*).*



(d) *Rewrite the integral.*

$$\sqrt{5-2x^2}=\sqrt{5}\,\cos\theta,$$

so

$$\int_0^{\sqrt{5/2}}\sqrt{5-2x^2}\,dx=\int_0^{\pi/2}(\sqrt{5}\,\cos\theta)\left(\sqrt{\frac{5}{2}}\cos\theta\right)\,d\theta=\frac{5}{\sqrt{2}}\int_0^{\pi/2}\cos^2\theta\,d\theta.$$

(e) *Integrate.*

$$\frac{5}{\sqrt{2}}\int_0^{\pi/2}\cos^2\theta\,d\theta=\frac{5}{\sqrt{2}}\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta=\frac{5}{2\sqrt{2}}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}.$$

(f) *Back-substitute.*

$$\frac{5}{2\sqrt{2}}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}=\frac{5}{2\sqrt{2}}\left(\frac{\pi}{2}-0\right)=\frac{5\pi}{4\sqrt{2}}.$$

(g) *Final answer.*

$$\boxed{\int_0^{\sqrt{5/2}}\sqrt{5-2x^2}\,dx=\frac{5\pi}{4\sqrt{2}}.}$$

13. $\displaystyle\int_0^2\frac{dx}{\sqrt{16+x^2}}$

(a) *Identify the form.*

$$\sqrt{16+x^2}=\sqrt{x^2+4^2}\quad\Rightarrow\quad\sqrt{x^2+a^2}\ \text{ with } a=4,\ \text{so use } x=4\tan\theta.$$

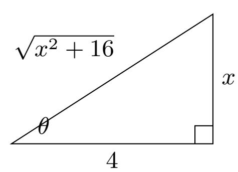
(b) *Trig substitution.*

$$x=4\tan\theta,\qquad dx=4\sec^2\theta\,d\theta,\qquad \sqrt{16+x^2}=4\sec\theta.$$

Change bounds:

$$x=0\Rightarrow\theta_1=0,\qquad x=2\Rightarrow\theta_2=\arctan\left(\frac{1}{2}\right).$$

(c) *Reference triangle (for* $\tan\theta=\frac{x}{4}$*).*



(d) *Rewrite the integral.*

$$\int_0^2\frac{dx}{\sqrt{16+x^2}}=\int_0^{\theta_2}\frac{4\sec^2\theta}{4\sec\theta}\,d\theta=\int_0^{\theta_2}\sec\theta\,d\theta.$$

(e) *Integrate.*

$$\int_0^{\theta_2}\sec\theta\,d\theta=\left[\ln|\sec\theta+\tan\theta|\right]_0^{\theta_2}.$$

(f) *Back-substitute.* At $\theta=\theta_2$, we have $\tan\theta_2=\frac{1}{2}$, so

$$\sec\theta_2=\sqrt{1+\tan^2\theta_2}=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2},\qquad \sec 0+\tan 0=1.$$

Therefore

$$\left[\ln|\sec\theta+\tan\theta|\right]_0^{\theta_2}=\ln\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)-\ln(1)=\ln\left(\frac{\sqrt{5}+1}{2}\right).$$

(g) *Final answer.*

$$\boxed{\int_0^2\frac{dx}{\sqrt{16+x^2}}=\ln\left(\frac{\sqrt{5}+1}{2}\right).}$$

14. $\displaystyle\int\frac{x^2}{(9+2x^2)^{3/2}}\,dx$

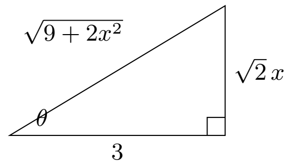
(a) *Identify the form.*

$$(9+2x^2)^{3/2}=\left(3^2+(\sqrt{2}\,x)^2\right)^{3/2}\quad\Rightarrow\quad\sqrt{a^2+u^2}\ \text{ with } u=\sqrt{2}\,x,\ a=3,\ \text{so use } u=a\tan\theta.$$

(b) *Trig substitution.*

$$\sqrt{2}\,x = 3\tan\theta \quad\Rightarrow\quad x = \frac{3}{\sqrt{2}}\tan\theta, \qquad dx = \frac{3}{\sqrt{2}}\sec^2\theta\,d\theta.$$

(c) *Reference triangle (for* $\tan\theta = \dfrac{\sqrt{2}\,x}{3}$*).*



(d) *Rewrite the integral.*

$$x^2 = \frac{9}{2}\tan^2\theta, \qquad (9+2x^2)^{3/2} = (9\sec^2\theta)^{3/2} = 27\sec^3\theta,$$

so

$$\int \frac{x^2}{(9+2x^2)^{3/2}}\,dx = \int \frac{\frac{9}{2}\tan^2\theta}{27\sec^3\theta}\left(\frac{3}{\sqrt{2}}\sec^2\theta\right)d\theta$$
$$= \frac{1}{2\sqrt{2}}\int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \frac{1}{2\sqrt{2}}\int (\sec\theta - \cos\theta)\,d\theta.$$

(e) *Integrate.*

$$\frac{1}{2\sqrt{2}}\int (\sec\theta - \cos\theta)\,d\theta = \frac{1}{2\sqrt{2}}\left(\ln|\sec\theta + \tan\theta| - \sin\theta\right) + C.$$

(f) *Back-substitute.* From the triangle,

$$\tan\theta = \frac{\sqrt{2}\,x}{3}, \qquad \sec\theta = \frac{\sqrt{9+2x^2}}{3}, \qquad \sin\theta = \frac{\sqrt{2}\,x}{\sqrt{9+2x^2}},$$

so

$$\ln|\sec\theta + \tan\theta| = \ln\left|\frac{\sqrt{9+2x^2} + \sqrt{2}\,x}{3}\right|.$$

(g) *Final answer.*

$$\boxed{\int \frac{x^2}{(9+2x^2)^{3/2}}\,dx = \frac{1}{2\sqrt{2}}\ln\left|\sqrt{9+2x^2} + \sqrt{2}\,x\right| - \frac{x}{2\sqrt{9+2x^2}} + C.}$$

15. $\displaystyle\int \frac{1}{(25+3x^2)^{3/2}}\,dx$

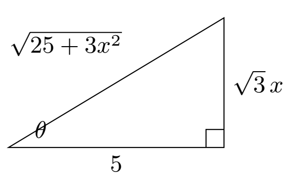
(a) *Identify the form.*

$$(25+3x^2)^{3/2} = \left(5^2 + (\sqrt{3}\,x)^2\right)^{3/2} \quad\Rightarrow\quad \sqrt{a^2+u^2} \text{ with } u = \sqrt{3}\,x,\ a = 5, \text{ so use } u = a\tan\theta.$$

(b) *Trig substitution.*

$$\sqrt{3}\,x = 5\tan\theta \quad\Rightarrow\quad x = \frac{5}{\sqrt{3}}\tan\theta, \qquad dx = \frac{5}{\sqrt{3}}\sec^2\theta\,d\theta.$$

(c) *Reference triangle (for* $\tan\theta = \dfrac{\sqrt{3}\,x}{5}$*).*



(d) *Rewrite the integral.*

$$(25+3x^2)^{3/2} = (25\sec^2\theta)^{3/2} = 125\sec^3\theta,$$

so

$$\int \frac{1}{(25+3x^2)^{3/2}}\,dx = \int \frac{1}{125\sec^3\theta}\left(\frac{5}{\sqrt{3}}\sec^2\theta\right)d\theta = \frac{1}{25\sqrt{3}}\int \cos\theta\,d\theta.$$

(e) *Integrate.*

$$\frac{1}{25\sqrt{3}}\int \cos\theta\,d\theta = \frac{1}{25\sqrt{3}}\sin\theta + C.$$

(f) *Back-substitute.* From the triangle,

$$\sin\theta = \frac{\sqrt{3}\,x}{\sqrt{25+3x^2}}.$$

(g) *Final answer.*

$$\boxed{\int \frac{1}{(25+3x^2)^{3/2}}\,dx = \frac{x}{25\sqrt{25+3x^2}} + C.}$$