7.1 Integration by Parts (Solutions)

1. $\displaystyle\int_1^e xe^x\,dx$

$$u = x,\ dv = e^x\,dx \ \Rightarrow\ du = dx,\ v = e^x.$$

$$\int_1^e xe^x\,dx = \Big[xe^x - \int e^x\,dx\Big]_1^e = \Big[e^x(x-1)\Big]_1^e = \boxed{e^e(e-1)}.$$

2. $\displaystyle\int x\ln x\,dx$

$$u = \ln x,\ dv = x\,dx \ \Rightarrow\ du = \frac{1}{x}\,dx,\ v = \frac{x^2}{2}.$$

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}.$$

3. $\displaystyle\int_0^\pi x^2\sin x\,dx$

$$u = x^2,\ dv = \sin x\,dx \ \Rightarrow\ du = 2x\,dx,\ v = -\cos x.$$

$$\int x^2\sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx.$$

For $\int 2x\cos x\,dx$, take $u = 2x$, $dv = \cos x\,dx$, so $du = 2\,dx$, $v = \sin x$:

$$\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx = 2x\sin x + 2\cos x.$$

$$\int_0^\pi x^2\sin x\,dx = \Big[-x^2\cos x + 2x\sin x + 2\cos x\Big]_0^\pi = \boxed{\pi^2 - 4}.$$

4. $\displaystyle\int x\cos x\,dx$

$$u = x,\ dv = \cos x\,dx \ \Rightarrow\ du = dx,\ v = \sin x.$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = \boxed{x\sin x + \cos x + C}.$$

5. $\displaystyle\int_1^2 x^3\ln x\,dx$

$$u = \ln x,\ dv = x^3\,dx \ \Rightarrow\ du = \frac{1}{x}\,dx,\ v = \frac{x^4}{4}.$$

$$\int_1^2 x^3\ln x\,dx = \Big[\frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx\Big]_1^2 = \Big[\frac{x^4}{4}\ln x - \frac{x^4}{16}\Big]_1^2 = \boxed{4\ln 2 - \frac{15}{16}}.$$

6. $\displaystyle\int x^2\ln(x^2)\,dx$

Use $\ln(x^2) = 2\ln|x|$:

$$\int x^2\ln(x^2)\,dx = 2\int x^2\ln|x|\,dx.$$

$$u = \ln|x|,\ dv = x^2\,dx \ \Rightarrow\ du = \frac{1}{x}\,dx,\ v = \frac{x^3}{3}.$$

$$\int x^2\ln(x^2)\,dx = 2\Big(\frac{x^3}{3}\ln|x| - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx\Big) = 2\Big(\frac{x^3}{3}\ln|x| - \int \frac{x^2}{3}\,dx\Big) = \boxed{\frac{2x^3}{3}\ln|x| - \frac{2x^3}{9} + C}.$$

7. $\displaystyle\int x^2\cos(2x)\,dx$

$$u = x^2,\ dv = \cos(2x)\,dx \ \Rightarrow\ du = 2x\,dx,\ v = \frac{1}{2}\sin(2x).$$

$$\int x^2\cos(2x)\,dx = \frac{x^2}{2}\sin(2x) - \int x\sin(2x)\,dx.$$

For $\int x\sin(2x)\,dx$, take $u = x$, $dv = \sin(2x)\,dx$, so $du = dx$, $v = -\frac{1}{2}\cos(2x)$:

$$\int x\sin(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x).$$

$$\int x^2\cos(2x)\,dx = \boxed{\frac{x^2}{2}\sin(2x) + \frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C}.$$

8. $\displaystyle\int \ln(x^2+1)\,dx$

$$u = \ln(x^2+1),\ dv = dx \ \Rightarrow\ du = \frac{2x}{x^2+1}\,dx,\ v = x.$$

$$\int \ln(x^2+1)\,dx = x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx = x\ln(x^2+1) - \int \Big(2 - \frac{2}{x^2+1}\Big)\,dx = \boxed{x\ln(x^2+1) - 2x + 2\arctan x + C}.$$

7.1 Boomerang Integrals (Solutions)

1. $\int e^{3x}\cos(3x)\,dx$

   Let $I = \int e^{3x}\cos(3x)\,dx$.

   $$I = \frac{e^{3x}\cos(3x)}{3} + \int e^{3x}\sin(3x)\,dx.$$

   Let $J = \int e^{3x}\sin(3x)\,dx$. Then

   $$\begin{aligned} J &= \frac{e^{3x}\sin(3x)}{3} - \int e^{3x}\cos(3x)\,dx \\ &= \frac{e^{3x}\sin(3x)}{3} - I. \end{aligned}$$

   $$\begin{aligned} I &= \frac{e^{3x}\cos(3x)}{3} + J \\ &= \frac{e^{3x}\cos(3x)}{3} + \frac{e^{3x}\sin(3x)}{3} - I, \end{aligned}$$

   so

   $$\begin{aligned} 2I &= \frac{e^{3x}}{3}\big(\sin(3x)+\cos(3x)\big) \\ I &= \boxed{\frac{e^{3x}\big(\sin(3x)+\cos(3x)\big)}{6} + C}. \end{aligned}$$

2. $\int e^{-x}\sin x\,dx$

   Let $I = \int e^{-x}\sin x\,dx$.

   $$I = -e^{-x}\sin x + \int e^{-x}\cos x\,dx.$$

   Let $J = \int e^{-x}\cos x\,dx$. Then

   $$\begin{aligned} J &= -e^{-x}\cos x - \int e^{-x}\sin x\,dx \\ &= -e^{-x}\cos x - I. \end{aligned}$$

   $$\begin{aligned} I &= -e^{-x}\sin x + J \\ &= -e^{-x}\sin x - e^{-x}\cos x - I, \end{aligned}$$

   so

   $$\begin{aligned} 2I &= -e^{-x}\big(\sin x + \cos x\big) \\ I &= \boxed{-\frac{e^{-x}\big(\sin x+\cos x\big)}{2} + C}. \end{aligned}$$