6.4 Work

Definition. In physics, a **force** is defined as a push or pull on an object. For example, a horizontal push of a book across a table, or the downward pull of gravity on a ball. Mathematically, if an object moves along a straight line with position function $s(t)$, the force F acting on the object is given by Newton's Second Law of Motion:

$$F = ma = m\frac{d^2 s}{dt^2},$$

where m is the mass of the object, and a is the acceleration of the object.

Definition. **Work** is defined as the product of the force F acting on an object and the distance d the object moves:

$$W = Fd \quad \text{(work = force} \times \text{distance).}$$

- If F is measured in newtons (N) and d in meters (m), W is measured in *joules* (J):

$$1 \text{ J} = 1 \text{ N} \cdot \text{m}.$$

- If F is measured in pounds (lb) and d in feet (ft), W is measured in *foot-pounds* (ft-lb):

$$1 \text{ ft-lb} \approx 1.36 \text{ J}.$$

Example.

(a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground? ← Here, a force is given

(a) $W = F \cdot d$
$= (m \cdot g) \cdot d$
$= (1.2)(9.8)(0.7)$
$= 8.232$ J

(b) $W = F \cdot d$
$= 20 \cdot 6$
$= 120$ ft-lb

When the force is constant, work is simply the product of force and distance: $W = F \cdot d$. What if the force is not constant? How can we calculate the work done over an interval $[a, b]$?

- Approximate the work done over a small subinterval:
  - Divide $[a, b]$ into n subintervals of equal width Δx.
  - Let $x_0, x_1, \ldots, x_n$ be the endpoints of these subintervals.
  - Choose a sample point x_i^* in the ith subinterval $[x_{i-1}, x_i]$.
  - The force at x_i^* is $f(x_i^*)$, so the work W_i done over the interval $[x_{i-1}, x_i]$ is:

$$W_i \approx \text{force} \times \text{distance} = f(x_i^*) \cdot \Delta x$$

- Approximate the total work over the entire interval $[a, b]$.

$$W \approx \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{force}} \cdot \underbrace{\Delta x}_{\text{distance}}$$

- How can we calculate the exact total work for a variable force?

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x \underset{\text{def. of integral}}{=} \int_a^b f(x)\, dx$$

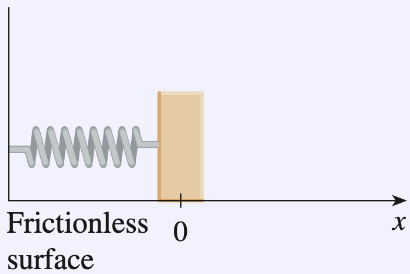
Example. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

$$W = \int_1^3 x^2 + 2x \, dx = \left[\frac{x^3}{3} + x^2 \right]_1^3 = \frac{50}{3} \text{ ft-lb}$$

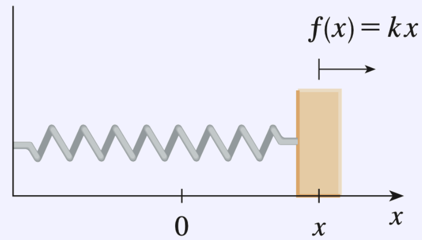
Theorem (Hooke's Law)**.** The force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant called the spring constant. Hooke's Law holds provided that x is not too large.



(a) Natural position of spring



(b) Stretched position of spring

Example. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

- The spring is stretched by $5\text{cm} = 0.05\,\text{m}$ from its natural length
- By Hooke's Law, $f(x) = kx$. Since $f(0.05) = 40$ N, we have

$$f(0.05) = k \cdot 0.05 \quad \Rightarrow \quad k = \frac{f(0.05)}{0.05} = \frac{40}{0.05} = 800$$

- Hence, $f(x) = 800x$.

- The interval of interest corresponds to stretching the spring from $15\text{ cm} = 0.15\text{m}$ to $18\text{cm} = 0.18\text{m}$. This means that x ranges from 0.05m to 0.08m (relative to resting position)

$$W = \int_{0.05}^{0.08} f(x)\,dx = \int_{0.05}^{0.08} 800x\,dx = 800\left[\frac{x^2}{2}\right]_{0.05}^{0.08} = 1.56\text{ J}$$

General Strategy for Solving Work Problems

1. Understand the Physical Setup
   - Identify the object/system involved (e.g., tank, spring, rope).
   - Determine the motion or action taking place (e.g., lifting, stretching, pumping).
   - Gather all relevant dimensions, forces, and physical constants (e.g., gravitational constant g, density of the material).
2. Define a Coordinate System:
   - Choose a coordinate system that simplifies the problem.
   - Specify the variable of integration (e.g., height x measured from a reference point).
   - Clearly define the bounds of the motion (e.g., from $x = a$ to $x = b$).
3. Divide the Object/System into Small Pieces:
   - Divide the system (e.g., water in a tank, a rope) into thin slices or segments.
   - Represent each segment by a point (e.g., x^*) to approximate force and distance.
   - Relate dimensions of the slices (e.g., radius or volume) to the integration variable.
4. Write an Expression for the Force on Each Segment:
   - Relate the force to the context of the problem:
     - For a spring: Use Hooke's Law, $f(x) = kx$.
     - For gravity: Use $f = mg$, where m depends on the density and volume.
   - Use any geometric relationships (e.g., similar triangles) to express quantities in terms of the integration variable.
5. Write an Expression for the Work on Each Segment:
   - Work for each segment is $W_i = F_i \cdot \text{distance}$.
   - Substitute the expressions for force and distance into the formula.
6. Sum the Work Over All Segments:
   - Approximate the total work as a Riemann sum:

$$W \approx \sum_{i=1}^{n} F(x_i^*) \cdot \text{distance}(x_i^*) \cdot \Delta x.$$

7. Convert the Sum to an Integral:
   - Take the limit as $n \to \infty$ to find the exact work:

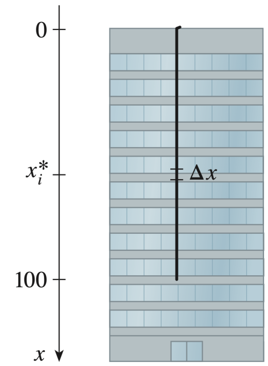
$$W = \int_a^b F(x) \cdot x \, dx.$$

   - Verify that the units (e.g., joules for work) are correct.

Example. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building.

(a) How much work is required to lift the cable to the top of the building?

(b) How much work is required to pull up only 20 feet of the cable?



Part (a):

1. Understand the Physical Setup
   - The cable hangs vertically from the top of a building.
   - The cable weighs 200 lb and is 100 ft long.
   - The weight per unit length is $\frac{200}{100} = 2$ lb/ft.
2. Define a Coordinate System

   Let x represent the distance from the top of the building

3. Divide the Cable into Small Pieces

   - Divide the cable into small parts, each of length Δx
   - The weight of each part is approximately $2\Delta x$ lb

4. Write an Expression for the Force on Each Segment

   $F_i = 2\Delta x$ (force = weight)

5. Write an Expression for the Work on Each Segment

Let x_i^* be a sample point in the ith subinterval representing the small segment

$$W_i = F_i \cdot x_i^* = 2x_i^* \Delta x$$

6. Sum the Work Over All Segments

def. of integral

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^* \Delta x = \int_0^{100} 2x \, dx$$

7. Simplify and Solve the Integral

$$W = \int_0^{100} 2x \, dx = \left[x^2\right]_0^{100} = 10{,}000 \text{ ft-lb}$$

Part (b):

1. The work required to move the top 20 ft of cable to the top of the building is computed in the same way as in part (a):

$$W_1 = \int_0^{20} 2x \, dx = \left[x^2\right]_0^{20} = 400 \text{ ft-lb}$$

2. The work required to lift the remaining 80 feet of cable by 20 feet is:

The lower 80ft of the cable weighs $80 \cdot 2 = 160$ lb

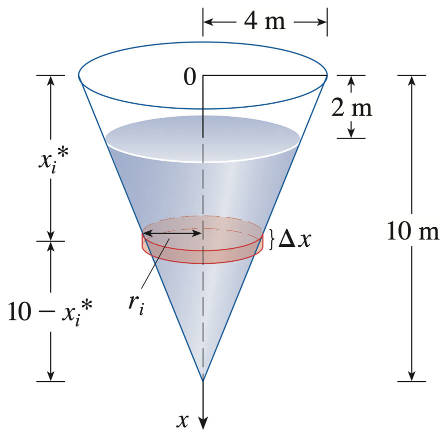
This portion is lifted uniformly by 20 ft, so

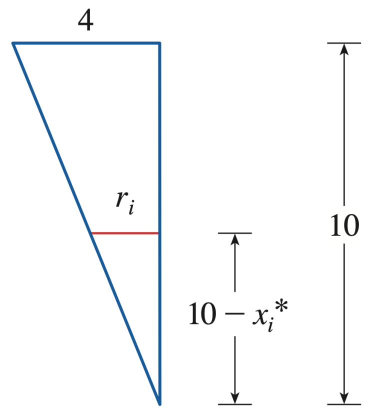
$$W_2 = 160 \cdot 20 = 3200 \text{ ft-lb}$$

3. The total work is:

$$W_{total} = W_1 + W_2 = 400 + 3200 = 3600 \text{ ft-lb}$$

Example. A tank has the shape of an inverted circular cone with a height of 10 m and a base radius of 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)





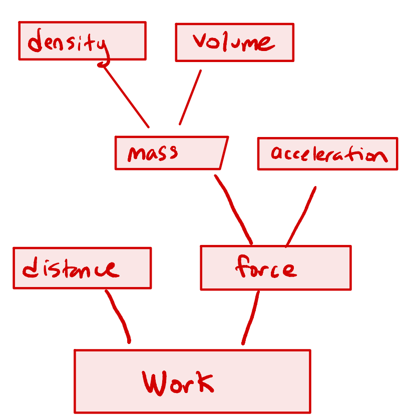
1. Understand the Physical Setup
    - The tank is an inverted circular cone.
    - The height of the tank is 10 m, and the base radius is 4 m.
    - The tank is filled with water up to a height of 8 m.
    - Water must be pumped to the top of the tank.
    - The density of water is 1000 kg/m^3, and the acceleration due to gravity is $g = 9.8$ m/s^2.
2. Define a Coordinate System

    Measure depth from the top of the tank, where $x = 0$

    The water lies between depths of $x = 2$ and $x = 10$

3. Divide the Object/System into Small Pieces

    - Divide $[2, 10]$ into subintervals so that the water is divided into thin horizontal layers of thickness Δx
    - Approximate the volume of each layer by $V_i = \pi r_i^2 \Delta x$
    - Choose sample points x_i^* in $[x_{i-1}, x_i]$ to represent the depth of the layer
    - By similar triangles, $\dfrac{r_i}{10 - x_i^*} = \dfrac{4}{10} \Rightarrow r_i = \frac{2}{5}(10 - x_i^*)$



• The volume of a thin layer is

$$V_i = \pi r_i^2 \Delta x = \pi \left(\frac{2}{5}(10-x_i^*)\right)^2 \Delta x = \frac{4\pi}{25}(10-x_i^*)^2 \Delta x$$

• The mass of a thin layer is

$$m_i = \text{density} \times \text{volume} = 1000 \cdot \frac{4\pi}{25}(10-x_i^*)^2 \Delta x = 160\pi (10-x_i^*)^2 \Delta x$$

4. Write an Expression for the Force on Each Layer ← i.e. the force required to lift each layer

9.8 m/s²

$$F_i = m_i \cdot g = 1568\pi (10-x_i^*)^2 \Delta x$$

5. Write an Expression for the Work on Each Layer

Each layer must be lifted a distance of x_i^* to the top.

$$W_i = F_i \cdot x_i^* = 1568\pi x_i^* (10-x_i^*)^2 \Delta x$$

6. Sum the Work Over All Layers

def. of integral

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 1568\pi x_i^* (10-x_i^*)^2 \Delta x = \int_2^{10} 1568\pi x (10-x)^2 dx$$

7. Simplify and Solve the Integral

$$W = 1568\pi \int_2^{10} 100x - 20x^2 + x^3 \, dx$$

$$= 1568\pi \left[50x^2 - \frac{20}{3}x^3 + \frac{1}{4}x^4 \right]_2^{10}$$

$$= 1568\pi \cdot \frac{20480}{3}$$

$$\approx 3.4 \times 10^6 \text{ J}$$