7.3 Trigonometric Substitution

Trigonometric substitution is a way to evaluate integrals that involve square roots of quadratic expressions. By substituting a trigonometric function for the variable x, the integral can be transformed into a simpler form using the fundamental Pythagorean identities. This method is especially useful when dealing with integrals of the following forms:

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2}.$$

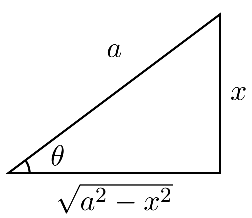
Common Substitutions

The table below summarizes the three standard trigonometric substitutions:

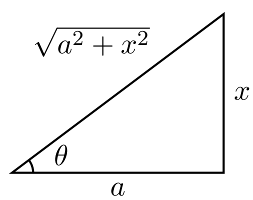
Expression	**Substitution**	**Identity**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Geometric Interpretation

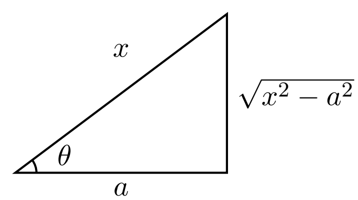
Each substitution corresponds to the sides of a right triangle:



Substitution: $x = a\sin\theta$



Substitution: $x = a\tan\theta$



Substitution: $x = a\sec\theta$

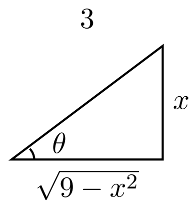
Steps for Using Trigonometric Substitution

1. Identify the form of the square root and the corresponding substitution from the table.
2. Replace x with the chosen trigonometric expression (e.g., $x = a\sin\theta$) and compute dx.
3. Simplify the square root using the trigonometric identity.
4. Rewrite the integral in terms of θ and solve.
5. If the integral is indefinite, return to the original variable x using the inverse trigonometric function or a reference triangle.

Example. Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution:

- **Identify the substitution:**



  Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\,d\theta$.

- **Simplify the square root:**

$$\begin{aligned}\sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta}\\ &= \sqrt{9\cos^2\theta}\\ &= 3|\cos\theta| = 3\cos\theta.\end{aligned}$$

  Since $\cos\theta \ge 0$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we take the positive root.

- **Rewrite the integral in terms of θ:**

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{(3\sin\theta)^2}(3\cos\theta)\,d\theta\\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta\\ &= \int \cot^2\theta\,d\theta.\end{aligned}$$

  Using $\cot^2\theta = \csc^2\theta - 1$, we integrate:

$$\begin{aligned}\int (\csc^2\theta - 1)\,d\theta &= \int \csc^2\theta\,d\theta - \int 1\,d\theta\\ &= -\cot\theta - \theta + C.\end{aligned}$$

- **Convert back to x:** The reference triangle gives:

  1. $\cot\theta = \frac{\sqrt{9-x^2}}{x}$.
  2. $x = 3\sin\theta$ implies $\theta = \sin^{-1}\frac{x}{3}$.

- **Final answer:**

$$-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$$

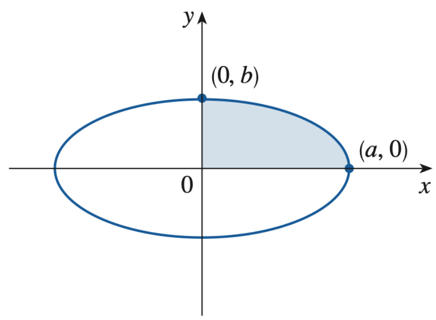
Example. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution:

Solving for y, we have:

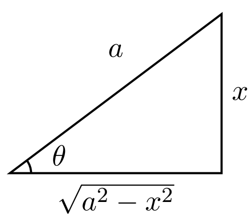
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \qquad \text{or} \qquad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}.$$

The total area is four times the area in the first quadrant:



The part of the ellipse in the first quadrant is given by $y = \frac{b}{a}\sqrt{a - x^2}$ for $0 \le x \le a$. So

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx.$$



Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$.

Then
$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta| = a\cos\theta$$
since $0 \le \theta \le \pi/2$.

The limits of integration change as follows:

$$x = 0 \implies \sin\theta = 0 \implies \theta = 0, \quad x = a \implies \sin\theta = 1 \implies \theta = \frac{\pi}{2}.$$

Substituting:

$$
\begin{aligned}
A &= 4\int_0^{\pi/2} \frac{b}{a}(a\cos\theta)(a\cos\theta)\,d\theta \\
&= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta. \\
&= 4ab\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta \\
&= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= 2ab\left[(\frac{\pi}{2}+0)-(0+0)\right] \\
&= \pi ab.
\end{aligned}
$$

Thus, the area of the ellipse is:

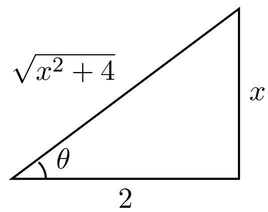
$$A = \pi ab.$$

Example. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Solution:

1. **Identify the form of the square root and the substitution:**

   The triangle for this substitution is shown below:



2. Let $x = 2\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.. Then: $dx = 2\sec^2\theta\,d\theta$.

3. Compute the square root:

$$\begin{aligned}\sqrt{x^2+4} &= \sqrt{4\tan^2\theta+4}\\ &= \sqrt{4(\tan^2\theta+1)}\\ &= \sqrt{4\sec^2\theta}\\ &= 2\sec\theta.\end{aligned}$$

4. Substitute into the integral:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \int \frac{1}{(2\tan\theta)^2\cdot 2\sec\theta}\cdot 2\sec^2\theta\,d\theta\\ &= \int \frac{2\sec^2\theta}{4\tan^2\theta\cdot 2\sec\theta}\,d\theta\\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta.\\ &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.\end{aligned}$$

   Note:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos\theta}{\sin^2\theta}.$$

5. Let $u = \sin\theta$, so $du = \cos\theta\,d\theta$. Substituting:

$$\begin{aligned}\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta &= \frac{1}{4}\int \frac{1}{u^2}\,du.\\ &= \frac{1}{4}\left(-\frac{1}{u}\right) + C\\ &= -\frac{1}{4u} + C.\end{aligned}$$

6. **Return to the original variable x:** From the triangle:

$$\sin\theta = \frac{x}{\sqrt{x^2+4}}.$$

Substituting back:

$$-\frac{1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C.$$

Example. Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$.

Solution:

It is possible to use the trigonometric substitution $x = 2\tan\theta$ here.

The direct substitution $u = x^2 + 4$ is simpler, because then $du = 2x\,dx$ and

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+4}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= \sqrt{u} + C \\ &= \sqrt{x^2+4} + C\end{aligned}$$

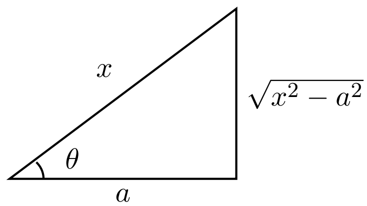
Remark. This example illustrates that even when trigonometric substitutions are possible, they may not always be the simplest approach. Direct substitution is often more efficient, so it is important to consider all options before proceeding.

Example. Evaluate $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$, where $a > 0$.

Solution:

1. **Identify the form of the square root and the substitution:**

   The triangle for this substitution is shown below:

2. **Substitute $x = a \sec \theta$:** Let $x = a \sec \theta$, where $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$. Then:

$$dx = a \sec \theta \tan \theta \, d\theta.$$

3. **Simplify the square root using the trigonometric identity:** Substituting $x = a \sec \theta$ into $\sqrt{x^2 - a^2}$:

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta| = a \tan \theta.$$

4. **Rewrite the integral in terms of θ and solve:** Substituting $x = a \sec \theta$, $dx = a \sec \theta \tan \theta \, d\theta$, and $\sqrt{x^2 - a^2} = a \tan \theta$, the integral becomes:

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \int \frac{1}{a \tan \theta} \cdot a \sec \theta \tan \theta \, d\theta = \int \sec \theta \, d\theta.$$

   The integral of $\sec \theta$ is:

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C.$$

5. **Return to the original variable x:** From the substitution $x = a \sec \theta$, we know:

$$\sec \theta = \frac{x}{a}, \quad \tan \theta = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{\sqrt{x^2 - a^2}}{a}.$$

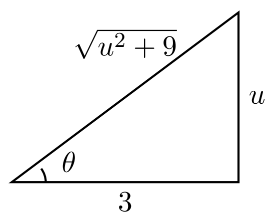
   Substituting these back, the solution becomes:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - a^2}}\, dx &= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C \\ &= \ln \left| x + \sqrt{x^2 - a^2} \right| - \ln a + C \\ &= \ln \left| x + \sqrt{x^2 - a^2} \right| + C_1\end{aligned}$$

Example. Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$.

Solution:

1. The square root $\sqrt{4x^2+9}$ is not directly in the form of the trigonometric substitution table. Let:
$$u = 2x \quad \Rightarrow \quad du = 2\,dx.$$
Bounds: when $x = 0$, $u = 0$. When $x = \frac{3\sqrt{3}}{2}$, $u = 3\sqrt{3}$. The integral becomes:
$$\int_0^{3\sqrt{3}} \frac{\left(\frac{u}{2}\right)^3}{(u^2+9)^{3/2}} \cdot \frac{du}{2} = \frac{1}{16}\int_0^{3\sqrt{3}} \frac{u^3}{(\sqrt{u^2+9})^3}\,du.$$



2. Let $u = 3\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $du = 3\sec^2\theta\,d\theta$.

3. $$\sqrt{u^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9(\tan^2\theta+1)} = \sqrt{9\sec^2\theta} = 3\sec\theta.$$

4. When $u = 0$, $\theta = \tan^{-1}(0) = 0$. When $u = 3\sqrt{3}$, $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$. (because $\theta = \tan^{-1}(\frac{u}{3})$.)
$$\begin{aligned}\frac{1}{16}\int_0^{\pi/3} \frac{(3\tan\theta)^3}{(3\sec\theta)^3} \cdot 3\sec^2\theta\,d\theta &= \frac{1}{16}\int_0^{\pi/3} \frac{27\tan^3\theta}{27\sec^3\theta} \cdot 3\sec^2\theta\,d\theta \\ &= \frac{1}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta} \cdot 3\,d\theta \\ &= \frac{3}{16}\int_0^{\pi/3} \tan^3\theta\cos\theta\,d\theta. \\ &= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta. \\ &= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\,d\theta.\end{aligned}$$

Let $v = \cos\theta$ and $dv = -\sin\theta\, d\theta$. When $\theta = 0, v = 1$. When $\theta = \pi/3, v = \frac{1}{2}$. Hence

$$\begin{aligned}
&= -\frac{3}{16}\int_1^{1/2} \frac{1-v^2}{v^2}\, dv \\
&= \frac{3}{16}\int_1^{1/2} 1 - v^{-2}\, dv \\
&= \frac{3}{16}\left[v + \frac{1}{v}\right]_1^{1/2} \\
&= \frac{3}{16}\left[(\frac{1}{2}+2) - (1+1)\right] \\
&= \frac{3}{32}
\end{aligned}$$

Example. Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

Solution:

- **Complete the square:** The expression inside the square root is rewritten as:

$$\begin{aligned} 3-2x-x^2 &= 3-(x^2+2x) \\ &= 3+1-(x^2+2x+1) \\ &= 4-(x+1)^2. \end{aligned}$$

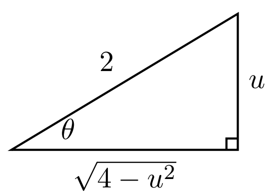
  Substituting into the integral:

$$\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx.$$

- **Introduce a substitution:** Let $u = x+1$, so that $du = dx$ and $x = u-1$. The integral becomes:

$$\int \frac{u-1}{\sqrt{4-u^2}}\,du.$$

- **Reference Triangle:**



- **Trigonometric Substitution:** Let $u = 2\sin\theta$, so $du = 2\cos\theta\,d\theta$ and $\sqrt{4-u^2} = 2\cos\theta$. The integral transforms into:

$$\begin{aligned} &\int \frac{2\sin\theta - 1}{2\cos\theta}\cdot 2\cos\theta\,d\theta \\ &= \int (2\sin\theta - 1)\,d\theta \\ &= -2\cos\theta - \theta + C. \end{aligned}$$

- **Convert back to x:** Since $u = 2\sin\theta$, we have:

$$\sin\theta = \frac{u}{2}, \quad \theta = \sin^{-1}\left(\frac{u}{2}\right).$$

  Also,

$$\cos\theta = \sqrt{1-\sin^2\theta} = \frac{\sqrt{4-u^2}}{2}.$$

- **Final Answer:** Substituting back $u = x+1$:

$$-\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$