6.4 Work (Solutions)

Spring Problems (Solutions)

1. A spring has a natural length of 20 m. A force of 12 N is required to stretch the spring to 25 m. Determine the work required to stretch the spring from 20 m to 30 m.

   - By Hooke's Law, the force required to stretch a spring is:
   $$F = kx,$$
   where k is the spring constant and x is the displacement from the natural length.
   - Given that a force of 12 N is required to stretch the spring to 25 m:
   $$12 = k(5).$$
   Solving for k:
   $$k = \frac{12}{5} = 2.4.$$
   - The work done to stretch the spring from $x = a$ to $x = b$ is given by:
   $$W = \int_a^b kx\,dx.$$
   - Here, we compute the work to stretch from 20 m to 30 m, which corresponds to $x = 0$ to $x = 10$ m:
   $$W = \int_0^{10} 2.4x\,dx.$$
   - Computing the integral:
   $$\begin{aligned} W &= 2.4\int_0^{10} x\,dx \\ &= 2.4\left[\frac{x^2}{2}\right]_0^{10} \\ &= 2.4\left(\frac{100}{2} - \frac{0}{2}\right) \\ &= 2.4 \times 50 \\ &= 120 \text{ J}. \end{aligned}$$

2. A spring has a natural length of 15 m. A force of 10 N is required to stretch the spring to 18 m. Determine the work required to stretch the spring from 16 m to 22 m.

   - Hooke's Law states that $F = kx$, where k is the spring constant and x is the displacement from the natural length.
   - Given that a force of 10 N stretches the spring to 18 m, we find k:
   $$\begin{aligned} 10 &= k(18 - 15) \\ 10 &= k(3) \\ k &= \frac{10}{3} \text{ N/m} \end{aligned}$$
   - The work done to stretch the spring from $x_1 = 16$ m to $x_2 = 22$ m is given by:
   $$\begin{aligned} W &= \int_{x_1}^{x_2} kx\,dx \\ &= \int_1^7 \frac{10}{3}x\,dx \\ &= \frac{10}{3}\left[\frac{x^2}{2}\right]_1^7 \\ &= \frac{10}{3}\left(\frac{49}{2} - \frac{1}{2}\right) \\ &= \frac{10}{3} \times \frac{48}{2} \\ &= \frac{10}{3} \times 24 \\ &= 80 \text{ J} \end{aligned}$$

3. A spring has a natural length of 30 m. A force of 8 N is required to stretch the spring to 35 m. Determine the work required to compress the spring from 30 m to 20 m.

    - Using Hooke's Law, we first determine the spring constant k:

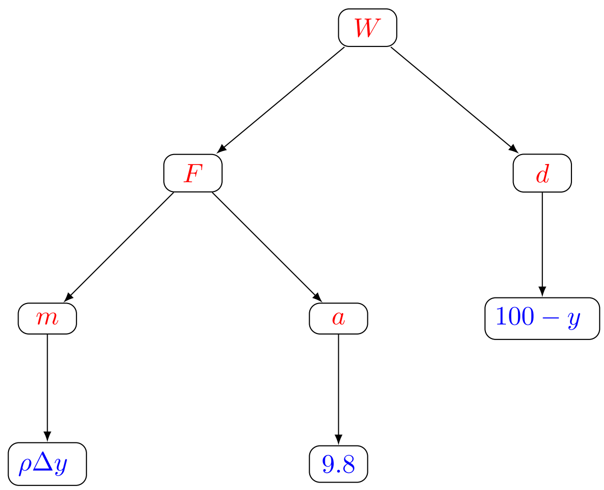
$$\begin{aligned} 8 &= k(35 - 30) \\ 8 &= k(5) \\ k &= \frac{8}{5} \text{ N/m} \end{aligned}$$

    - The work required to compress the spring from $x_1 = 30$ m to $x_2 = 20$ m is:

$$\begin{aligned} W &= \int_{x_1}^{x_2} kx\,dx \\ &= \int_0^{-10} \frac{8}{5}x\,dx \\ &= \frac{8}{5}\left[\frac{x^2}{2}\right]_0^{-10} \\ &= \frac{8}{5}\left(\frac{100}{2} - 0\right) \\ &= \frac{8}{5} \times 50 \\ &= 80 \text{ J} \end{aligned}$$

Cable Problems (Solutions)

1. A **100-meter-long** cable with a **linear density of 5 kg/m** is hanging from a winch at the top of a well. The cable is initially fully extended into the well and is lifted **to the top**. Compute the work required to lift the entire cable.



Step 1: Define Variables and Divide the Cable into Slices

- Let y be the height above the **bottom of the well**, with $y = 0$ at the bottom and $y = 100$ at the top.
- Partition the interval $[0, 100]$ into n slices of equal height Δy.
- The mass of a slice of cable at height y_i^* is:

$$m_i = \rho \Delta y = 5\Delta y.$$

- The force due to gravity acting on the slice is:

$$F_i = g \cdot m_i = 9.8 \cdot (5\Delta y).$$

Step 2: Compute the Work on Each Slice

- Each slice at height y_i^* must be lifted from its original position y_i^* to the top ($y = 100$).
- The lifting distance for the slice is:

$$d_i = 100 - y_i^*.$$

- The work done to lift the ith slice is:

$$W_i = F_i \cdot d_i = (9.8 \cdot 5\Delta y) \cdot (100 - y_i^*).$$

Step 3: Express Work as an Integral

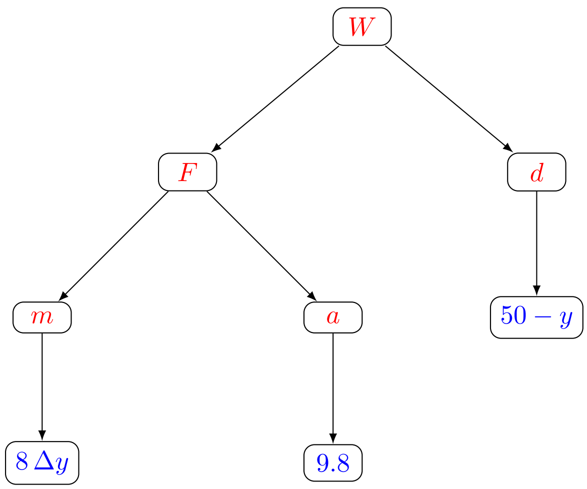
- Summing over all slices and taking the limit as $n \to \infty$, the total work is:

$$W = \int_0^{100} 9.8 \cdot 5 \cdot (100 - y)\, dy.$$

- Evaluating the integral:

$$\begin{aligned} W &= 49 \int_0^{100} (100 - y)\, dy \\ &= 49 \left[100y - \frac{y^2}{2} \right]_0^{100} \\ &= 49 \left(100(100) - \frac{100^2}{2} \right) \\ &= 49\,(10000 - 5000) \\ &= 49 \times 5000 \\ &= 245000 \text{ J.} \end{aligned}$$

2. A **50-meter-long** chain with a **linear density of 8 kg/m** is hanging from a pulley at the top of a mine shaft. The chain is initially fully extended into the shaft and is lifted **to the top**. Compute the work required to lift the entire chain.



Step 1: Define Variables and Divide the Chain into Slices

- Let y denote the height above the **bottom of the shaft** (with $y = 0$ at the bottom and $y = 50$ at the top).
- Divide the interval $[0, 50]$ into n slices of equal thickness Δy.
- The mass of a slice at height y_i^* is:

$$m_i = \rho\,\Delta y = 8\,\Delta y.$$

- The weight (force due to gravity) on the slice is:

$$F_i = m_i\,g = 8(9.8)\,\Delta y = 78.4\,\Delta y.$$

Step 2: Compute the Work on Each Slice

- A slice at height y_i^* is lifted to the top, a distance of:

$$d_i = 50 - y_i^*.$$

- The work done on this slice is approximately:

$$W_i = F_i \cdot d_i = 78.4\,(50 - y_i^*)\,\Delta y.$$

Step 3: Express Work as an Integral

- Summing over all slices and taking the limit as $n \to \infty$ gives:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 78.4\,(50 - y_i^*)\,\Delta y \\ &= 78.4 \int_0^{50} (50 - y)\,dy. \end{aligned}$$

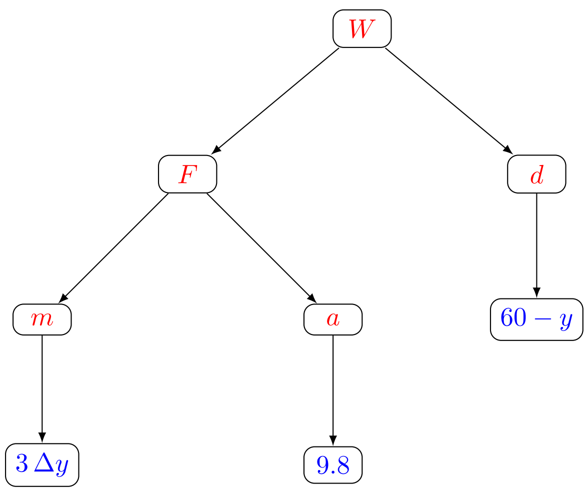
- Evaluate the integral:

$$\begin{aligned} \int_0^{50} (50 - y)\,dy &= \left[50y - \frac{y^2}{2}\right]_0^{50} \\ &= 50(50) - \frac{50^2}{2} \\ &= 2500 - 1250 \\ &= 1250. \end{aligned}$$

- Thus, the total work is:

$$W = 78.4 \times 1250 = 98\,000 \text{ J}.$$

3. A **60-meter-long** rope with a **linear density of 3 kg/m** is hanging over the edge of a cliff, with one end secured at the top and the other end dangling freely. The rope is slowly lifted until it is fully coiled at the top of the cliff. Compute the work required to lift the rope.



Step 1: Define Variables and Divide the Rope into Slices

- Let y denote the height above the **bottom of the rope** (with $y = 0$ at the bottom and $y = 60$ at the top).
- Divide the interval $[0, 60]$ into n slices of equal length Δy.
- The mass of a slice is:

$$m_i = \rho\,\Delta y = 3\,\Delta y.$$

- The weight on the slice is:

$$F_i = m_i\,g = 3(9.8)\,\Delta y = 29.4\,\Delta y.$$

Step 2: Compute the Work on Each Slice

- Each slice at height y_i^* is lifted a distance:

$$d_i = 60 - y_i^*.$$

- The work done on the slice is:

$$W_i = F_i \cdot d_i = 29.4\,(60 - y_i^*)\,\Delta y.$$

Step 3: Express Work as an Integral

- The total work is:

$$W = 29.4 \int_0^{60} (60 - y)\,dy.$$

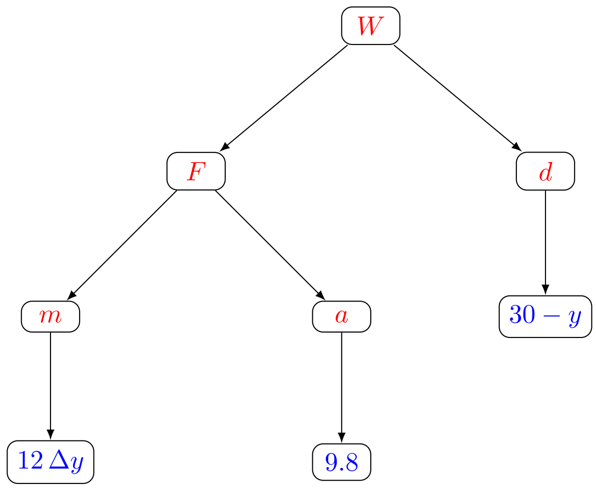
- Evaluate the integral:

$$\int_0^{60} (60 - y)\,dy = \left[60y - \frac{y^2}{2}\right]_0^{60} = 3600 - 1800 = 1800.$$

- Hence, the work is:

$$W = 29.4 \times 1800 = 52\,920 \text{ J}.$$

4. A **30-meter-long** anchor chain with a **linear density of 12 kg/m** is hanging from the side of a ship, with one end attached to the ship and the other submerged in the water. The chain is hoisted onto the deck of the ship. Compute the work required to lift the entire chain onto the ship.



Step 1: Define Variables and Divide the Chain into Slices

- Let y denote the height above the **bottom of the chain** (with $y = 0$ at the submerged end and $y = 30$ at the deck).
- Divide the interval $[0, 30]$ into n slices of thickness Δy.
- The mass of a slice is:

$$m_i = \rho\,\Delta y = 12\,\Delta y.$$

- The weight on the slice is:

$$F_i = m_i\,g = 12(9.8)\,\Delta y = 117.6\,\Delta y.$$

Step 2: Compute the Work on Each Slice

- A slice at height y_i^* is lifted a distance:

$$d_i = 30 - y_i^*.$$

- The work done on the slice is:

$$W_i = F_i \cdot d_i = 117.6\,(30 - y_i^*)\,\Delta y.$$

Step 3: Express Work as an Integral

- The total work is:

$$W = 117.6 \int_0^{30} (30 - y)\,dy.$$

- Evaluate the integral:

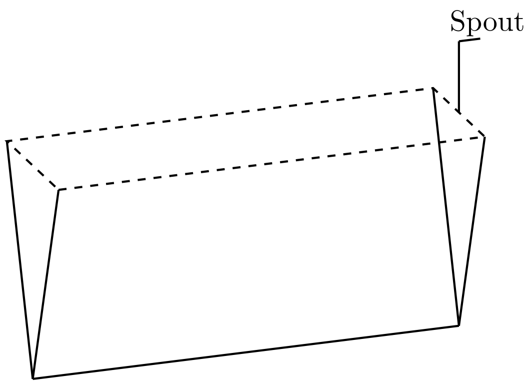
$$\int_0^{30} (30 - y)\,dy = \left[30y - \frac{y^2}{2}\right]_0^{30}$$
$$= 900 - 450$$
$$= 450.$$

- Thus, the work is:

$$W = 117.6 \times 450 = 52\,920 \text{ J}.$$

Tank Problems (Solutions)

1. **Rectangular Tank with Triangular Ends**. A tank is 6 m long (into the page) and its end view is an isosceles triangle with a base of 2 m and a height of 3 m. The tank is filled with water, and the water is pumped out through a spout located 0.5 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the bottom (vertex) and $y = 3$ at the top.

Partition the interval $[0, 3]$ into n subintervals of equal width Δy. For the ith subinterval, choose a representative point y_i^*.

By similar triangles, the width of the tank at y_i^* is

$$w_i = \frac{2}{3}\, y_i^*.$$

Since the tank is 6 m long, the cross-sectional area of a slice is

$$A_i = 6 \cdot w_i = 6\left(\frac{2}{3}\, y_i^*\right) = 4\, y_i^*.$$

Thus, the volume of the ith slice is given by

$$V_i = A_i\, \Delta y = 4\, y_i^*\, \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is obtained by multiplying the mass by gravitational acceleration:

$$F_i = \rho\, g\, V_i = 1000 \cdot 9.8 \cdot (4\, y_i^*\, \Delta y).$$

Each slice must be lifted to the spout, which is at a height of $3 + 0.5 = 3.5$ m. Hence, the lifting distance for the ith slice is

$$d_i = 3.5 - y_i^*.$$

Thus, the work done on the ith slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot 4\, y_i^*\, (3.5 - y_i^*)\, \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

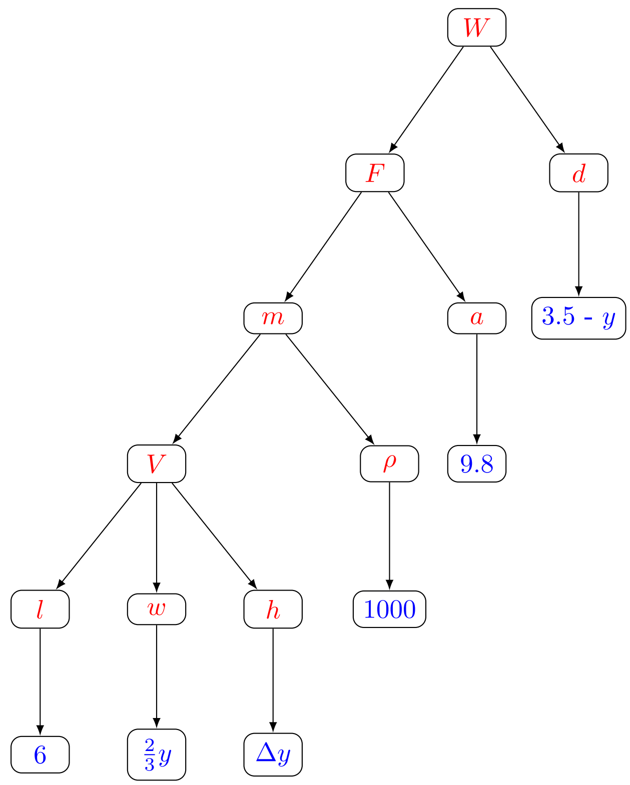
The total work required to pump the water is given by the Riemann sum

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1000 \cdot 9.8 \cdot 4\, y_i^* (3.5 - y_i^*)\, \Delta y\right].$$

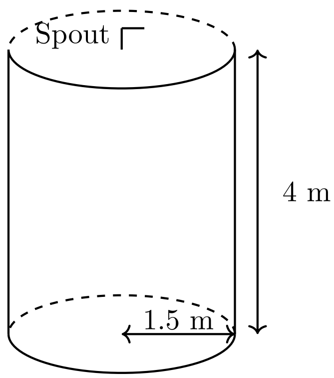
This is equivalent to the integral

$$W = \int_0^3 1000 \cdot 9.8 \cdot 4y\, (3.5 - y)\, dy.$$

Tree Diagram Representation:



2. **Cylindrical Tank with a Spout**. A vertical cylindrical tank is 4 m high with a radius of 1.5 m. The tank is completely full of water, and the water is pumped out through a spout located 0.3 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Let $y = 0$ at the bottom of the cylinder and $y = 4$ at the top. Partition $[0, 4]$ into n subintervals of equal width Δy. In the ith subinterval, pick a representative point y_i^*.

The cross-section at height y_i^* is a circle of radius 1.5. Hence, its area is

$$A = \pi \times (1.5)^2 = 2.25\pi.$$

The volume of the ith slice is

$$V_i = A\,\Delta y = 2.25\pi\,\Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is

$$F_i = \rho\,g\,V_i = 1000 \cdot 9.8 \cdot (2.25\pi\,\Delta y).$$

Each slice must be lifted to the spout, which is at $y = 4 + 0.3 = 4.3$. Hence, the lifting distance is

$$d_i = 4.3 - y_i^*.$$

Thus, the work for the ith slice is

$$W_i = F_i \cdot d_i \;=\; 1000 \cdot 9.8 \cdot 2.25\pi\,(4.3 - y_i^*)\,\Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

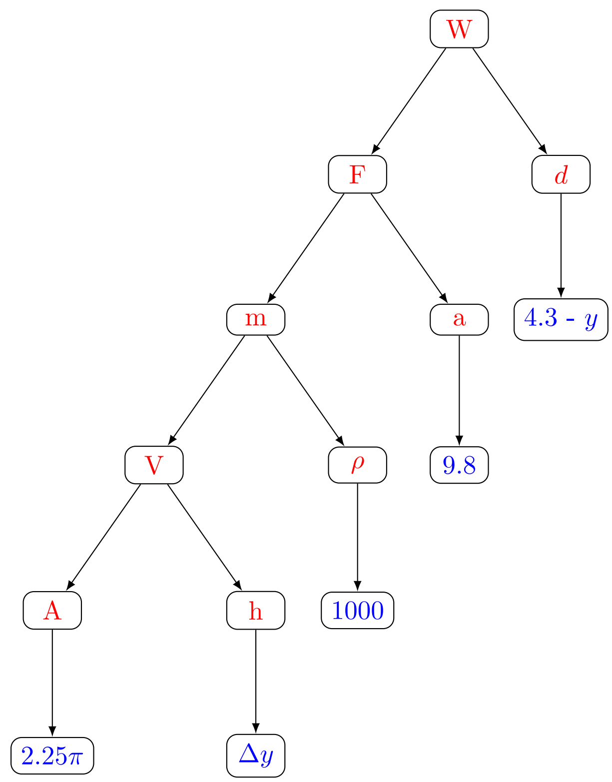
Summing over all slices,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \big[\,1000 \cdot 9.8 \cdot 2.25\pi\,(4.3 - y_i^*)\,\Delta y\big].$$

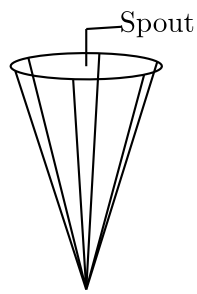
Equivalently, in integral form,

$$W = \int_0^4 1000 \cdot 9.8 \cdot 2.25\pi\,(4.3 - y)\,dy.$$

Tree Diagram Representation:



3. **Inverted Conical Tank**. An inverted conical tank has a height of 3 m and a top (open) radius of 1 m. The tank is completely full of water, and the water is pumped out through a spout located 0.2 m above the top of the tank.



Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the tip (bottom) and $y = 3$ at the top. Partition the interval $[0, 3]$ into n equal subintervals of width Δy; in the ith subinterval choose a representative point y_i^*.

By similar triangles, the radius at height y_i^* is

$$r_i = \frac{1}{3}\, y_i^*,$$

so the cross-sectional area is

$$A_i = \pi\big(r_i\big)^2 = \pi\left(\frac{y_i^*}{3}\right)^2 = \frac{\pi(y_i^*)^2}{9}.$$

Thus, the volume of the ith slice is

$$V_i = A_i\,\Delta y = \frac{\pi(y_i^*)^2}{9}\,\Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is given by

$$F_i = \rho\, g\, V_i = 1000 \cdot 9.8 \cdot \frac{\pi(y_i^*)^2}{9}\,\Delta y.$$

The water must be pumped up to the spout, which is located 0.2 m above the top. Since the top is at $y = 3$, the spout is at $y = 3.2$. Therefore, the lifting distance is

$$d_i = 3.2 - y_i^*.$$

Thus, the work done on the ith slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot \frac{\pi(y_i^*)^2}{9}\,(3.2 - y_i^*)\,\Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

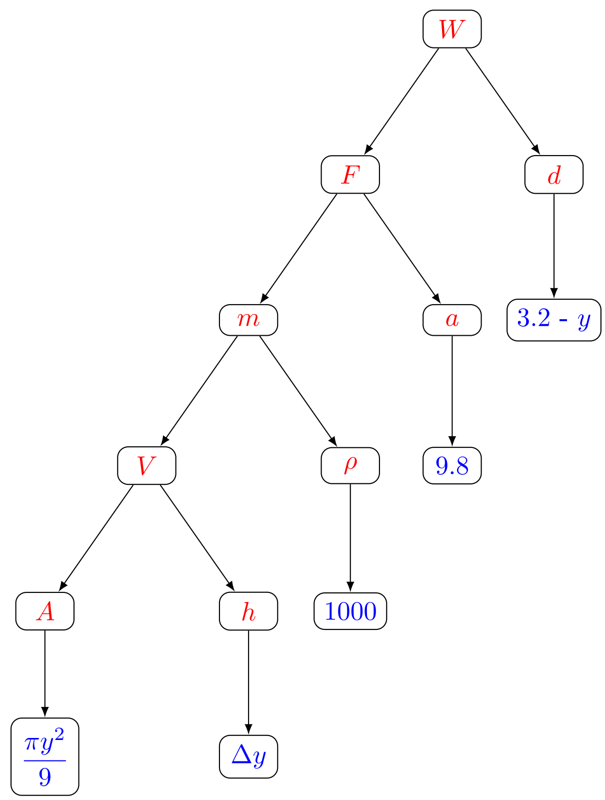
Summing over all slices, the total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1000 \cdot 9.8 \cdot \frac{\pi(y_i^*)^2}{9}\,(3.2 - y_i^*)\,\Delta y\right],$$

which is equivalent to the integral

$$W = \int_0^3 1000 \cdot 9.8 \cdot \frac{\pi y^2}{9}\,(3.2 - y)\,dy.$$

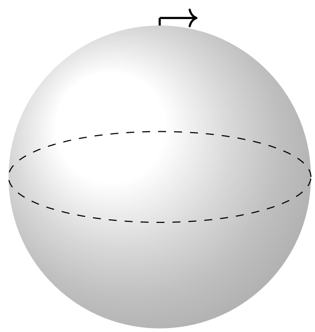
Tree Diagram Representation:



Thus, the work required to pump the water is

$$W = \int_0^3 1000 \cdot 9.8 \cdot \frac{\pi y^2}{9}\,(3.2 - y)\,dy.$$

4. **Spherical Tank**. A spherical tank of radius 2 m is completely filled with water. A spout is located 0.1 m above the top of the sphere.



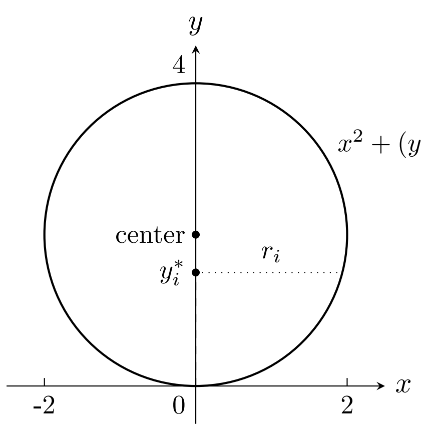
Step 1: Divide the Tank into Slices.

Define a vertical coordinate y with $y = 0$ at the bottom of the sphere, and $y = 4$ at the top. Partition the interval $[0, 4]$ into n subintervals of equal width Δy; in the ith subinterval, choose a representative point y_i^*.

At height y_i^*, the horizontal cross section of the sphere is a circle whose radius is given by

$$r_i = \sqrt{2^2 - (y_i^* - 2)^2} = \sqrt{4 - (y_i^* - 2)^2}.$$

To see this, consider the side view of the spherical tank:



Then at height y_i^*, the radius of the cross section is the corresponding x-coordinate on the graph of $x^2 + (y - 2)^2 = 4$. Plugging in y_i^* for y, we solve

$$r_i = x = \sqrt{4 - (y_i^* - 2)^2}$$

Thus, the area of the slice is

$$A_i = \pi \left(r_i\right)^2 = \pi \left[4 - (y_i^* - 2)^2\right].$$

The volume of the ith slice is then

$$V_i = A_i \, \Delta y = \pi \left[4 - (y_i^* - 2)^2\right] \Delta y.$$

Step 2: Compute the Work on Each Slice.

The weight (force) on the ith slice is

$$F_i = \rho \, g \, V_i = 1000 \cdot 9.8 \cdot \pi \left[4 - (y_i^* - 2)^2\right] \Delta y.$$

The spout is 0.1 m above the top of the sphere (top is at $y = 4$), so the spout is at $y = 4.1$. Hence, the lifting distance for the ith slice is

$$d_i = 4.1 - y_i^*.$$

Thus, the work done on the ith slice is

$$W_i = F_i \cdot d_i = 1000 \cdot 9.8 \cdot \pi \left[4 - (y_i^* - 2)^2\right] (4.1 - y_i^*) \, \Delta y.$$

Step 3: Write the Total Work as a Riemann Sum.

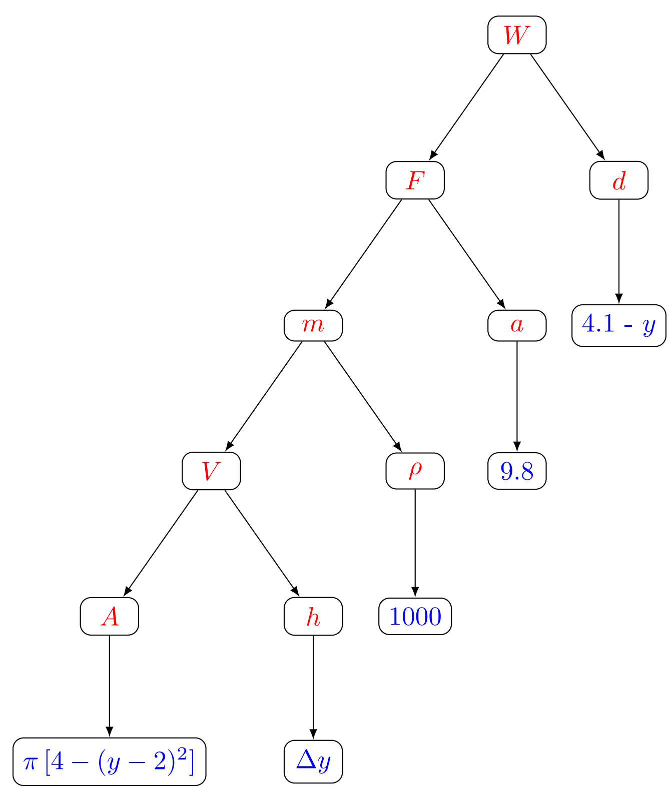
The total work required to pump the water is given by the Riemann sum

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left[1000 \cdot 9.8 \cdot \pi \left[4 - (y_i^* - 2)^2\right] (4.1 - y_i^*) \, \Delta y\right],$$

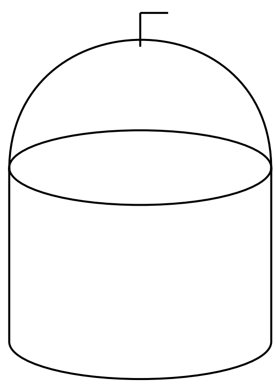
which is equivalent to the integral

$$W = \int_0^4 1000 \cdot 9.8 \cdot \pi \left[4 - (y - 2)^2\right] (4.1 - y) \, dy.$$

Tree Diagram Representation:



5. **Composite Tank – Cylinder with Hemispherical Top**. A tank consists of a cylindrical section 3 m high with a circular cross section of radius 1 m, topped by a hemispherical dome of radius 1 m. Water is pumped out through a spout located 0.15 m above the dome.



Step 1: Divide the Tank into Slices.

(a) Cylindrical Section (for $0 \leq y \leq 3$):

- Partition $[0, 3]$ into n subintervals of width Δy; choose a representative point y_i^* in each.
- The cross-sectional area is constant:

$$A_{\text{cyl}} = \pi(1)^2 = \pi.$$

- The volume of the ith slice is

$$V_i^{\text{cyl}} = A_{\text{cyl}}\,\Delta y = \pi\,\Delta y.$$

(b) Hemispherical Dome (for $3 \leq y \leq 4$):

- Partition $[3, 4]$ into subintervals of width Δy; choose a representative point y_i^* in each.
- The hemisphere is the upper half of a sphere of radius 1.
- At height y_i^*, the horizontal cross-sectional radius is

$$r_i = \sqrt{1 - (y_i^* - 3)^2},$$

(see the previous problem for an explanation), so the area is

$$A_i^{\text{hemi}} = \pi\,(r_i)^2 = \pi\Big[1 - (y_i^* - 3)^2\Big].$$

- The volume of the ith slice is then

$$V_i^{\text{hemi}} = A_i^{\text{hemi}}\,\Delta y.$$

Step 2: Compute the Work on Each Slice.

Let the density $\rho = 1000$ and gravitational acceleration $g = 9.8$.

(a) Cylindrical Section:

$$F_i^{\text{cyl}} = \rho\, g\, V_i^{\text{cyl}} = 1000 \cdot 9.8 \cdot \pi\,\Delta y.$$

Each slice must be lifted to the spout at $y = 4.15$; therefore, the lifting distance is

$$d_i^{\text{cyl}} = 4.15 - y_i^*.$$

Thus, the work on the ith cylindrical slice is

$$W_i^{\text{cyl}} = F_i^{\text{cyl}} \cdot d_i^{\text{cyl}} = 1000 \cdot 9.8 \cdot \pi\,(4.15 - y_i^*)\,\Delta y.$$

(b) Hemispherical Dome:

$$F_i^{\text{hemi}} = \rho\, g\, V_i^{\text{hemi}} = 1000 \cdot 9.8 \cdot \pi\Big[1 - (y_i^* - 3)^2\Big]\,\Delta y.$$

The lifting distance is the same:

$$d_i^{\text{hemi}} = 4.15 - y_i^*.$$

Thus, the work on the ith hemispherical slice is

$$\begin{aligned} W_i^{\text{hemi}} &= F_i^{\text{hemi}} \cdot d_i^{\text{hemi}} \\ &= 1000 \cdot 9.8 \cdot \pi\Big[1 - (y_i^* - 3)^2\Big]\,(4.15 - y_i^*)\,\Delta y. \end{aligned}$$

Step 3: Write the Total Work as a Riemann Sum.

The total work is the sum of the work on the cylindrical section and the hemispherical dome:

$$W = \lim_{n\to\infty} \left[\sum_{i=1}^{n^{\text{cyl}}} W_i^{\text{cyl}} + \sum_{i=1}^{n^{\text{hemi}}} W_i^{\text{hemi}}\right],$$

which is equivalent to

$$\begin{aligned} W = &\int_0^3 1000 \cdot 9.8 \cdot \pi\,(4.15 - y)\,dy \\ &+ \int_3^4 1000 \cdot 9.8 \cdot \pi\Big[1 - (y - 3)^2\Big]\,(4.15 - y)\,dy. \end{aligned}$$

Tree Diagram Representation: