

7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called *integration by parts*.

Theorem (Integration by Parts, Formula 1). If f and g are differentiable functions, then

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx.$$

Recall the product rule for differentiation:

$$\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Integrate both sides:

$$\int \frac{d}{dx} [f(x)g(x)] dx = \int f'(x)g(x) + f(x)g'(x) dx$$

We obtain

$$f(x)g(x) + C = \int f'(x)g(x) dx + \int f(x)g'(x) dx$$

Rearranging, we get

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx + C$$

Note: We typically treat the constant of integration as implied and neglect writing it in the formula. (After integrating, combine all the "+C" terms into one)

The formula for integration by parts is often written in the following form:

Theorem (Integration by Parts, Formula 2). Let $u = f(x)$ and $v = g(x)$ in the theorem above. Then $du = f'(x) dx$ and $dv = g'(x) dx$. We obtain:

$$\int u dv = uv - \int v du.$$

Example. Find $\int x \sin x \, dx$.

Formula #1. Let $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$

and $g(x) = \int \sin x \, dx = -\cos x$

$$\begin{aligned}\int x \sin x \, dx &= \int f(x) \cdot g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx \\ &= x \cdot (-\cos x) - \int 1 \cdot (-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C\end{aligned}$$

Formula #2. Let $u = x$ and $dv = \sin x \, dx$. Then $du = dx$
and $v = -\cos x$.

$$\begin{aligned}\int x \sin x \, dx &= \int u \cdot dv = u \cdot v - \int v \cdot du \\ &= x \cdot (-\cos x) - \int -\cos x \cdot dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C\end{aligned}$$

Question. In the above example, what would have happened if we had instead chosen $u = \sin x$ and $dv = x \, dx$?

In this case, $du = \cos x \, dx$ and $v = \int x \, dx = \frac{x^2}{2}$. We obtain

$$\int \overset{u \cdot dv}{x \sin x \, dx} = \overset{u \cdot v}{\sin x \cdot \frac{x^2}{2}} - \int \overset{v \cdot du}{\frac{x^2}{2} \cdot \cos x \, dx}$$

This is more complicated than the original problem!

Example. Evaluate $\int \ln x \, dx$.

$u = \ln x$	$v = x$
$du = \frac{1}{x} dx$	$dv = dx$

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$$\begin{aligned}
 \int \ln x \, dx &= \overset{u \cdot dv}{\ln x \cdot x} - \overset{u \cdot v}{\int x \cdot \overset{v \cdot du}{\frac{1}{x} dx}} \\
 &= x \ln x - \int 1 \, dx \\
 &= x \ln x - x + C
 \end{aligned}$$

Example. Find $\int t^2 e^t \, dt$.

$u = t^2$	$v = e^t$
$du = 2t \, dt$	$dv = e^t \, dt$

① $\int t^2 \cdot e^t \, dt = t^2 e^t - \int e^t \cdot 2t \, dt$

$u = 2t$	$v = e^t$
$du = 2 \, dt$	$dv = e^t \, dt$

② $\int e^t \cdot 2t \, dt = 2t \cdot e^t - \int e^t \cdot 2 \, dt$

$$= 2t \cdot e^t - 2e^t + C$$

In total,

$$\begin{aligned}
 \int t^2 \cdot e^t \, dt &= t^2 \cdot e^t - (2t e^t - 2e^t + C) \\
 &= t^2 e^t - 2t e^t + 2e^t + C \\
 &= e^t (t^2 - 2t + 2) + C
 \end{aligned}$$

"Boomerang
Integral"

Example. Evaluate $\int e^x \sin x \, dx$.

$u = \sin x$	$v = e^x$
$du = \cos x \, dx$	$dv = e^x \, dx$

$$\int e^x \sin x \, dx = e^x \cdot \sin x - \int e^x \cos x \, dx$$

$u = \cos x$	$v = e^x$
$du = -\sin x \, dx$	$dv = e^x \, dx$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \cdot \cos x - \int e^x \cdot (-\sin x) \, dx \\ &= e^x \cos x + \int e^x \sin x \, dx \end{aligned}$$

In total,

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - (e^x \cos x + \int e^x \sin x \, dx) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \end{aligned}$$

So

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\Rightarrow \int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

* Note: remember to add the constant of integration.

Definite Integration by Parts

If we combine the formula for integration by parts with the Evaluation Theorem, we can evaluate definite integrals by parts:

Theorem (Definite Integration by Parts).

$$\int_a^b \overset{u \cdot dv}{f(x) g'(x) dx} = \left[\overset{u \cdot v}{f(x) g(x)} \right]_a^b - \int_a^b \overset{v \cdot du}{f'(x) g(x) dx}.$$

Example. Calculate $\int_0^1 \tan^{-1}(x) dx$.

①

$u = \tan^{-1}(x)$	$v = x$
$du = \frac{1}{1+x^2} dx$	$dv = dx$

$$\textcircled{2} \quad \int_0^1 \overset{u \cdot dv}{\tan^{-1}(x) dx} = \left[\overset{u \cdot v}{\tan^{-1}(x) \cdot x} \right]_0^1 - \int_0^1 \overset{v \cdot du}{\frac{x}{1+x^2} dx}$$

$$\textcircled{3} \quad \text{We have } \left[\tan^{-1}(x) \cdot x \right]_0^1 = \tan^{-1}(1) \cdot 1 - \tan^{-1}(0) \cdot 0 = \frac{\pi}{4}$$

$$\textcircled{4} \quad \text{To compute } \int_0^1 \frac{x}{1+x^2} dx. \quad \text{Let } u=1+x^2, \text{ so } du=2x dx \\ \Rightarrow x dx = \frac{1}{2} du$$

$$\text{Then } \int_0^1 \frac{x}{1+x^2} dx = \int_{u=1}^{u=2} \frac{1}{u} \cdot \frac{1}{2} du$$

when $x=0$, $u=1$
when $x=1$, $u=2$

$$= \left[\frac{1}{2} \cdot \ln|u| \right]_1^2 = \frac{1}{2} \ln 2$$

$$\textcircled{5} \quad \text{In total, } \int_0^1 \tan^{-1}(x) dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$$