

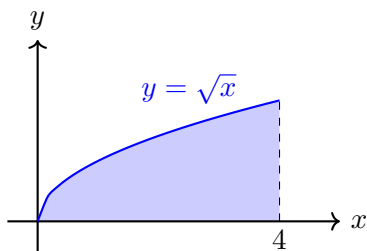
6.2-6.3 Solids of Revolution (Solutions)

6.2-6.3 Disk Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x -axis.

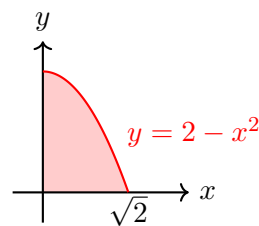


$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{x})^2 dx \\ &= \pi \int_0^4 x dx \\ &= \pi \left[\frac{x^2}{2} \right]_0^4 \\ &= \pi \left(\frac{16}{2} \right) \\ &= 8\pi. \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x^2, \quad y = 0, \quad x \in [0, \sqrt{2}]$$

about the x -axis.

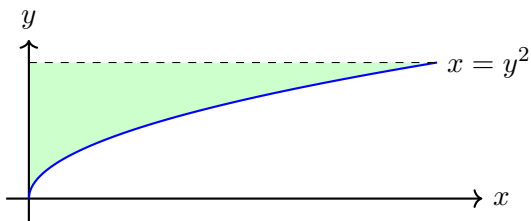


$$\begin{aligned} V &= \pi \int_0^{\sqrt{2}} (2 - x^2)^2 dx \\ &= \pi \int_0^{\sqrt{2}} (4 - 4x^2 + x^4) dx \\ &= \pi \left[4x - \frac{4x^3}{3} + \frac{x^5}{5} \right]_0^{\sqrt{2}} \\ &= \pi \left(4\sqrt{2} - \frac{4(\sqrt{2})^3}{3} + \frac{(\sqrt{2})^5}{5} \right) \\ &= \pi\sqrt{2} \left(4 - \frac{8}{3} + \frac{4}{5} \right) \\ &= \pi\sqrt{2} \left(\frac{60 - 40 + 12}{15} \right) \\ &= \pi\sqrt{2} \cdot \frac{32}{15} \\ &= \frac{32\sqrt{2}\pi}{15}. \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 0, \quad y = 3$$

about the y -axis.

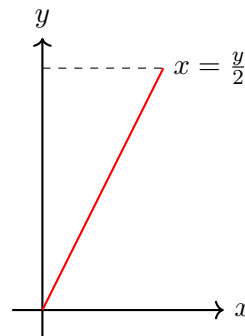


$$\begin{aligned} V &= \pi \int_0^3 (y^2)^2 dy \\ &= \pi \int_0^3 y^4 dy \\ &= \pi \left[\frac{y^5}{5} \right]_0^3 \\ &= \pi \cdot \frac{3^5}{5} \\ &= \frac{243\pi}{5}. \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$x = \frac{y}{2}, \quad x = 0, \quad y = 4$$

about the y -axis.

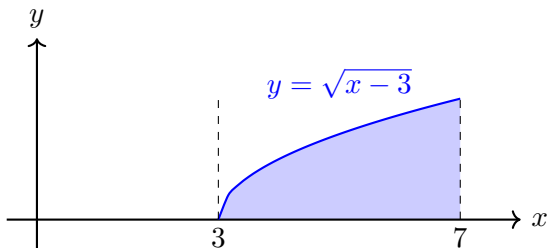


$$\begin{aligned} V &= \pi \int_0^4 \left(\frac{y}{2} \right)^2 dy \\ &= \pi \int_0^4 \frac{y^2}{4} dy \\ &= \frac{\pi}{4} \int_0^4 y^2 dy \\ &= \frac{\pi}{4} \left[\frac{y^3}{3} \right]_0^4 \\ &= \frac{\pi}{4} \cdot \frac{64}{3} \\ &= \frac{16\pi}{3}. \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x-3}, \quad y = 0, \quad x = 7$$

about the vertical line $x = 7$.



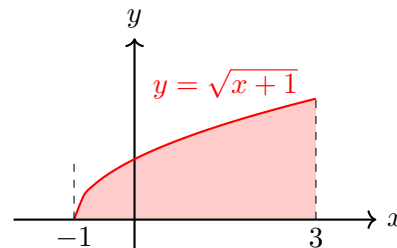
First, rewrite $y = \sqrt{x-3}$ as $x = y^2 + 3$. When $x = 7$, $y = \sqrt{7-3} = 2$, so using horizontal slices:

$$\begin{aligned} V &= \pi \int_0^2 [(7 - (y^2 + 3))^2] dy \\ &= \pi \int_0^2 (4 - y^2)^2 dy \\ &= \pi \int_0^2 16 - 8y^2 + y^4 dy \\ &= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\ &= \pi \cdot \left[32 - \frac{64}{3} + \frac{32}{5} \right] \\ &= \frac{256\pi}{15}. \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x+1}, \quad y = 0, \quad x = 3$$

about the vertical line $x = 3$.



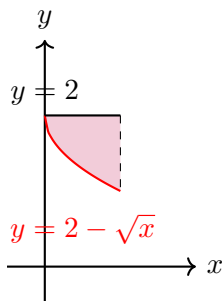
Rewrite $y = \sqrt{x+1}$ as $x = y^2 - 1$. When $x = 3$, $y = \sqrt{3+1} = 2$. Then,

$$\begin{aligned} V &= \pi \int_0^2 [(3 - (y^2 - 1))^2] dy \\ &= \pi \int_0^2 (4 - y^2)^2 dy \\ &= \pi \int_0^2 16 - 8y^2 + y^4 dy \\ &= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\ &= \pi \cdot \left[32 - \frac{64}{3} + \frac{32}{5} \right] \\ &= \frac{256\pi}{15}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - \sqrt{x}, \quad y = 2, \quad x = 1$$

about the horizontal line $y = 2$.

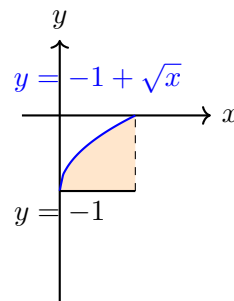


$$\begin{aligned} V &= \pi \int_0^1 [2 - (2 - \sqrt{x})]^2 dx \\ &= \pi \int_0^1 (\sqrt{x})^2 dx \\ &= \pi \int_0^1 x dx \\ &= \pi \left[\frac{x^2}{2} \right]_0^1 \\ &= \frac{\pi}{2} \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = -1 + \sqrt{x}, \quad y = -1, \quad x = 1$$

about the horizontal line $y = -1$.



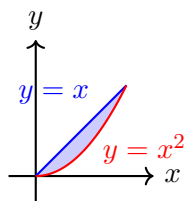
$$\begin{aligned} V &= \pi \int_0^1 [(-1 + \sqrt{x}) - (-1)]^2 dx \\ &= \pi \int_0^1 (\sqrt{x})^2 dx \\ &= \pi \int_0^1 x dx \\ &= \pi \left[\frac{x^2}{2} \right]_0^1 \\ &= \frac{\pi}{2} \end{aligned}$$

6.2-6.3 Washer Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = x^2$$

about the x -axis.

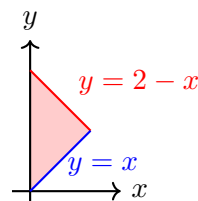


$$\begin{aligned} V &= \pi \int_0^1 [(R(x))^2 - (r(x))^2] dx \\ &= \pi \int_0^1 [(x)^2 - (x^2)^2] dx \\ &= \pi \int_0^1 [x^2 - x^4] dx \\ &= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \pi \left(\frac{5-3}{15} \right) \\ &= \frac{2\pi}{15} \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = x, \quad x = 0$$

about the x -axis.

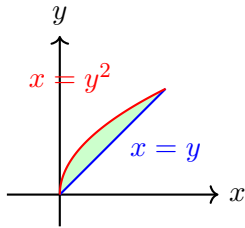


$$\begin{aligned} V &= \pi \int_0^1 [(R(x))^2 - (r(x))^2] dx \\ &= \pi \int_0^1 [(2-x)^2 - (x)^2] dx \\ &= \pi \int_0^1 [(4-4x+x^2) - x^2] dx \\ &= \pi \int_0^1 (4-4x) dx \\ &= \pi [4x - 2x^2]_0^1 \\ &= \pi(4-2) \\ &= 2\pi \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$x = y, \quad x = y^2$$

about the y -axis.

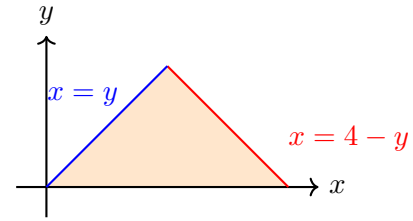


$$\begin{aligned} V &= \pi \int_0^1 \left[(R(y))^2 - (r(y))^2 \right] dy \\ &= \pi \int_0^1 \left[(y)^2 - (y^2)^2 \right] dy \\ &= \pi \int_0^1 \left[y^2 - y^4 \right] dy \\ &= \pi \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{5} \right) \\ &= \frac{2\pi}{15} \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \quad y = 4 - x, \quad \text{and} \quad y = 0$$

about the y -axis.

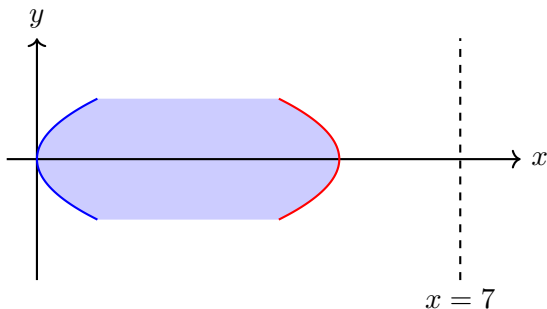


$$\begin{aligned} V &= \pi \int_0^2 \left[(R(y))^2 - (r(y))^2 \right] dy \\ &= \pi \int_0^2 \left[((4 - y))^2 - (y)^2 \right] dy \\ &= \pi \int_0^2 \left[(16 - 8y + y^2) - y^2 \right] dy \\ &= \pi \int_0^2 (16 - 8y) dy \\ &= \pi \left[16y - 4y^2 \right]_0^2 \\ &= \pi (32 - 16) \\ &= 16\pi \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 5 - y^2, \quad y = -1, \quad \text{and} \quad y = 1$$

about the vertical line $x = 7$.



For a given y , the distances from the vertical line $x = 7$ to the curves are:

$$\begin{aligned} r(y) &= 7 - (5 - y^2) = 2 + y^2 \\ R(y) &= 7 - y^2 \end{aligned}$$

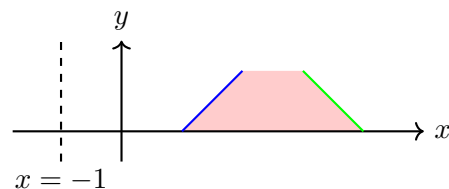
Thus, the volume is

$$\begin{aligned} V &= \pi \int_{-1}^1 \left[(7 - y^2)^2 - (2 + y^2)^2 \right] dy \\ &= \pi \int_{-1}^1 \left[(49 - 14y^2 + y^4) - (4 + 4y^2 + y^4) \right] dy \\ &= \pi \int_{-1}^1 (45 - 18y^2) dy \\ &= \pi \left[45y - 18 \left(\frac{y^3}{3} \right) \right]_{-1}^1 \\ &= \pi \left[45y - 6y^3 \right]_{-1}^1 \\ &= \pi \left(\left[45(1) - 6(1)^3 \right] - \left[45(-1) - 6(-1)^3 \right] \right) \\ &= \pi \left((45 - 6) - (-45 + 6) \right) \\ &= \pi (39 + 39) \\ &= 78\pi \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x - 1, \quad y = 4 - x, \quad y = 0, \quad \text{and} \quad y = 1$$

about the vertical line $x = -1$.



For a fixed y with $0 \leq y \leq 1$, the region extends horizontally from

$$x = y + 1 \quad \text{to} \quad x = 4 - y.$$

Since the axis of rotation is $x = -1$, the distances from $x = -1$ to these curves are:

$$\begin{aligned} r(y) &= (y + 1) - (-1) = y + 2 \\ R(y) &= (4 - y) - (-1) = 5 - y. \end{aligned}$$

Thus, the cross-sectional area (washer) is:

$$A(y) = \pi \left[(5 - y)^2 - (y + 2)^2 \right].$$

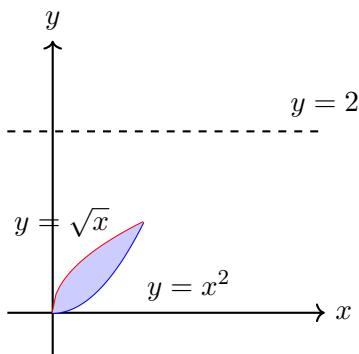
Hence, the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left[(5 - y)^2 - (y + 2)^2 \right] dy \\ &= \pi \int_0^1 \left[(25 - 10y + y^2) - (y^2 + 4y + 4) \right] dy \\ &= \pi \int_0^1 \left[25 - 10y + y^2 - y^2 - 4y - 4 \right] dy \\ &= \pi \int_0^1 (21 - 14y) dy \\ &= \pi \left[21y - \frac{14y^2}{2} \right]_0^1 \\ &= \pi \left[21y - 7y^2 \right]_0^1 \\ &= \pi \left((21 - 7) - 0 \right) \\ &= 14\pi \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the horizontal line $y = 2$.



Using the washer method with vertical slices, note that the region in x runs from $x = 0$ to $x = 1$. For a fixed x in this interval, the y -values range from $y = x^2$ (lower curve) to $y = \sqrt{x}$ (upper curve).

When this slice is rotated about the horizontal line $y = 2$, the distances from $y = 2$ to the curves are:

$$\begin{aligned} \text{Outer radius: } R(x) &= 2 - x^2 \\ \text{Inner radius: } r(x) &= 2 - \sqrt{x}. \end{aligned}$$

Thus, the cross-sectional area is:

$$A(x) = \pi \left[(2 - x^2)^2 - (2 - \sqrt{x})^2 \right].$$

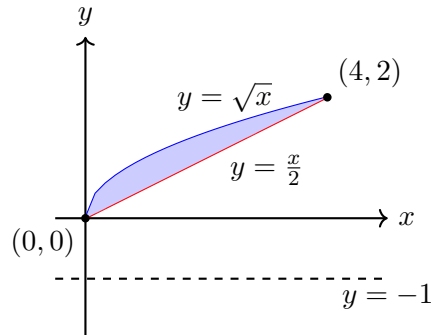
Hence, the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left[(2 - x^2)^2 - (2 - \sqrt{x})^2 \right] dx \\ &= \pi \int_0^1 \left[(4 - 4x^2 + x^4) - (4 - 4\sqrt{x} + x) \right] dx \\ &= \pi \int_0^1 \left(-4x^2 + x^4 + 4\sqrt{x} - x \right) dx \\ &= \pi \left[-\frac{4x^3}{3} + \frac{x^5}{5} + \frac{8x^{3/2}}{3} - \frac{x^2}{2} \right]_0^1 \\ &= \pi \left(-\frac{4}{3} + \frac{1}{5} + \frac{8}{3} - \frac{1}{2} \right) \\ &= \frac{31\pi}{30}. \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -1$.



For a fixed x in $[0, 4]$, the region extends vertically from

$$y = \frac{x}{2} \quad \text{to} \quad y = \sqrt{x}.$$

When this slice is rotated about $y = -1$, it forms a washer with

$$\begin{aligned} \text{Outer radius: } R(x) &= \sqrt{x} - (-1) = \sqrt{x} + 1 \\ \text{Inner radius: } r(x) &= \frac{x}{2} - (-1) = \frac{x}{2} + 1. \end{aligned}$$

Thus, the cross-sectional area is

$$A(x) = \pi \left[(\sqrt{x} + 1)^2 - \left(\frac{x}{2} + 1 \right)^2 \right].$$

Hence, the volume is

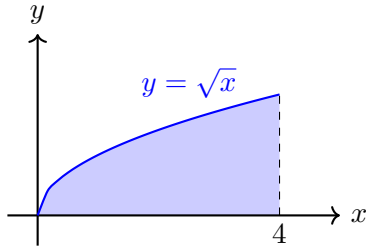
$$\begin{aligned} V &= \pi \int_0^4 \left[(\sqrt{x} + 1)^2 - \left(\frac{x}{2} + 1 \right)^2 \right] dx \\ &= \pi \int_0^4 \left[(x + 2\sqrt{x} + 1) - \left(\frac{x^2}{4} + x + 1 \right) \right] dx \\ &= \pi \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx \\ &= \pi \left[\frac{4}{3} x^{3/2} - \frac{x^3}{12} \right]_0^4 \\ &= \pi \left[\frac{4}{3} (4^{3/2}) - \frac{4^3}{12} \right] \\ &= \pi \left[\frac{32}{3} - \frac{16}{3} \right] \\ &= \frac{16\pi}{3}. \end{aligned}$$

6.2-6.3 Cylindrical Shells Method (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 4$$

about the x -axis.



Using horizontal shells (with y as the variable), note that:

- Express $y = \sqrt{x}$ as $x = y^2$.
- For a given horizontal slice at height y , the shell extends from $x = y^2$ to $x = 4$. Thus, the *height* is $h(y) = 4 - y^2$.
- The *radius* of the shell is the distance from y to the x -axis, i.e. $r(y) = y$.

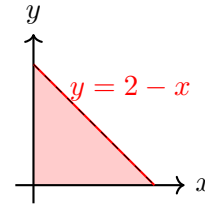
Then,

$$\begin{aligned} V &= 2\pi \int_0^2 (\text{radius}) \cdot (\text{height}) \, dy \\ &= 2\pi \int_0^2 y(4 - y^2) \, dy \\ &= 2\pi \left[\int_0^2 (4y - y^3) \, dy \right] \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi [8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x, \quad y = 0, \quad x \in [0, 2]$$

about the x -axis.



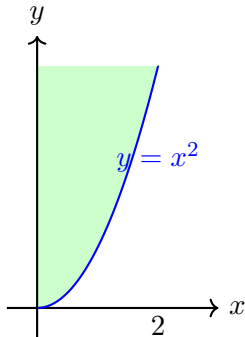
Rewriting $y = 2 - x$ as $x = 2 - y$, the region in y is from $y = 0$ to $y = 2$ with horizontal length $h(y) = 2 - y$ and radius $r(y) = y$. Then,

$$\begin{aligned} V &= 2\pi \int_0^2 y(2 - y) \, dy \\ &= 2\pi \int_0^2 (2y - y^2) \, dy \\ &= 2\pi \left[y^2 - \frac{y^3}{3} \right]_0^2 \\ &= 2\pi \left[4 - \frac{8}{3} \right] \\ &= 2\pi \left(\frac{12 - 8}{3} \right) \\ &= 2\pi \left(\frac{4}{3} \right) \\ &= \frac{8\pi}{3} \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 4, \quad x \in [0, 2]$$

about the y -axis.



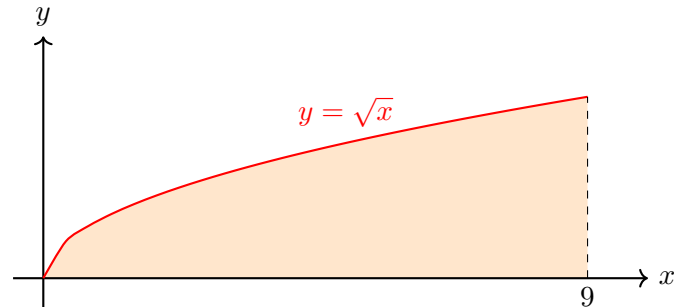
Using vertical shells, the radius is $r(x) = x$ and the height is $h(x) = 4 - x^2$. Thus,

$$\begin{aligned} V &= 2\pi \int_0^2 x(4 - x^2) dx \\ &= 2\pi \left[\int_0^2 (4x - x^3) dx \right] \\ &= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi [8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 9$$

about the y -axis.



For vertical shells, the radius is $r(x) = x$ and the height is $h(x) = \sqrt{x}$. Then,

$$\begin{aligned} V &= 2\pi \int_0^9 x \cdot \sqrt{x} dx \\ &= 2\pi \int_0^9 x^{3/2} dx \\ &= 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^9 \\ &= 2\pi \cdot \frac{2}{5} \cdot 9^{5/2} \\ &= \frac{4\pi}{5} \cdot (9^{5/2}). \end{aligned}$$

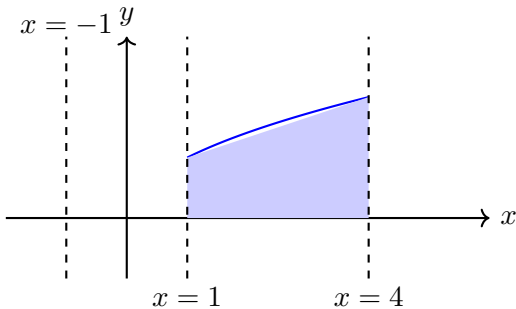
Since $9^{5/2} = (\sqrt{9})^5 = 3^5 = 243$, we have

$$V = \frac{4\pi}{5} \cdot 243 = \frac{972\pi}{5}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the vertical line $x = -1$.



Since the region is bounded vertically by $y = 0$ and $y = \sqrt{x}$ and horizontally by $x = 1$ and $x = 4$, we use the *shell method* with vertical slices. For a typical slice at position x , the height is

$$h(x) = \sqrt{x} - 0 = \sqrt{x},$$

and the radius (distance from x to the axis $x = -1$) is

$$r(x) = x - (-1) = x + 1.$$

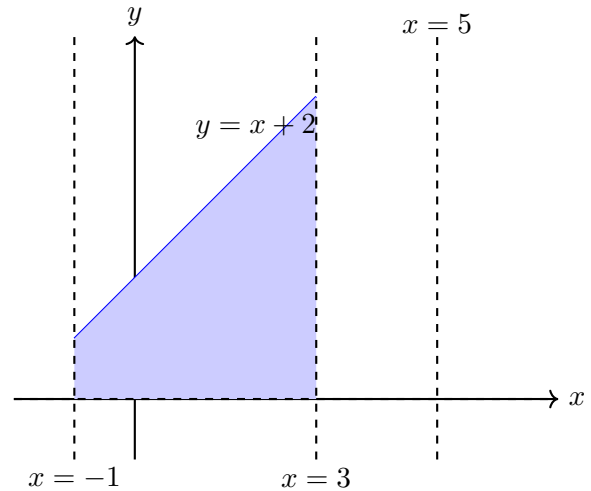
Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 r(x) h(x) dx \\ &= 2\pi \int_1^4 (x+1)\sqrt{x} dx \\ &= 2\pi \int_1^4 (x^{3/2} + x^{1/2}) dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} \right]_{x=1}^4 \\ &= 2\pi \left\{ \left[\frac{2}{5}(4^{5/2}) + \frac{2}{3}(4^{3/2}) \right] - \left[\frac{2}{5}(1^{5/2}) + \frac{2}{3}(1^{3/2}) \right] \right\} \\ &= \frac{512\pi}{15}. \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x + 2, \quad y = 0, \quad x = -1, \quad x = 3$$

about the vertical line $x = 5$.



Since the region is defined by $-1 \leq x \leq 3$ and $0 \leq y \leq x+2$, we use the *cylindrical shells* method. A typical vertical slice at x has:

$$\text{Height: } h(x) = x + 2,$$

and its distance from the axis $x = 5$ is

$$\text{Radius: } r(x) = 5 - x.$$

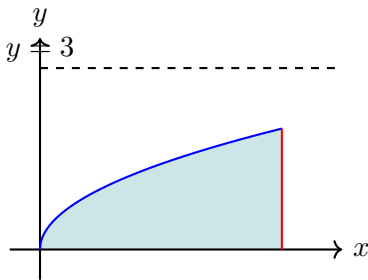
Thus, the volume is given by

$$\begin{aligned} V &= 2\pi \int_{-1}^3 r(x) h(x) dx \\ &= 2\pi \int_{-1}^3 (5-x)(x+2) dx \\ &= 2\pi \int_{-1}^3 [5(x+2) - x(x+2)] dx \\ &= 2\pi \int_{-1}^3 [5x + 10 - x^2 - 2x] dx \\ &= 2\pi \int_{-1}^3 [-x^2 + 3x + 10] dx \\ &= 2\pi \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 10x \right]_{x=-1}^3 \\ &= \frac{256\pi}{3}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4, \quad y = 0$$

about the horizontal line $y = 3$.



For a horizontal slice at y (with $0 \leq y \leq 2$):

- The *radius* is the distance from y to the line $y = 3$: $r(y) = 3 - y$.
- The *height* is the horizontal length: $h(y) = 4 - x$ where x runs from $x = y^2$ to $x = 4$; thus, $h(y) = 4 - y^2$.

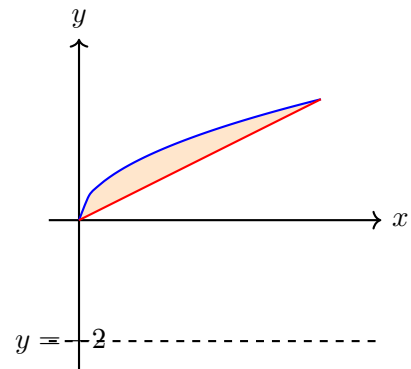
Then,

$$\begin{aligned} V &= 2\pi \int_{y=0}^2 (3 - y)(4 - y^2) dy \\ &= 2\pi \int_0^2 [12 - 3y^2 - 4y + y^3] dy \\ &= 2\pi \left[12y - \frac{3y^3}{3} - 2y^2 + \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left[12(2) - y^3|_2 - 2(2)^2 + \frac{(2)^4}{4} \right] \\ &= 2\pi [24 - 8 - 8 + 4] \\ &= 2\pi(12) \\ &= 24\pi \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2}$$

about the horizontal line $y = -2$.



Using horizontal shells, for a slice at height y (with y from the intersection of the curves, $0 \leq y \leq 2$):

- The *radius* is the distance from y to $y = -2$: $r(y) = y - (-2) = y + 2$.
- The *height* is the horizontal length between the curves: $h(y) = 2y - y^2$.

Then,

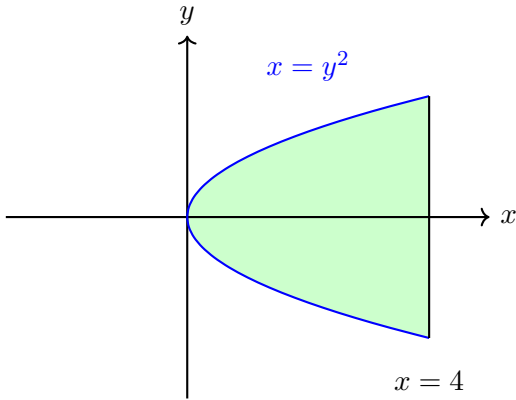
$$\begin{aligned} V &= 2\pi \int_{y=0}^2 (y + 2) [2y - y^2] dy \\ &= 2\pi \int_0^2 [(y + 2)(2y - y^2)] dy \\ &= 2\pi \int_0^2 [2y^2 - y^3 + 4y - 2y^2] dy \\ &= 2\pi \int_0^2 (4y - y^3) dy \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left[2(4) - \frac{16}{4} \right] \\ &= 2\pi [8 - 4] \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

6.2-6.3 Additional Problems (Solutions)

1. Find the volume of the solid obtained by rotating the region bounded by

$$x = y^2, \quad x = 4$$

about the y -axis.



Solution via Washer Method:

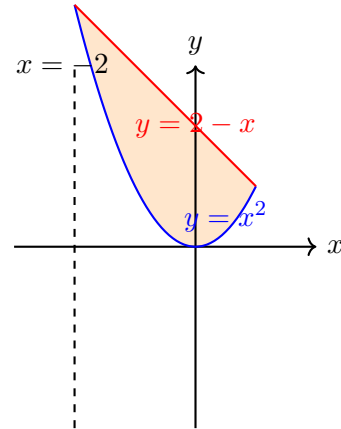
Using the Washer Method with y -limits $y = -2$ to 2 , outer radius $R(y) = 4$ and inner radius $r(y) = y^2$:

$$\begin{aligned} V &= \pi \int_{-2}^2 [4^2 - (y^2)^2] dy = \pi \int_{-2}^2 (16 - y^4) dy \\ &= 2\pi \int_0^2 (16 - y^4) dy = 2\pi \left[16y - \frac{y^5}{5} \right]_0^2 \\ &= 2\pi \left(32 - \frac{32}{5} \right) = 2\pi \left(\frac{160 - 32}{5} \right) = \frac{256\pi}{5}. \end{aligned}$$

2. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x$$

about the vertical line $x = -2$.



Solution via the Shell Method:

Using vertical slices, for x from -2 to 1 the height of a typical slice is

$$h(x) = (2 - x) - x^2,$$

and its distance from the axis $x = -2$ (the shell radius) is

$$r(x) = x - (-2) = x + 2.$$

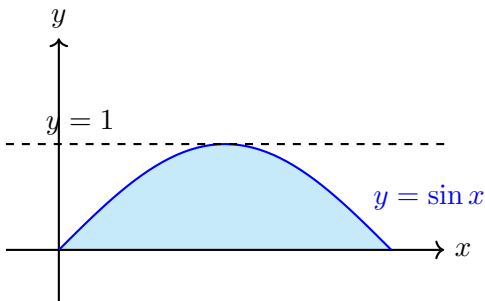
Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_{-2}^1 [(x + 2)((2 - x) - x^2)] dx \\ &= 2\pi \int_{-2}^1 (x + 2)(2 - x - x^2) dx \\ &= 2\pi \int_{-2}^1 [4 - 3x^2 - x^3] dx \\ &= 2\pi \left[4x - x^3 - \frac{x^4}{4} \right]_{x=-2}^1 \\ &= 2\pi \left[\frac{11}{4} - (-4) \right] = 2\pi \left(\frac{11}{4} + 4 \right) = 2\pi \left(\frac{11 + 16}{4} \right) \\ &= 2\pi \left(\frac{27}{4} \right) = \frac{27\pi}{2} \end{aligned}$$

3. Find the volume of the solid obtained by rotating the region bounded by

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

about the line $y = 1$.



Solution via the Washer Method:

Since the region lies entirely below $y = 1$, a vertical slice at a fixed x (with $0 \leq x \leq \pi$) extends from $y = 0$ to $y = \sin x$. When rotated about $y = 1$, this slice generates a washer with

$$\text{Outer radius: } R = 1 - 0 = 1$$

$$\text{Inner radius: } r = 1 - \sin x$$

Thus, the area of the washer is

$$A(x) = \pi [R^2 - r^2] = \pi [1^2 - (1 - \sin x)^2].$$

Expanding the inner square,

$$(1 - \sin x)^2 = 1 - 2 \sin x + \sin^2 x,$$

so that

$$A(x) = \pi [1 - (1 - 2 \sin x + \sin^2 x)] = \pi (2 \sin x - \sin^2 x).$$

Hence, the volume is given by

$$\begin{aligned} V &= \int_{x=0}^{\pi} A(x) dx \\ &= \pi \int_0^{\pi} (2 \sin x - \sin^2 x) dx. \end{aligned}$$

We evaluate the integrals separately:

$$\int_0^{\pi} 2 \sin x dx = 2 [-\cos x]_0^{\pi} = 2 [-(-1) + 1] = 4,$$

and

$$\int_0^{\pi} \sin^2 x dx = \frac{\pi}{2}.$$

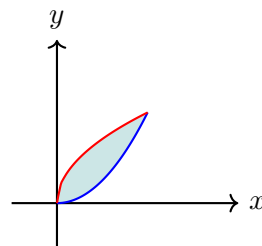
Thus,

$$V = \pi \left(4 - \frac{\pi}{2} \right) = \frac{\pi}{2} (8 - \pi).$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = \sqrt{x}$$

about the y -axis.



Solution via the Washer Method:

$$\begin{aligned} V_{\text{washers}} &= \int_{y=0}^1 A(y) dy \\ &= \pi \int_0^1 (y - y^4) dy \\ &= \pi \left[\frac{y^2}{2} - \frac{y^5}{5} \right]_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{5} \right) \\ &= \pi \left(\frac{5 - 2}{10} \right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

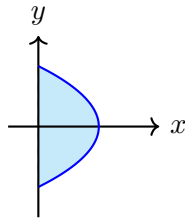
Solution via the Shell Method:

$$\begin{aligned} V_{\text{shells}} &= 2\pi \int_0^1 (x^{3/2} - x^3) dx \\ &= 2\pi \left[\frac{2}{5} x^{5/2} - \frac{x^4}{4} \right]_0^1 \\ &= 2\pi \left(\frac{2}{5} - \frac{1}{4} \right) \\ &= 2\pi \left(\frac{8 - 5}{20} \right) \\ &= 2\pi \left(\frac{3}{20} \right) \\ &= \frac{3\pi}{10}. \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by

$$x = 1 - y^2, \quad x = 0, \quad y \in [-1, 1]$$

about the y -axis.



Solution Using the Disk Method:

Since rotation is about the y -axis, the radius of a typical disk at height y is

$$R(y) = 1 - y^2.$$

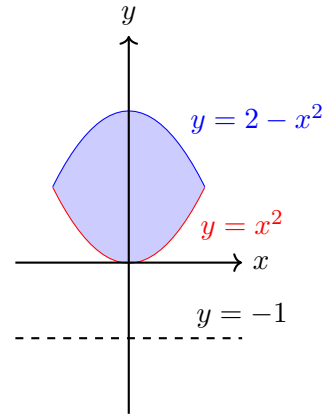
Thus, the volume is

$$\begin{aligned} V &= \pi \int_{y=-1}^1 (1 - y^2)^2 dy \\ &= \pi \int_{-1}^1 (1 - 2y^2 + y^4) dy \\ &= \pi \left[\int_{-1}^1 1 dy - 2 \int_{-1}^1 y^2 dy + \int_{-1}^1 y^4 dy \right] \\ &= \pi \left[2 - 2 \left(\frac{2}{3} \right) + \frac{2}{5} \right] \\ &= \pi \left[2 - \frac{4}{3} + \frac{2}{5} \right] \\ &= \pi \left[\frac{30}{15} - \frac{20}{15} + \frac{6}{15} \right] \\ &= \pi \left(\frac{16}{15} \right). \end{aligned}$$

6. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x^2$$

about the line $y = -1$.



Solution Using the Washer Method:

For a fixed x with $x \in [-1, 1]$, the region extends vertically from the lower curve $y = x^2$ to the upper curve $y = 2 - x^2$. When rotated about $y = -1$, the distances from $y = -1$ are:

$$\text{Inner radius: } r(x) = x^2 - (-1) = x^2 + 1$$

$$\text{Outer radius: } R(x) = (2 - x^2) - (-1) = 3 - x^2.$$

Thus, the washer cross-sectional area is

$$A(x) = \pi \left[(3 - x^2)^2 - (x^2 + 1)^2 \right].$$

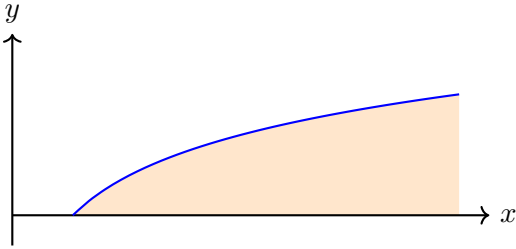
The volume is

$$\begin{aligned} V &= \pi \int_{x=-1}^1 \left[(3 - x^2)^2 - (x^2 + 1)^2 \right] dx \\ &= \pi \int_{-1}^1 (8 - 8x^2) dx \\ &= 8\pi \int_{-1}^1 (1 - x^2) dx \\ &= 8\pi \left[\int_{-1}^1 1 dx - \int_{-1}^1 x^2 dx \right] \\ &= 8\pi \left[(1 - (-1)) - \left(\frac{1^3 - (-1)^3}{3} \right) \right] \\ &= \frac{32\pi}{3}. \end{aligned}$$

7. Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln x, \quad y = 0, \quad x = 1, \quad x = e^2$$

about the y -axis.



Solution Using the Washer Method:

Since we are rotating about the y -axis, it is convenient to express x in terms of y . From

$$y = \ln x \implies x = e^y,$$

and note that when $x = e^2$ we have $y = \ln(e^2) = 2$. Thus, for a fixed y between 0 and 2, the region extends in x from the curve $x = e^y$ to the vertical line $x = e^2$. When rotated about the y -axis, the horizontal slice produces an annular cross section with

$$\text{Inner radius: } R_{\text{in}}(y) = e^y, \quad \text{Outer radius: } R_{\text{out}}(y) = e^2.$$

The area of the washer is then

$$A(y) = \pi \left[(e^2)^2 - (e^y)^2 \right] = \pi (e^4 - e^{2y}).$$

Thus, the volume by washers is

$$\begin{aligned} V_{\text{washers}} &= \pi \int_{y=0}^2 (e^4 - e^{2y}) dy \\ &= \pi \left[e^4 y - \frac{e^{2y}}{2} \right]_{y=0}^2 \\ &= \pi \left[e^4(2) - \frac{e^4}{2} - \left(0 - \frac{1}{2} \right) \right] \\ &= \pi \left(2e^4 - \frac{e^4}{2} + \frac{1}{2} \right) \\ &= \pi \left(\frac{4e^4 - e^4 + 1}{2} \right) \\ &= \frac{\pi(3e^4 + 1)}{2}. \end{aligned}$$

Solution Using the Shell Method:

Using vertical slices, the region is described by x from 1 to e^2 and, for each x , y runs from $y = 0$ to $y = \ln x$. A typical vertical slice at x has height

$$h(x) = \ln x,$$

and its distance (radius) from the y -axis is

$$r(x) = x.$$

Thus, the volume by shells is given by

$$V_{\text{shells}} = 2\pi \int_{x=1}^{e^2} x (\ln x) dx.$$

We compute the integral using integration by parts. Let

$$\begin{aligned} u = \ln x &\implies du = \frac{1}{x} dx \\ dv = x dx &\implies v = \frac{x^2}{2} \end{aligned}$$

Then,

$$\begin{aligned} \int x \ln x dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4}. \end{aligned}$$

Evaluating from $x = 1$ to $x = e^2$:

$$\begin{aligned} &\left. \frac{x^2}{2} \ln x - \frac{x^2}{4} \right|_{x=1}^{e^2} \\ &= \left[\frac{(e^2)^2}{2} \ln(e^2) - \frac{(e^2)^2}{4} \right] - \left[\frac{1}{2} \ln 1 - \frac{1}{4} \right] \\ &= \left[\frac{e^4}{2} \cdot 2 - \frac{e^4}{4} \right] - \left[0 - \frac{1}{4} \right] \\ &= \left[e^4 - \frac{e^4}{4} \right] + \frac{1}{4} \\ &= \frac{4e^4 - e^4 + 1}{4} \\ &= \frac{3e^4 + 1}{4}. \end{aligned}$$

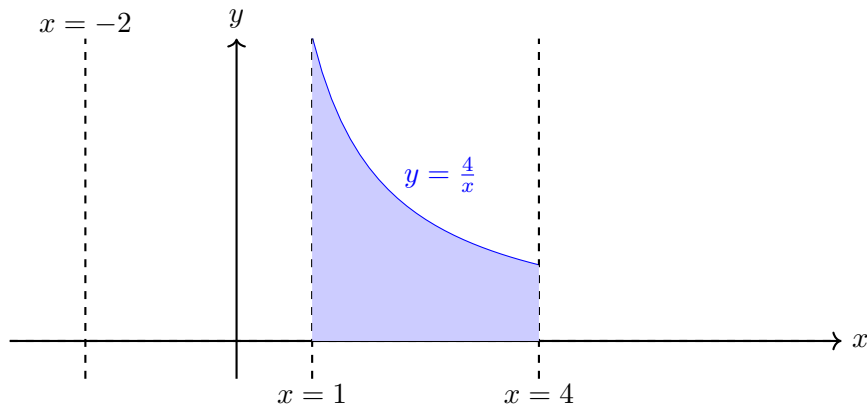
Thus,

$$\begin{aligned} V_{\text{shells}} &= 2\pi \cdot \frac{3e^4 + 1}{4} \\ &= \frac{\pi(3e^4 + 1)}{2}. \end{aligned}$$

8. Find the volume of the solid obtained by rotating the region bounded by

$$y = \frac{4}{x}, \quad y = 0, \quad x = 1, \quad x = 4$$

about the line $x = -2$.



Solution Using the Shell Method:

A vertical slice at position x (with $1 \leq x \leq 4$) has height

$$h(x) = \frac{4}{x},$$

and its distance (radius) from the axis $x = -2$ is

$$r(x) = x - (-2) = x + 2.$$

Thus, the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 r(x) h(x) dx \\ &= 2\pi \int_1^4 (x + 2) \left(\frac{4}{x}\right) dx \\ &= 2\pi \int_1^4 \frac{4(x + 2)}{x} dx \\ &= 8\pi \int_1^4 \left(1 + \frac{2}{x}\right) dx \\ &= 8\pi \left[\int_1^4 1 dx + 2 \int_1^4 \frac{1}{x} dx \right] \\ &= 8\pi \left[(4 - 1) + 2 \ln x \Big|_1^4 \right] \\ &= 8\pi [3 + 2 \ln(4)] \end{aligned}$$