7.2 Trigonometric Integrals

Integrals of Powers of Sine and Cosine

We begin by considering integrals in which the integrand is a power of sine, a power of cosine, or a product of these.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$:

1. If the power of cosine is odd, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. Substitute $u = \sin x$.
2. If the power of sine is odd, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine. Substitute $u = \cos x$.
3. If the powers of both sine and cosine are even, use the half-angle identities:
$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \quad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$
4. It is sometimes helpful to use the identity $\sin x \cos x = \frac{1}{2}\sin(2x)$.

Example. Evaluate $\int \cos^3 x \, dx$.

Example. Find $\int \sin^5 x \cos^2 x \, dx$.

Example. Evaluate $\int_0^{\pi} \sin^2 x \, dx$.

Example. Find $\int \sin^4 x \, dx$.

Integrals of Powers of Secant and Tangent

Now we consider integrals in which the integrand is a power of tangent, a power of secant, or a product of these.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

1. The derivatives of tangent and secant:

$$\frac{d}{dx}[\tan x] = \sec^2 x, \quad \frac{d}{dx}[\sec x] = \sec x \tan x.$$

2. If the power of secant is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to rewrite the remaining powers of secant in terms of tangent.

3. If the power of tangent is odd, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite the remaining powers of tangent in terms of secant.

Example. Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Example. Find $\int \tan^5\theta \sec^7\theta \, d\theta$.

Remark. In some cases, the guidelines for integrating powers of $\tan x$ and $\sec x$ are not as straightforward. We may need to use trigonometric identities, integration by parts, or creative problem-solving techniques. Two important integrals to remember for these cases are:

$$\int \tan x \, dx = \ln|\sec x| + C$$

and

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

Example. Find $\int \tan^3 x \, dx$.

Example. Find $\int \sec^3 x \, dx$.