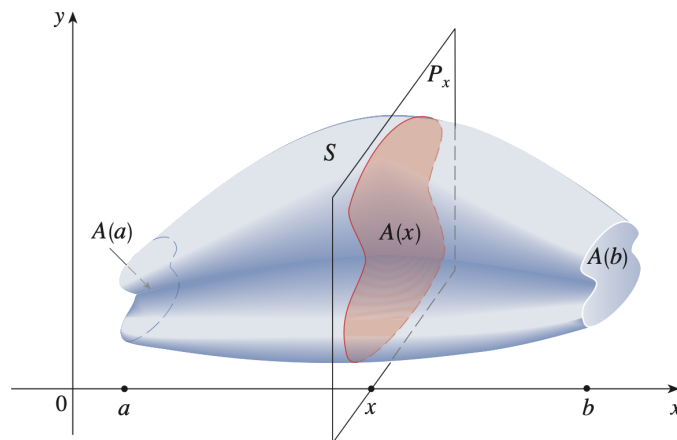


6.2 Volume by Known Cross-Sections

When a solid has cross-sectional areas that are easy to compute, we can find its volume by “stacking” these slices.



Let S be a solid lying between $x = a$ and $x = b$. Suppose that for each x in $[a, b]$ the cross-sectional area perpendicular to the x -axis is given by $A(x)$. Then the volume V of S is defined by

$$V = \int_a^b A(x) dx.$$

Remark. This definition generalizes the familiar formula for a cylinder. For a cylinder with constant base area A and height h , we have $V = Ah$.

Common Cross-Section Shapes

1. Square Cross Sections. If the cross section perpendicular to the x -axis is a square with side length $s(x)$, then the area is

$$A(x) = [s(x)]^2.$$

2. Rectangular Cross Sections. For a rectangle with base $b(x)$ and height $h(x)$, the area is

$$A(x) = b(x) h(x).$$

3. Isosceles Right Triangle Cross Sections. If each cross section is an isosceles right triangle with leg length $s(x)$, then its area is given by

$$A(x) = \frac{1}{2} [s(x)]^2.$$

4. Equilateral Triangle Cross Sections. For an equilateral triangle with side length $s(x)$, the area is

$$A(x) = \frac{\sqrt{3}}{4} [s(x)]^2.$$

5. Circular Cross Sections (Disks). If the cross section is a circle (or disk) with radius $r(x)$, then

$$A(x) = \pi [r(x)]^2.$$

6. Semicircular Cross Sections. For a semicircular cross section with radius $r(x)$, the area is

$$A(x) = \frac{1}{2} \pi [r(x)]^2.$$

Practice

1. A solid has a circular base of radius 2 centered at the origin in the xy -plane. The cross-sections perpendicular to the x -axis (i.e., vertical slices) are squares. If the side of each square lies in the plane of the cross-section, find the volume of this solid.
2. A solid has a right isosceles triangular base in the xy -plane with vertices at $(0,0)$, $(4,0)$, and $(0,4)$. The cross-sections perpendicular to the x -axis are semicircles. Find the volume of this solid.
3. A solid has a square base in the xy -plane with corners at $(0,0)$, $(3,0)$, $(3,3)$, and $(0,3)$. The cross-sections perpendicular to the x -axis are equilateral triangles. Each triangle's base runs from $y = 0$ to $y = 3$. Find the volume of the solid.
4. A solid has a rectangular base in the xy -plane: $0 \leq x \leq 5$, $0 \leq y \leq 2$. Cross-sections perpendicular to the y -axis are right triangles whose legs lie along the base plane. Specifically, for each fixed y , the base of the right triangle is 5 (the full length in the x direction) and the height of the triangle is y . Find the volume of the solid.
5. A solid has a circular base of radius 1 (centered at the origin). Cross-sections perpendicular to the y -axis are equilateral triangles whose base is the chord of the circle at that y . Find the volume of this solid.
6. A solid has a right isosceles triangular base in the xy -plane with vertices at $(0,0)$, $(6,0)$, and $(0,6)$. For each fixed x in the interval $[0,6]$, the vertical line through the base (parallel to the y -axis) intersects the triangle from $y = 0$ up to $y = 6 - x$. At each such x , a rectangle is erected whose:
 - Base (lying in the xy -plane) is the segment from $y = 0$ to $y = 6 - x$, of length $6 - x$, and
 - Height (in the z -direction) is defined to be twice the length of that segment, i.e., $2(6 - x)$.

Find the volume of the solid.

7. A solid has an elliptical base given by $\frac{x^2}{9} + \frac{y^2}{4} \leq 1$. Cross-sections perpendicular to the y -axis are squares whose side length equals the x -span of the ellipse at that y . Find the volume.
8. A solid sits over a square base $[0,4] \times [0,4]$ in the xy -plane. The cross-sections perpendicular to the x -axis are isosceles trapezoids with:
 - One base along the y -range from 0 to 4 (i.e., length 4).
 - The other base has length 2, centered with respect to the first base.
 - The height (thickness in the z -direction) of each trapezoid is $h = 1$ (constant).

Find the volume of the solid.

9. A solid has a base in the xy -plane bounded by $y = 0$, $x = 2$, and $y = 4 - x^2$ (which intersects $x = 2$ at $y = 0$). The cross-sections perpendicular to the x -axis are rectangles. The width of each rectangle is from $y = 0$ to $y = 4 - x^2$, and its height (out of the plane) is twice the y -value at that slice. Find the volume.