6.2 Volume by Known Cross-Sections (Solutions)

1. **Circular Base, Square Cross-Sections.**

   - **Base:** The base is the circle
   $$x^2 + y^2 \leq 4.$$
   - **Slicing:** A slice perpendicular to the x-axis at a given x meets the circle at
   $$y = \pm\sqrt{4 - x^2},$$
   so the width is
   $$2\sqrt{4 - x^2}.$$
   - **Cross-Section:** Each slice is a square with side length
   $$s(x) = 2\sqrt{4 - x^2},$$
   hence its area is
   $$A(x) = \left(2\sqrt{4 - x^2}\right)^2 = 16 - 4x^2.$$
   - **Volume:** The solid extends from $x = -2$ to $x = 2$, so
   $$V = \int_{-2}^{2} (16 - 4x^2)\, dx.$$
   - **Computation:**
   $$\begin{aligned} V &= \int_{-2}^{2} 16\, dx - 4\int_{-2}^{2} x^2\, dx \\ &= 16\,[x]_{-2}^{2} - 4\left[\frac{x^3}{3}\right]_{-2}^{2} \\ &= 16(2 - (-2)) - 4\left(\frac{8}{3} - \left(-\frac{8}{3}\right)\right) \\ &= 16 \cdot 4 - 4\left(\frac{16}{3}\right) \\ &= 64 - \frac{64}{3} \\ &= \frac{192 - 64}{3} = \frac{128}{3}. \end{aligned}$$
   - **Answer:** $V = \frac{128}{3}$ cubic units.

2. **Triangular Base, Semicircular Cross-Sections.**

   - **Base:** The triangular base is bounded by

$$x = 0, \quad y = 0, \quad \text{and} \quad y = 4 - x.$$

   - **Slicing:** For a fixed x (with $0 \leq x \leq 4$), the vertical slice extends from $y = 0$ to $y = 4 - x$. Thus, the diameter of the semicircular cross-section is

$$D(x) = 4 - x.$$

   - **Cross-Section:**
     - The radius is

$$r(x) = \frac{4 - x}{2}.$$

     - The area of a semicircle is half that of a full circle:

$$A(x) = \frac{1}{2}\pi\,[r(x)]^2 = \frac{\pi(4 - x)^2}{8}.$$

   - **Volume:** The volume is

$$V = \int_0^4 A(x)\,dx = \frac{\pi}{8}\int_0^4 (4 - x)^2\,dx.$$

   - **Substitution:** Let $u = 4 - x$ so that $du = -dx$. When $x = 0$, $u = 4$; when $x = 4$, $u = 0$. Then,

$$V = \frac{\pi}{8}\int_4^0 u^2\,(-du) = \frac{\pi}{8}\int_0^4 u^2\,du.$$

   - **Compute the Integral:**

$$\int_0^4 u^2\,du = \left.\frac{u^3}{3}\right|_0^4 = \frac{64}{3}.$$

   - **Result:**

$$V = \frac{\pi}{8} \cdot \frac{64}{3} = \frac{8\pi}{3}.$$

   - **Answer:** $V = \frac{8\pi}{3}$ cubic units.

3. **Square Base, Equilateral Triangular Cross-Sections.**

- **Slicing:** Slicing perpendicular to the x-axis, each slice has a base (in the y-direction) from $y = 0$ to $y = 3$ (length 3).
- **Cross-Section:** For an equilateral triangle of side s, the area is

$$A = \frac{\sqrt{3}}{4} s^2.$$

With $s = 3$, the area becomes

$$A(x) = \frac{\sqrt{3}}{4}(3^2) = \frac{9\sqrt{3}}{4}.$$

- **Volume:**

$$V = \int_0^3 A(x)\, dx = \frac{9\sqrt{3}}{4} \cdot 3 = \frac{27\sqrt{3}}{4}.$$

- **Answer:** $V = \frac{27\sqrt{3}}{4}$ cubic units.

4. **Rectangular Base, Right Triangular Cross-Sections.**

- **Slicing:** We slice perpendicular to the y-axis. At a fixed y, the slice has a length of 5 in the x-direction.
- **Cross-Section:** Each slice is a right triangle with base 5 and height y. Thus, the area is

$$A(y) = \frac{1}{2} \cdot 5 \cdot y = \frac{5y}{2}.$$

- **Volume:**

$$V = \int_0^2 \frac{5y}{2}\, dy.$$

- **Compute:**

$$\begin{aligned} V &= \frac{5}{2} \int_0^2 y\, dy \\ &= \frac{5}{2} \left[\frac{y^2}{2} \right]_0^2 \\ &= \frac{5}{2} \cdot \frac{4}{2} = 5. \end{aligned}$$

- **Answer:** $V = 5$ cubic units.

5. **Circular Base, Equilateral Triangles.**

- **Base:** The base is the circle
$$x^2 + y^2 \leq 1.$$
- **Slicing:** At a fixed y, the chord in the x-direction runs from
$$x = -\sqrt{1-y^2} \quad \text{to} \quad x = \sqrt{1-y^2},$$
with length
$$2\sqrt{1-y^2}.$$
- **Cross-Section:** This chord forms the base of an equilateral triangle, so its area is
$$A(y) = \frac{\sqrt{3}}{4}\left(2\sqrt{1-y^2}\right)^2 = \sqrt{3}(1-y^2).$$
- **Volume:**
$$V = \int_{-1}^{1} \sqrt{3}(1-y^2)\,dy.$$
- **Compute the inner integral:**
$$\begin{aligned}\int_{-1}^{1}(1-y^2)\,dy &= \left[y - \frac{y^3}{3}\right]_{-1}^{1} \\ &= \left[(1-\tfrac{1}{3}) - (-1+\tfrac{1}{3})\right] \\ &= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.\end{aligned}$$
- **Result:**
$$V = \sqrt{3} \cdot \frac{4}{3} = \frac{4\sqrt{3}}{3}.$$
- **Answer:** $V = \frac{4\sqrt{3}}{3}$ cubic units.

6. **Triangular Base, Rectangular Cross-Sections.**

- **Base:** The triangular base is bounded by

$$x = 0, \quad y = 0, \quad x + y = 6.$$

For a fixed $x \in [0, 6]$, y runs from 0 to $6 - x$. Define

$$L(x) = 6 - x.$$

- **Cross-Section:** The rectangle at x has:
  - **Base (in the xy-plane):** $L(x) = 6 - x$,
  - **Height (in the z-direction):** $2(6 - x)$.

  Thus, its area is

$$A(x) = (6 - x) \cdot [2(6 - x)] = 2(6 - x)^2.$$

- **Volume:**

$$V = \int_0^6 2(6 - x)^2 \, dx.$$

- **Substitution:** Let $u = 6 - x$ so that $du = -dx$. Then,

$$\begin{aligned} V &= \int_{u=6}^{0} 2u^2 \, (-du) \\ &= \int_0^6 2u^2 \, du. \end{aligned}$$

- **Compute:**

$$\begin{aligned} \int_0^6 2u^2 \, du &= 2 \left[\frac{u^3}{3} \right]_0^6 \\ &= 2 \cdot \frac{216}{3} = 144. \end{aligned}$$

- **Answer:** $V = 144$ cubic units.

7. **Elliptical Base, Square Cross-Sections.**

- **Base:** The ellipse is given by
$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

- **Slicing:** For a fixed y, solve for x^2:
$$x^2 = 9\left(1 - \frac{y^2}{4}\right) = 9 - \frac{9y^2}{4}.$$
Hence,
$$x = \pm\sqrt{9 - \frac{9y^2}{4}} = \pm\frac{3}{2}\sqrt{4 - y^2}.$$
The total width is
$$2 \cdot \frac{3}{2}\sqrt{4 - y^2} = 3\sqrt{4 - y^2}.$$

- **Cross-Section:** Since the cross-sections are squares,
$$A(y) = \left[3\sqrt{4 - y^2}\right]^2 = 9(4 - y^2) = 36 - 9y^2.$$

- **Volume:** Integrate with respect to y (from -2 to 2):
$$V = \int_{-2}^{2} (36 - 9y^2)\,dy.$$

- **Computation:**
$$\int_{-2}^{2} 36\,dy = 36\,[y]_{-2}^{2} = 36(4) = 144,$$
$$\int_{-2}^{2} y^2\,dy = \left[\frac{y^3}{3}\right]_{-2}^{2} = \frac{8}{3} - \left(-\frac{8}{3}\right) = \frac{16}{3}.$$
Therefore,
$$V = 144 - 9\left(\frac{16}{3}\right) = 144 - 48 = 96.$$

- **Answer:** $V = 96$ cubic units.

8. **Isosceles Trapezoidal Cross-Sections over a Square Base.**

- **Slicing:** For a fixed x, the cross-section extends in y from 0 to 4. Thus, the longer base is 4 and the shorter base is 2; the trapezoid's height (perpendicular to the xy-plane) is 1.
- **Cross-Section:** Its area is
$$A = \frac{4+2}{2} \times 1 = 3.$$
- **Volume:** Since the area is constant,
$$V = \int_0^4 3\,dx = 12.$$
- **Answer:** $V = 12$ cubic units.

9. **Parabolic Region, Rectangular Cross-Sections (Height Proportional to y).**

- **Base:** For a given x, the region in y goes from 0 to $4 - x^2$; thus the width is
$$4 - x^2.$$
- **Cross-Section:** The rectangle has width $4 - x^2$ and (constant for the slice) height
$$2(4 - x^2).$$
Its area is
$$A(x) = (4 - x^2) \cdot 2(4 - x^2) = 2(4 - x^2)^2.$$
- **Volume:** The region in x is from 0 to 2, so
$$V = \int_0^2 2(4 - x^2)^2 \, dx.$$
- **Expand and Compute:** Note that
$$(4 - x^2)^2 = 16 - 8x^2 + x^4.$$
Then,
$$\begin{aligned} V &= 2 \int_0^2 \left(16 - 8x^2 + x^4\right) dx \\ &= 2 \left[\int_0^2 16 \, dx - 8 \int_0^2 x^2 \, dx + \int_0^2 x^4 \, dx \right]. \end{aligned}$$
- **Individual Integrals:**
$$\int_0^2 16 \, dx = 32, \quad \int_0^2 x^2 \, dx = \frac{8}{3}, \quad \int_0^2 x^4 \, dx = \frac{32}{5}.$$
- **Thus,**
$$\begin{aligned} V &= 2 \left(32 - 8 \cdot \frac{8}{3} + \frac{32}{5} \right) \\ &= 2 \left(32 - \frac{64}{3} + \frac{32}{5} \right) \\ &= 2 \left[\frac{32 \cdot 15 - 64 \cdot 5 + 32 \cdot 3}{15} \right] \\ &= 2 \left[\frac{480 - 320 + 96}{15} \right] \\ &= 2 \left[\frac{256}{15} \right] = \frac{512}{15}. \end{aligned}$$
- **Answer:** $V = \frac{512}{15}$ cubic units.