7.2 Trigonometric Integrals

Integrals of Powers of Sine and Cosine

We begin by considering integrals in which the integrand is a power of sine, a power of cosine, or a product of these.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$:

1. If the power of cosine is odd, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. Substitute $u = \sin x$.
2. If the power of sine is odd, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine. Substitute $u = \cos x$.
3. If the powers of both sine and cosine are even, use the half-angle identities:
$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}.$$
4. It is sometimes helpful to use the identity $\sin x \cos x = \frac{1}{2}\sin(2x)$.

Example. Evaluate $\int \cos^3 x \, dx$.

Note: Setting $u = \cos x$ isn't helpful, since $du = -\sin x \, dx$.

We would need an extra $\sin x$ factor.

We can use the identity $\cos^2 x = 1 - \sin^2 x$:

$$\int \cos^3 x \, dx = \int \cos x \cdot \cos^2 x \, dx = \int \cos x \cdot (1-\sin^2 x) \, dx$$

Let $u = \sin x$. Then $du = \cos x \, dx$. The integral becomes

$$\int (1-u^2) \, du = u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Example. Find $\int \sin^5 x \cos^2 x \, dx$.

We could convert $\cos^2 x$ to $1-\sin^2 x$, but this gives an expression in terms of $\sin x$ with no extra $\cos x$ factor. Instead, we save one $\sin x$ factor and convert the remaining $\sin x$ factors to $\cos x$.

$$\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cdot \sin x \cdot \cos^2 x \, dx$$

$$= \int (\sin^2 x)^2 \cdot \sin x \cdot \cos^2 x \, dx$$

$$= \int (1-\cos^2 x)^2 \cdot \sin x \cdot \cos^2 x \, dx$$

Let $u = \cos x$. Then $du = -\sin x \, dx$. The integral becomes

$$-\int (1-u^2)^2 \cdot u^2 \, du = -\int (1-2u^2+u^4)\cdot u^2 \, du$$

$$= -\int u^2 - 2u^4 + u^6 \, du$$

$$= -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C$$

$$= -\left(\frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7} \right) + C$$

Example. Evaluate $\int_0^\pi \sin^2 x\,dx$.

We can use the half angle identity $\sin^2 x = \frac{1-\cos(2x)}{2}$.

$$\int_0^\pi \sin^2 x\,dx = \int_0^\pi \frac{1-\cos(2x)}{2}\,dx$$

$$= \frac{1}{2}\int_0^\pi 1-\cos(2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^\pi$$

$$= \frac{1}{2}\left[(\pi - 0) - (0-0)\right]$$

$$= \frac{\pi}{2}$$

Example. Find $\int \sin^4 x\, dx$.

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$$

$$= \int \left(\frac{1-\cos(2x)}{2} \right)^2 dx$$

$$= \frac{1}{4} \int 1 - 2\cos(2x) + \cos^2(2x) \, dx$$

We can use the half angle identity $\cos^2(2x) = \frac{1+\cos(4x)}{2}$

$$= \frac{1}{2}\left(1+\cos(4x)\right)$$

The integral becomes

$$= \frac{1}{4} \int 1 - 2\cos(2x) + \frac{1}{2}\left(1+\cos(4x)\right) dx$$

$$= \frac{1}{4} \int \frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x) \, dx$$

$$= \frac{1}{4} \left(\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x) \right) + C$$

Integrals of Powers of Secant and Tangent

Now we consider integrals in which the integrand is a power of tangent, a power of secant, or a product of these.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

$\cos^2 x + \sin^2 x = 1$

$\Rightarrow \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$\Rightarrow 1 + \tan^2 x = \sec^2 x$

1. The derivatives of tangent and secant:

$$\frac{d}{dx}[\tan x] = \sec^2 x, \quad \frac{d}{dx}[\sec x] = \sec x \tan x.$$

2. If the power of secant is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to rewrite the remaining powers of secant in terms of tangent.

3. If the power of tangent is odd, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite the remaining powers of tangent in terms of secant.

Example. Evaluate $\int \tan^6 x \sec^4 x \, dx$.

The power of secant is even. Save a factor of $\sec^2 x$ and write the rest in terms of tangent.

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \cdot \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^6 x \cdot (1 + \tan^2 x) \cdot \sec^2 x \, dx$$

Let $u = \tan x$. Then $du = \sec^2 x \, dx$. We obtain

$$\int u^6 \cdot (1 + u^2) \, du = \int u^6 + u^8 \, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Example. Find $\int \tan^5\theta \sec^7\theta \, d\theta$.

The power of tangent is odd. Save a factor of $\sec\theta\tan\theta$ and write the rest in terms of secant.

$$\int \tan^5\theta \sec^7\theta \, d\theta = \int \tan^4\theta \sec^6\theta \cdot \sec\theta\tan\theta \, d\theta$$

$$= \int (\tan^2\theta)^2 \cdot \sec^6\theta \cdot \sec\theta\tan\theta \, d\theta$$

$$= \int (\sec^2\theta - 1)^2 \cdot \sec^6\theta \cdot \sec\theta\tan\theta \, d\theta$$

Let $u = \sec\theta$. Then $du = \sec\theta\tan\theta \, d\theta$

$$= \int (u^2-1)^2 \cdot u^6 \, du$$

$$= \int (u^4 - 2u^2 + 1) \cdot u^6 \, du$$

$$= \int u^{10} - 2u^8 + u^6 \, du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{\sec^{11}\theta}{11} - \frac{2\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C$$

Remark. In some cases, the guidelines for integrating powers of $\tan x$ and $\sec x$ are not as straightforward. We may need to use trigonometric identities, integration by parts, or creative problem-solving techniques. Two important integrals to remember for these cases are:

$$\int \tan x\, dx = \ln|\sec x| + C$$

and

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C.$$

Example. Find $\int \tan^3 x\, dx$.

We can't pull out a $\sec^2 x$ term or a $\sec x \tan x$ term.

Rewrite $\tan^3 x$ as $\tan x \cdot \tan^2 x$ and use $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}\int \tan^3 x\, dx &= \int \tan x \cdot \tan^2 x\, dx \\ &= \int \tan x \cdot (\sec^2 x - 1)\, dx \\ &= \int \tan x \cdot \sec^2 x - \tan x\, dx \\ &= \int \tan x \sec^2 x\, dx - \int \tan x\, dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C\end{aligned}$$

$u = \tan x$
$du = \sec^2 x$
$\int u\, du = \frac{1}{2}u^2$

Example. Find $\int \sec^3 x\, dx$.

① We can rewrite this as

$$\int \sec^2 x \cdot \sec x\, dx = \int (1+\tan^2 x)\cdot \sec x\, dx = \int \sec x\, dx + \int \sec x \cdot \tan^2 x\, dx$$

$$\overset{u\,\cdot\, dv}{\int \sec x \cdot \tan^2 x\, dx} = \overset{u\,\cdot\, v}{\sec x \tan x} - \overset{v\,\cdot\, du}{\int \sec^3 x\, dx}$$

$u = \tan x$	$v = \sec x$
$du = \sec^2 x\, dx$	$dv = \sec x \tan x\, dx$

$$\Rightarrow \int \sec^3 x\, dx = \int \sec x\, dx + \sec x \tan x - \int \sec^3 x\, dx$$

$$\Rightarrow 2\int \sec^3 x\, dx = \int \sec x\, dx + \sec x \tan x$$

$$\Rightarrow \int \sec^3 dx = \frac{1}{2}\left[\ln|\sec x \tan x| + \sec x \tan x\right] + C$$

② Alternatively, use integration by parts on $\int \sec^3 x\, dx$.

$u = \sec x$	$v = \tan x$
$du = \sec x \tan x$	$dv = \sec^2 x\, dx$

$$\begin{aligned}\overset{u\,\cdot\, dv}{\int \sec^3 x\, dx} &= \overset{u\,\cdot\, v}{\sec x \cdot \tan x} - \overset{v\,\cdot\, du}{\int \tan^2 x \sec x\, dx}\\ &= \sec x \cdot \tan x - \int (\sec^2 x - 1)\cdot \sec x\, dx\\ &= \sec x \cdot \tan x - \int \sec^3 x - \sec x\, dx\\ &= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx\end{aligned}$$

Solve for $\sec^3 x$:

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$\int \sec^3 x\, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

$$\begin{aligned}
\int \sec^4 x\,dx &= \int (\sec^2 x)(\sec^2 x)\,dx \\
&= \int (\tan^2 x + 1)(\sec^2 x)\,dx \\
&= \int u^2 + 1\,du \\
&= \frac{1}{3}u^3 + u + C \\
&= \frac{1}{3}\tan^3 x + \tan x + C
\end{aligned}$$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x\,dx \end{aligned}}$$

$$\begin{aligned}
\int \tan^2 x \sec^4 x\,dx &= \int \tan^2 x(\sec^2 x)(\sec^2 x)\,dx \\
&= \int \tan^2 x(\tan^2 x + 1)(\sec^2 x)\,dx \\
&= \int u^2(u^2 + 1)\,du \\
&= \int u^4 + u^2\,du \\
&= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C
\end{aligned}$$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x\,dx \end{aligned}}$$

$$\begin{aligned}
\int \tan^3 x \sec x\,dx &= \int \tan^2 x(\tan x \sec x)\,dx \\
&= \int (\sec^2 x - 1)(\tan x \sec x)\,dx \\
&= \int u^2 - 1\,du \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}\sec^3 x - \sec x + C
\end{aligned}$$

$$\boxed{\begin{aligned} u &= \sec x \\ du &= \sec x \tan x\,dx \end{aligned}}$$