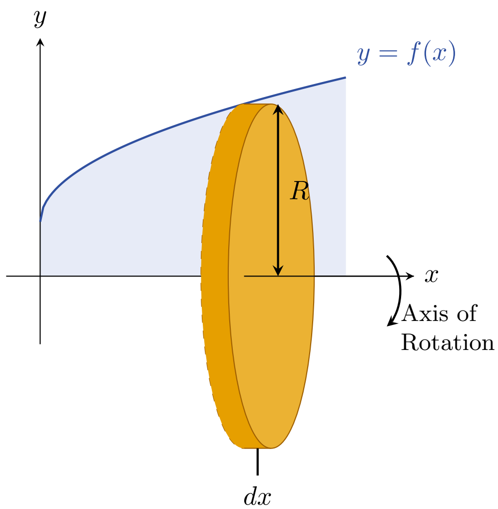
## 6.2 Volumes of Solids of Revolution

When a region is revolved around an axis, we can compute the volume by adding up thin circular cross-sections.

**1. Disk method:** cross-sections are *solid* circles.

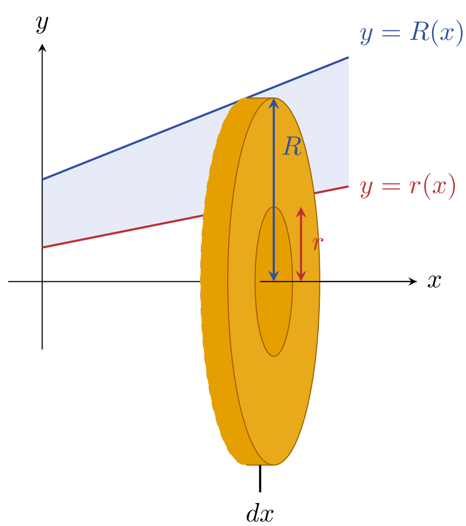
$$V = \pi \int_a^b \big(R(x)\big)^2\,dx \qquad \text{or} \qquad V = \pi \int_c^d \big(R(y)\big)^2\,dy$$




At each x, we get a disk with area $\pi \cdot [R(x)]^2$

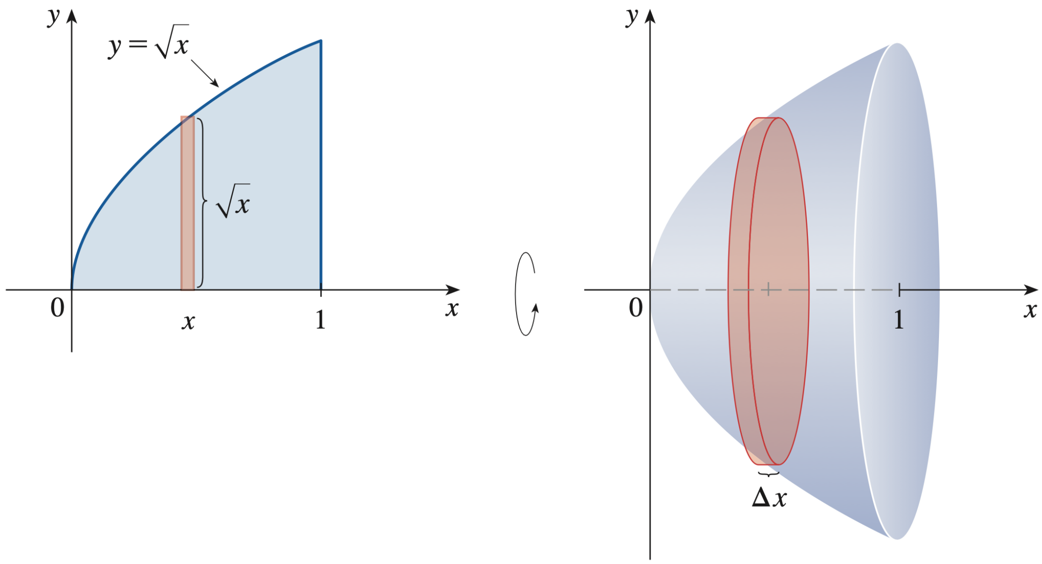
**2. Washer method:** cross-sections have a *hole*.

$$V = \pi \int_a^b \Big(\big(R(x)\big)^2 - \big(r(x)\big)^2\Big)\,dx \qquad \text{or} \qquad V = \pi \int_c^d \Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$




If there is space between the region and the axis of rotation, we get a washer with area $\pi\left[R(x)^2 - r(x)^2\right]$

**Example.** Find the volume of the solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

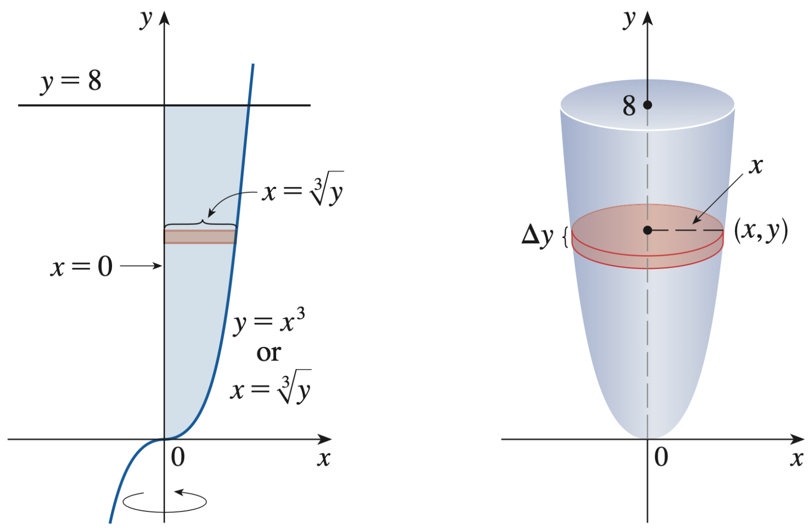



The slice at $x$ is a disk of radius $\sqrt{x}$

The area of the cross-section is $A(x) = \pi \big(\underbrace{\sqrt{x}}_{\text{radius}}\big)^2 = \pi x$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x \,dx = \pi \left[ \frac{x^2}{2} \right]_0^1 = \frac{\pi}{2}$$

**Example.** Find the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 8$, and $x = 0$ about the $y$-axis.



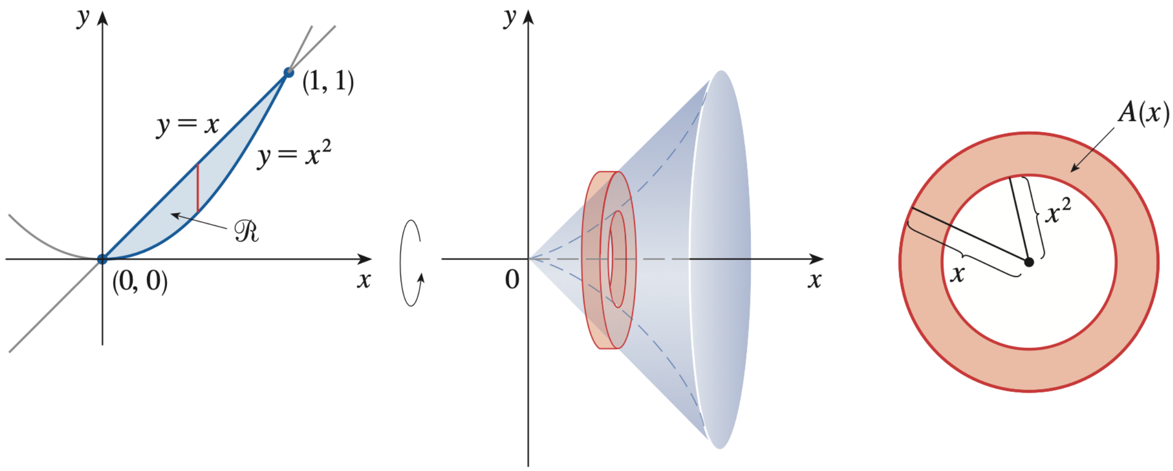

The region is being rotated about the y-axis ⇒ slice the solid perpendicular to the y-axis and integrate with respect to y.

At height y, the disk has radius $x = \sqrt[3]{y}$

The area of the disk is $A(y) = \pi \left( \underbrace{\sqrt[3]{y}}_{\text{radius}} \right)^2 = \pi y^{2/3}$

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi \left[ \frac{3}{5} y^{5/3} \right]_0^8 = \frac{96\pi}{5}$$

**Example.** The region $R$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the $x$-axis. Find the volume of the resulting solid.



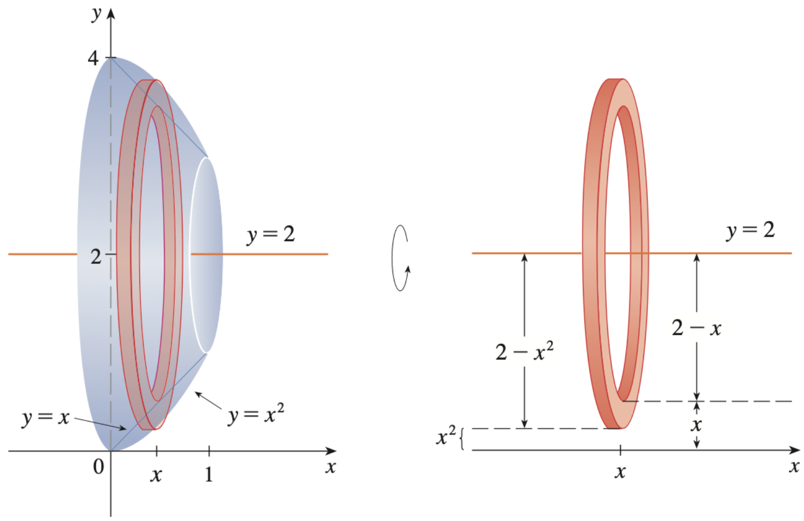

The cross-sectional area is

$$A(x) = \pi \underbrace{(x)}_{\text{outer radius}}{}^2 - \pi \underbrace{(x^2)}_{\text{inner radius}}{}^2 = \pi(x^2 - x^4)$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{2\pi}{15}$$

**Example.** Find the volume of the solid obtained by rotating the region $R$ enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.




The cross-sectional area is $A(x) = \pi \underbrace{(2-x^2)^2}_{\text{outer radius}} - \pi \underbrace{(2-x)^2}_{\text{inner radius}}$

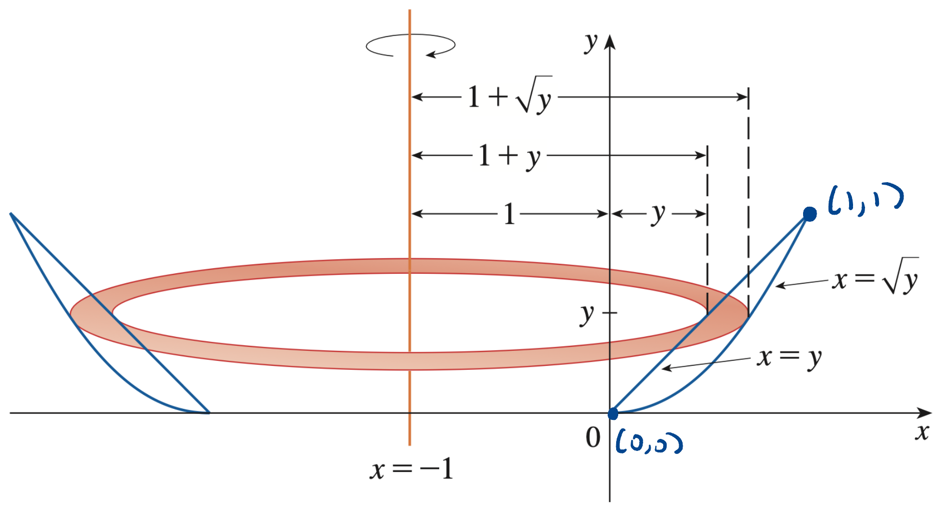
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi \left[(2-x^2)^2 - (2-x)^2\right] dx$$

$$= \pi \int_0^1 x^4 - 5x^2 + 4x \, dx$$

$$= \pi \left[ \frac{x^5}{5} - 5 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} \right]_0^1$$

$$= \frac{8\pi}{15}$$

**Example.** Find the volume of the solid obtained by rotating the region $R$ enclosed by the curves $y = x$ and $y = x^2$ about the line $x = -1$.




The cross-sectional area is

$$A(y) = \text{outer circle} - \text{hole}$$
$$= \pi(1+\sqrt{y})^2 - \pi(1+y)^2$$

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 (1+\sqrt{y})^2 - (1+y)^2\,dy$$

$$= \pi \int_0^1 2\sqrt{y} - y - y^2\,dy$$

$$= \pi \left[ \frac{4}{3} y^{3/2} - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= \frac{\pi}{2}$$