6.4 Work

Overview

Definition (Force and Newton's Second Law)**.** If an object of mass m moves along a straight line with position function $s(t)$, then its *acceleration* is

$$a(t) = \frac{d^2 s}{dt^2}.$$

By Newton's Second Law of Motion, the *force* F on the object is

$$F = m\,a = m\,\frac{d^2 s}{dt^2}.$$

Definition (Work for Constant Force)**.** If a constant force F acts on an object and moves it through a distance d in the direction of the force, then the *work* W done by the force is

$$W = F \cdot d \quad \text{(force} \times \text{distance)}.$$

Definition (Units of Work)**.**

- In the **SI (metric) system**, force is measured in *newtons* (N) and distance in *meters* (m). Hence work is measured in *newton-meters*, also called *joules* (J).

$$1\,\text{joule} = 1\,\text{newton} \cdot 1\,\text{meter}.$$

- In the **US Customary system**, force is measured in *pounds* (lb) and distance in *feet* (ft). Hence work is measured in *foot-pounds* (ft-lb).

$$1\,\text{ft-lb} \approx 1.36\,\text{joules}.$$

Definition (Weight vs. Mass)**.**

- An object's *mass* (in kilograms, kg) times the acceleration due to gravity ($\approx 9.8\,\text{m/s}^2$) gives its *weight* (in newtons).

$$\text{weight} = m \times 9.8 \text{ (in newtons)}.$$

- In the US system, the word "pound" itself denotes a force (weight). Hence an object that weighs W pounds has mass $m = W/g$ (in "slugs") if $g \approx 32\,\text{ft/s}^2$.

Theorem (Work for a Variable Force)**.** Let an object move along the x-axis from $x = a$ to $x = b$, acted upon by a *continuous* force $f(x)$ that depends on the position x. The *work* W done by this force is given by the definite integral

$$W = \int_a^b f(x)\,dx.$$

Interpretation: We partition the interval $[a, b]$ into small segments, approximate the (nearly constant) force on each segment, multiply by the small distance, and then let the partition become finer. The limit of these Riemann sums is the above integral.

Definition (Hooke's Law for Springs)**.** If a spring is stretched (or compressed) x units from its natural length (where x is not too large), the *force* F required to hold it there obeys

$$F(x) \;=\; k\,x,$$

where k is a positive constant called the *spring constant*. Consequently, the work required to stretch a spring from $x = a$ to $x = b$ (beyond its natural length) is

$$W \;=\; \int_a^b kx\,dx \;=\; \frac{k}{2}\left[b^2 - a^2\right].$$

Work Problems

Spring Problems

1. A spring has a natural length of 20 m. A force of 12 N is required to stretch the spring to 25 m. Determine the work required to stretch the spring from 20 m to 30 m.

2. A spring has a natural length of 15 m. A force of 10 N is required to stretch the spring to 18 m. Determine the work required to stretch the spring from 16 m to 22 m.

3. A spring has a natural length of 30 m. A force of 8 N is required to stretch the spring to 35 m. Determine the work required to compress the spring from 30 m to 20 m.

Cable Problems

1. A 100-meter-long cable with a linear density of 5 kg/m is hanging from a winch at the top of a well. The cable is initially fully extended into the well and is lifted to the top. Compute the work required to lift the entire cable.

2. A 50-meter-long chain with a linear density of 8 kg/m is hanging from a pulley at the top of a mine shaft. The chain is initially fully extended into the shaft and is lifted to the top. Compute the work required to lift the entire chain.

3. A 60-meter-long rope with a linear density of 3 kg/m is hanging over the edge of a cliff, with one end secured at the top and the other end dangling freely. The rope is slowly lifted until it is fully coiled at the top of the cliff. Compute the work required to lift the rope.

4. A 30-meter-long anchor chain with a linear density of 12 kg/m is hanging from the side of a ship, with one end attached to the ship and the other submerged in the water. The chain is hoisted onto the deck of the ship. Compute the work required to lift the entire chain onto the ship.

Tank Problems

For each problem, set up (but do not solve) the work integral for pumping water out of the tank. In all setups, use the density of water as 1000 (kg/m^3) and gravity as 9.8 (m/s^2).

1. **Rectangular Tank with Triangular Ends:** A tank is 6 m long, and its end view is an isosceles triangle with a base of 2 m and a height of 3 m. Water is pumped out through a spout located 0.5 m above the top of the tank.

2. **Cylindrical Tank with a Spout:** A vertical cylindrical tank is 4 m high with a circular cross section of radius 1.5 m. Water is pumped out through a spout that is 0.3 m above the top of the tank.

3. **Inverted Conical Tank:** An inverted conical tank has a height of 3 m and an open top with a radius of 1 m. Water is pumped out to a spout 0.2 m above the top.

4. **Spherical Tank:** A spherical tank of radius 2 m is completely filled with water. Water is pumped out through a spout located 0.1 m above the top of the sphere.

5. **Composite Tank – Cylinder with Hemispherical Top:** The tank consists of a cylindrical section 3 m high with a circular cross section of radius 1 m, topped by a hemispherical dome of radius 1 m. Water is pumped out through a spout located 0.15 m above the dome.