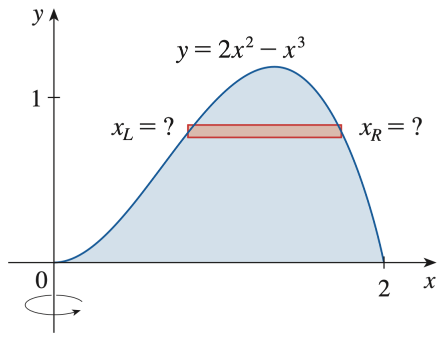
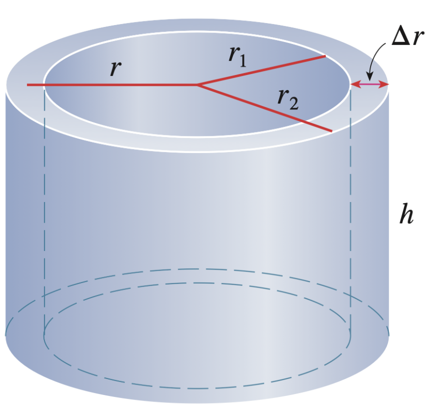
6.3 Volumes by Cylindrical Shells

Question. How can we find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$?



- If we slice perpendicular to the y-axis (as in 6.2) we get washers
- Computing the inner radius and outer radius would require us to solve $y = 2x^2 - x^3$ for x in terms of y. ← Not easy!
- Solution: use cylindrical shells

Question. Below is a cylindrical shell with inner radius r_1, outer radius r_2, and height h. What is the volume of the shell?



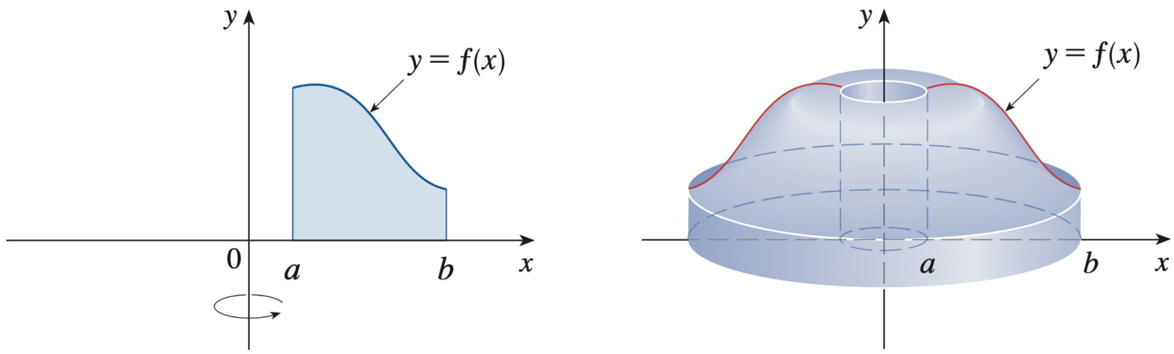
$V =$ circumference × height × thickness

$= 2\pi r \cdot h \cdot \Delta r$

$r = \frac{1}{2}(r_1 + r_2)$ is the average radius

$\Delta r = r_2 - r_1$ is thickness of the shell

Theorem. Let $f(x) \geq 0$, and let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$, where $0 \leq a < b$.



Show that the volume of S is $V = \int_a^b 2\pi x f(x)\,dx$.

- Divide $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx
- Let $\bar{x}_i$ be the midpoint of the ith subinterval
- The rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ produces a cylindrical shell with volume:

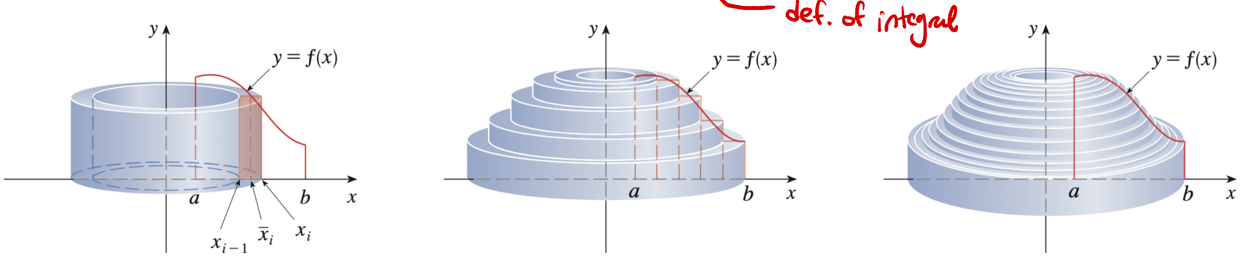
$$V_i = \text{circumference} \times \text{height} \times \text{thickness}$$
$$= 2\pi \bar{x}_i \cdot f(\bar{x}_i) \cdot \Delta x$$

- The volume of S is approximately $V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \bar{x}_i \cdot f(\bar{x}_i) \cdot \Delta x$
- As $n \to \infty$, the number of shells goes to ∞, and

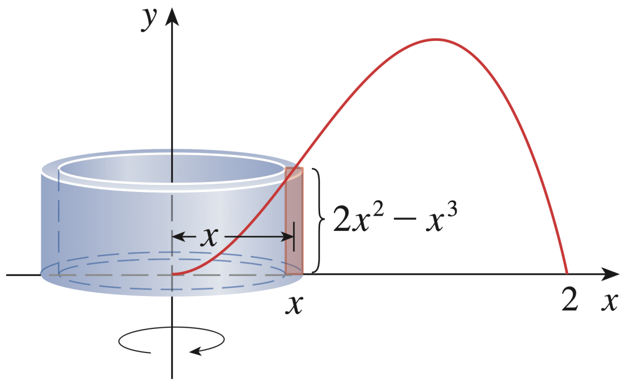
circumference, height, thickness

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i \cdot f(\bar{x}_i) \cdot \Delta x = \int_a^b 2\pi x \cdot f(x) \cdot dx$$

def. of integral



Example. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



Dimensions of the shell:

Radius : x

Circumference : $2\pi x$

Height : $f(x) = 2x^2 - x^3$

$$V = \int_0^2 \underbrace{2\pi x}_{\text{circumference}} \cdot \underbrace{(2x^2 - x^3)}_{\text{height}} \cdot \underbrace{dx}_{\text{thickness}}$$

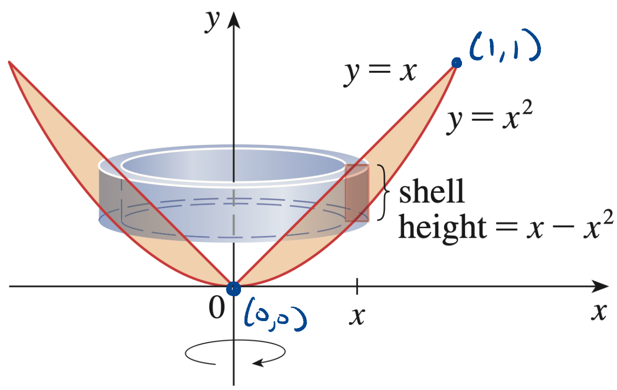
$$= 2\pi \int_0^2 (2x^3 - x^4)\, dx$$

$$= 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5 \right]_0^2$$

$$= 2\pi \left[8 - \frac{32}{5} \right]$$

$$= \frac{16\pi}{5}$$

Example. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



Dimensions of the shell:

Circumference: $2\pi x$

Height: $x - x^2$

$$V = \int_0^1 2\pi x \cdot (x - x^2)\, dx$$

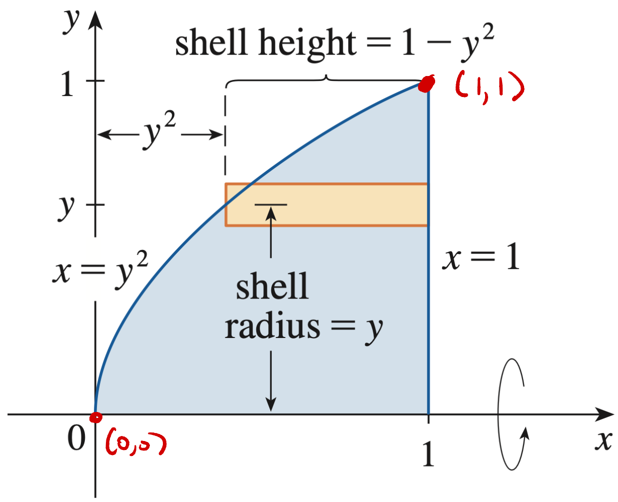
$$= 2\pi \int_0^1 x^2 - x^3\, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{\pi}{6}$$

Example. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

* We did this in §6.2 using disks, which is probably easier



Dimensions of shell:

Circumference: $2\pi y$

Height : $1-y^2$

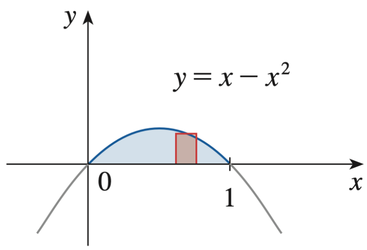
$$V = \int_0^1 2\pi y \cdot (1-y^2)\, dy$$

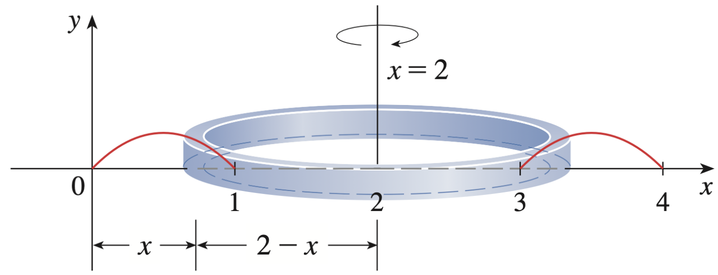
$$= 2\pi \int_0^1 y - y^3\, dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1$$

$$= \frac{\pi}{2}$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.





Dimensions of the shell:

Radius: $2 - x$

Circumference: $2\pi(2-x)$

Height: $x - x^2$

$$V = \int_0^1 2\pi(2-x)\cdot(x - x^2)\,dx$$

$$= 2\pi \int_0^1 x^3 - 3x^2 + 2x\,dx$$

$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1$$

$$= \frac{\pi}{2}$$

Comparison of Washer and Shell Methods

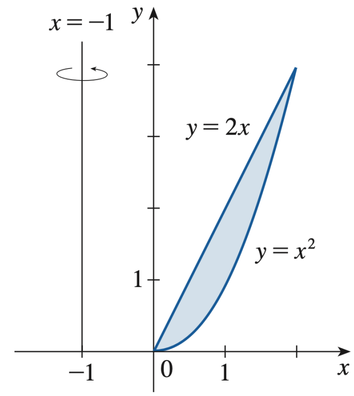
Aspect	Washer Method	Shell Method
Definition	Used when the volume is computed by revolving a region and forming disks or washers (hollow disks) perpendicular to the axis of rotation.	Used when the volume is computed by revolving a region and forming cylindrical shells parallel to the axis of rotation.
Integration Variable	Typically x or y, corresponding to the axis perpendicular to the cross-section of the washer.	Typically x or y, corresponding to the height of the shell parallel to the axis of rotation.
Region Description	The region is described in terms of top and bottom boundaries (y_T and y_B or x_R and x_L).	The region is described in terms of radii and heights of cylindrical shells.
Formula	$V = \pi \int_a^b \left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] dx$ (or dy).	$V = 2\pi \int_a^b (\text{Radius}) \times (\text{Height})\, dx$ (or dy).
Axis of Rotation	Typically suited for regions rotated around the x-axis or y-axis.	Typically suited for regions rotated around vertical or horizontal lines other than the x-axis or y-axis.
Cross-Section Shape	Perpendicular slices of the region form disks or washers.	Parallel slices of the region form cylindrical shells.
Visual Representation	Imagine cutting the solid into thin disks or washers perpendicular to the axis of rotation.	Imagine peeling the solid into thin cylindrical shells by cutting parallel to the axis of rotation.
Advantages	Simple to apply for solids symmetric about the axis of rotation.	Useful for solving problems where the washer method requires splitting into multiple integrals.
Disadvantages	May require splitting into multiple integrals if the region changes structure along the axis of rotation.	May involve slightly more complex visualizations and formulas.

Example. Consider the region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x$. A solid is formed by rotating the region about the line $x = -1$.

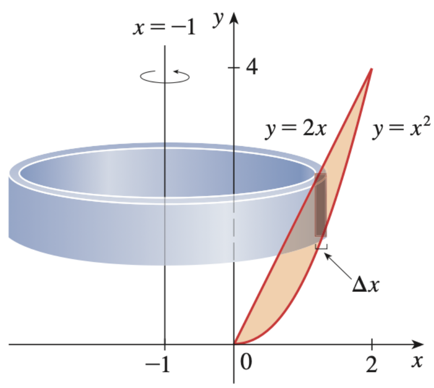
Find the volume of the solid using:

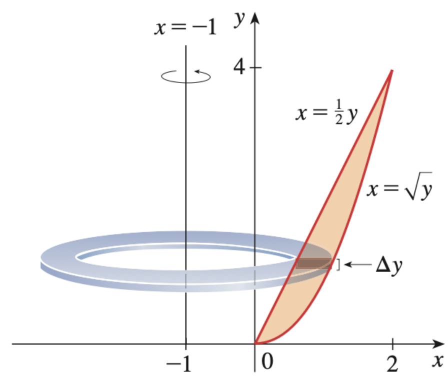
(a) x as the variable of integration.

(b) y as the variable of integration.



Solution:





- To use x, draw a vertical rectangle with base Δx
- Rectangle is parallel to axis of rotation, so we get cylindrical shells.

$$V = \int_0^2 \underbrace{2\pi(x+1)}_{\text{circumference}} \underbrace{(2x-x^2)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi \int_0^2 x^2 + 2x - x^3 \, dx$$

$$= 2\pi \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4} \right]_0^2$$

$$= \frac{16\pi}{3}$$

- To use y, draw a horizontal rectangle with base Δy
- Rectangle is perpendicular to axis of rotation so we get washers

$$V = \int_0^4 \pi \underbrace{(\sqrt{y}+1)}_{\text{outer radius}}{}^2 - \pi \underbrace{\left(\tfrac{1}{2}y+1\right)}_{\text{inner radius}}{}^2 \, dy$$

$$= \pi \int_0^4 2\sqrt{y} - \frac{1}{4}y^2 \, dy$$

$$= \pi \left[\frac{4}{3} y^{3/2} - \frac{1}{12} y^3 \right]_0^4$$

$$= \frac{16\pi}{3}$$