6.5 Average Value of a Function

Theorem. The average value of f on the interval $[a, b]$ is $f_{\text{avg}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

- The average of n numbers $x_1, x_2, \ldots, x_n$ is given by:

$$\text{Average} = \frac{\text{Total sum}}{\text{number of values}} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

- For functions, the idea of "average" extends to infinitely many values (since a function takes on a value at every point in the interval $[a, b]$). We use an integral to represent the total sum.

$$\text{Total sum} = \int_a^b f(x)\,dx$$

- The average value requires dividing the total sum by the "number of values." For functions over $[a, b]$, the equivalent is dividing by the length of the interval $b - a$.

$$f_{\text{avg}} = \frac{\text{Total sum}}{\text{number of values}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

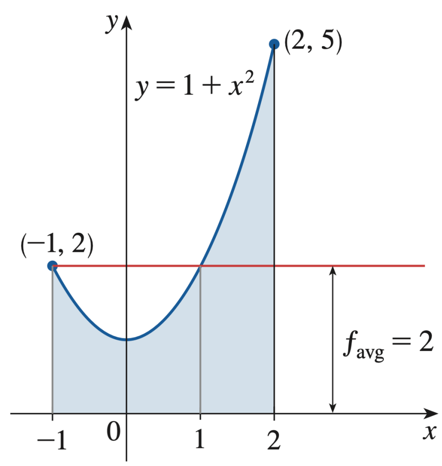
Example. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2} 1 + x^2\,dx$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

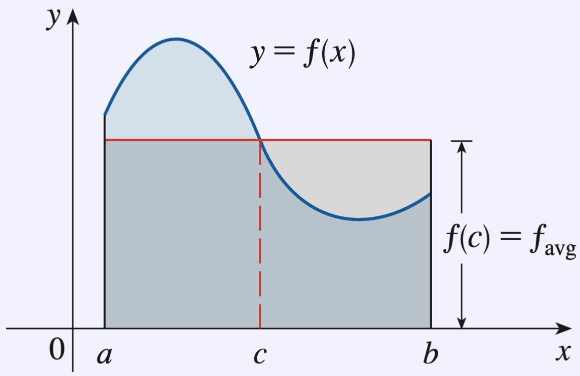
$$= 2$$



Theorem (The Mean Value Theorem for Integrals)**.** If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f\text{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is, $\int_a^b f(x)\,dx = f(c)(b-a)$



You can always chop off the top of a mountain at a certain height (namely, f_{avg}), and use it to fill in the valleys so that the mountain becomes completely flat.

Example. Determine the value of c that satisfies the conclusion of the Mean Value Theorem for the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

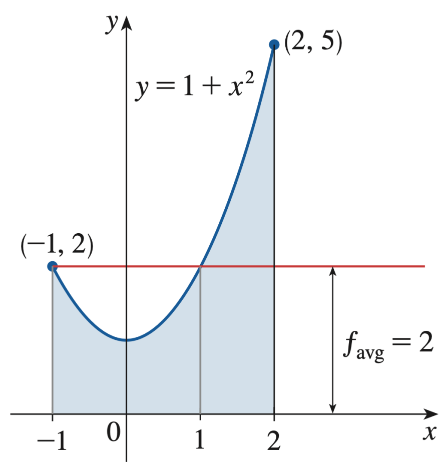
By the Mean Value Theorem, there is some c in $[-1,2]$ so that

$$f(c) = f_{avg} = \frac{1}{2-(-1)}\int_{-1}^{2} 1+x^2\,dx$$

We know $f_{avg} = 2$, so

$$1 + c^2 = 2 \Rightarrow c^2 = 1$$

$$\Rightarrow c = \pm 1$$



Example. Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

Let $s(t)$ be the displacement of the car at time t. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

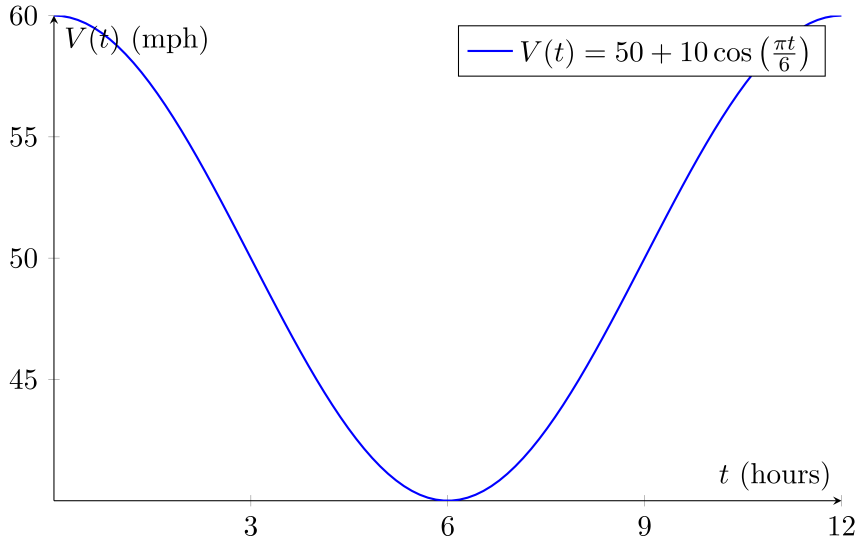
The average value of the velocity function is

$$\begin{aligned} V_{avg} &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\, dt \\ &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s'(t)\, dt \\ &= \frac{1}{t_2 - t_1} \Big[s(t) \Big]_{t_1}^{t_2} \\ &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \end{aligned}$$

Conclude: Taking the averages of instantaneous velocities (e.g. readings from a speedometer) agrees with the theoretical definition of average velocity, which is change in position over change in time.

Example (Monitoring Traffic Flow)**.** Traffic engineers are analyzing vehicle speeds on a 6-mile stretch of highway during a 12-hour observation period. The speed of traffic (in miles per hour) at time t (in hours) is modeled as:

$$V(t) = 50 + 10\cos\left(\frac{\pi t}{6}\right), \quad t \in [0, 12].$$



Question. Knowing the average speed helps ensure that traffic flows smoothly and within safe limits, preventing dangerous speed fluctuations. What is the average speed of vehicles over the 12-hour period?

$$V_{avg} = \frac{1}{b-a}\int_a^b V(t)\,dt = \frac{1}{12}\int_0^{12} 50 + 10\cos\left(\frac{\pi t}{6}\right) dt$$

$$= \frac{1}{12}\int_0^{12} 50\,dt + \frac{1}{12}\int_0^{12} 10\cos\left(\frac{\pi t}{6}\right) dt$$

$$= \frac{1}{12}\Big[50t\Big]_0^{12} + \frac{1}{12}\left[\frac{60}{\pi}\sin\left(\frac{\pi t}{6}\right)\right]_0^{12}$$

$$= 50 \text{ mph}$$

Question. At what times does the instantaneous speed match the average speed? These moments indicate when traffic flow is most representative of overall conditions, helping engineers optimize traffic signals and safety measures.

- By the Mean Value Theorem, there is some c in $[0, 12]$ so that $V(c) = V_{avg} = 50$

$$50 + 10\cos\left(\frac{\pi c}{6}\right) = 50$$

$$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = 0$$

$$\Rightarrow \frac{\pi c}{6} = \frac{\pi}{2} + n\pi \quad \text{for all integers } n$$

$$\Rightarrow c = 3 + 6n \quad \text{for all integers } n$$

- In $[0, 12]$, $c = 3, 9$
- If this is afternoon traffic, the instantaneous speed matches the average speed at 3pm and 9pm
- This tells engineers when to observe traffic to get representative conditions.