7.2 Trigonometric Integrals (Solutions)

1. $\displaystyle\int \sin^3 x\,dx$

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin x\,(1-\cos^2 x)\,dx\\ &= -\int (1-u^2)\,du \qquad (u=\cos x,\ du=-\sin x\,dx)\\ &= -\left(u-\frac{u^3}{3}\right)+C\\ &= -\cos x+\frac{\cos^3 x}{3}+C.\end{aligned}$$

2. $\displaystyle\int \sin^6 x\,dx$

$$\begin{aligned}\int \sin^6 x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right)^3\,dx\\ &= \frac{1}{8}\int \Big(1-3\cos 2x+3\cos^2 2x-\cos^3 2x\Big)\,dx\\ &= \frac{1}{8}\int \Big(1-3\cos 2x+\tfrac{3}{2}(1+\cos 4x)-\tfrac{1}{4}(3\cos 2x+\cos 6x)\Big)\,dx\\ &= \int \left(\frac{5}{16}-\frac{15}{32}\cos 2x+\frac{3}{16}\cos 4x-\frac{1}{32}\cos 6x\right)\,dx\\ &= \frac{5}{16}x-\frac{15}{64}\sin 2x+\frac{3}{64}\sin 4x-\frac{1}{192}\sin 6x+C.\end{aligned}$$

3. $\displaystyle\int \cos^5 x\,dx$

$$\begin{aligned}\int \cos^5 x\,dx &= \int \cos x\,(1-\sin^2 x)^2\,dx\\ &= \int (1-2u^2+u^4)\,du \qquad (u=\sin x,\ du=\cos x\,dx)\\ &= u-\frac{2}{3}u^3+\frac{1}{5}u^5+C\\ &= \sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C.\end{aligned}$$

4. $\displaystyle\int \cos^4 x\,dx$

$$\begin{aligned}\int \cos^4 x\,dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2\,dx\\ &= \frac{1}{4}\int \Big(1+2\cos 2x+\cos^2 2x\Big)\,dx\\ &= \frac{1}{4}\int \Big(1+2\cos 2x+\tfrac{1}{2}(1+\cos 4x)\Big)\,dx\\ &= \int \left(\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)\,dx\\ &= \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.\end{aligned}$$

5. $\displaystyle\int \sin^2 x\,\cos x\,dx$

$$\begin{aligned}\int \sin^2 x\,\cos x\,dx &= \int u^2\,du \qquad (u=\sin x,\ du=\cos x\,dx)\\ &= \frac{u^3}{3}+C\\ &= \frac{\sin^3 x}{3}+C.\end{aligned}$$

6. $\int \sin x \, \cos^2 x \, dx$

$$\begin{aligned}\int \sin x \, \cos^2 x \, dx &= -\int u^2 \, du \qquad (u = \cos x, \; du = -\sin x \, dx) \\ &= -\frac{u^3}{3} + C \\ &= -\frac{\cos^3 x}{3} + C.\end{aligned}$$

7. $\int \sin^4 x \, \cos^3 x \, dx$

$$\begin{aligned}\int \sin^4 x \, \cos^3 x \, dx &= \int \sin^4 x \, \cos^2 x \, \cos x \, dx \\ &= \int \sin^4 x \, (1 - \sin^2 x) \, \cos x \, dx \\ &= \int (u^4 - u^6) \, du \qquad (u = \sin x, \; du = \cos x \, dx) \\ &= \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.\end{aligned}$$

8. $\int \sin^3 x \, \cos^4 x \, dx$

$$\begin{aligned}\int \sin^3 x \, \cos^4 x \, dx &= \int \sin^2 x \, \cos^4 x \, \sin x \, dx \\ &= \int (1 - \cos^2 x) \cos^4 x \, \sin x \, dx \\ &= -\int (u^4 - u^6) \, du \qquad (u = \cos x, \; du = -\sin x \, dx) \\ &= -\frac{u^5}{5} + \frac{u^7}{7} + C \\ &= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.\end{aligned}$$

9. $\int \sin^2 x \, \cos^2 x \, dx$

$$\begin{aligned}\int \sin^2 x \, \cos^2 x \, dx &= \int \frac{1}{4} \sin^2(2x) \, dx \\ &= \int \frac{1}{4} \cdot \frac{1 - \cos 4x}{2} \, dx \\ &= \int \frac{1}{8}\big(1 - \cos 4x\big) \, dx \\ &= \frac{x}{8} - \frac{1}{32} \sin 4x + C.\end{aligned}$$

10. $\int \tan^2 x \, dx$

$$\begin{aligned}\int \tan^2 x \, dx &= \int (\sec^2 x - 1) \, dx \\ &= \tan x - x + C.\end{aligned}$$

11. $\int \sec^4 x \, dx$

$$\begin{aligned}\int \sec^4 x \, dx &= \int \sec^2 x (1 + \tan^2 x) \, dx \\ &= \int (1 + u^2) \, du \qquad \big(u = \tan x, \; du = \sec^2 x \, dx\big) \\ &= u + \frac{u^3}{3} + C \\ &= \tan x + \frac{\tan^3 x}{3} + C.\end{aligned}$$

12. $\displaystyle\int \tan^4 x \sec^2 x\, dx$

$$\begin{aligned}\int \tan^4 x \sec^2 x\, dx &= \int u^4\, du \qquad (u = \tan x,\ du = \sec^2 x\, dx)\\ &= \frac{u^5}{5} + C\\ &= \frac{\tan^5 x}{5} + C.\end{aligned}$$

13. $\displaystyle\int \tan^3 x \sec^3 x\, dx$

$$\begin{aligned}\int \tan^3 x \sec^3 x\, dx &= \int \tan^2 x \sec^2 x\,(\sec x \tan x)\, dx\\ &= \int (\sec^2 x - 1)\sec^2 x\,(\sec x \tan x)\, dx\\ &= \int (u^2 - 1)u^2\, du \qquad (u = \sec x,\ du = \sec x \tan x\, dx)\\ &= \int (u^4 - u^2)\, du\\ &= \frac{u^5}{5} - \frac{u^3}{3} + C\\ &= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.\end{aligned}$$

14. $\displaystyle\int \tan^2 x \sec^3 x\, dx$

$$\begin{aligned}\int \tan^2 x \sec^3 x\, dx &= \int (\sec^2 x - 1)\sec^3 x\, dx\\ &= \int \sec^5 x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Compute $\int \sec^5 x\, dx$ by integration by parts with $u = \sec^3 x$ and $dv = \sec^2 x\, dx$:

$$\begin{aligned}\int \sec^5 x\, dx &= \int \sec^3 x \sec^2 x\, dx\\ &= \sec^3 x \tan x - \int \tan x\,(3\sec^3 x \tan x)\, dx\\ &= \sec^3 x \tan x - 3\int \sec^3 x \tan^2 x\, dx\\ &= \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1)\, dx\\ &= \sec^3 x \tan x - 3\left(\int \sec^5 x\, dx - \int \sec^3 x\, dx\right).\end{aligned}$$

Solve for $\int \sec^5 x\, dx$:

$$\begin{aligned}4\int \sec^5 x\, dx &= \sec^3 x \tan x + 3\int \sec^3 x\, dx\\ \int \sec^5 x\, dx &= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\, dx.\end{aligned}$$

Substitute back to finish:

$$\begin{aligned}\int \tan^2 x \sec^3 x\, dx &= \left(\frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\, dx\right) - \int \sec^3 x\, dx\\ &= \frac{1}{4}\sec^3 x \tan x - \frac{1}{4}\int \sec^3 x\, dx + C.\end{aligned}$$