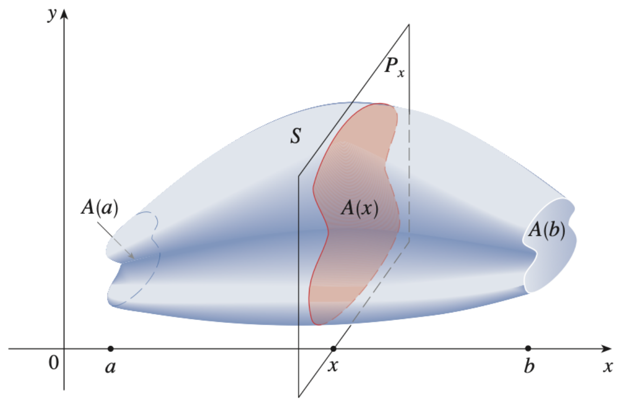
6.2 Volumes by Known Cross-Sections

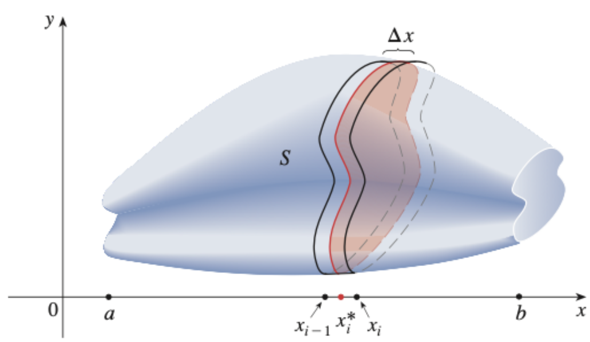
Question. How can we find the volume of a solid region S?

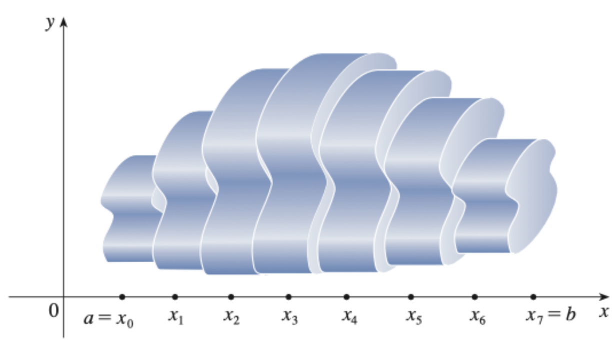


- Let $A(x)$ be the area of the cross-section in the plane P_x, which is the plane perpendicular to the x-axis, passing through x.
  (think about slicing S with a knife at x).

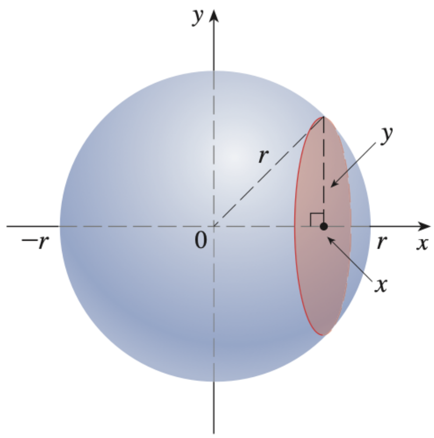
- The area $A(x)$ varies as x increases from a to b

- The total volume of S is $\int_a^b A(x)\,dx$





Example. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.



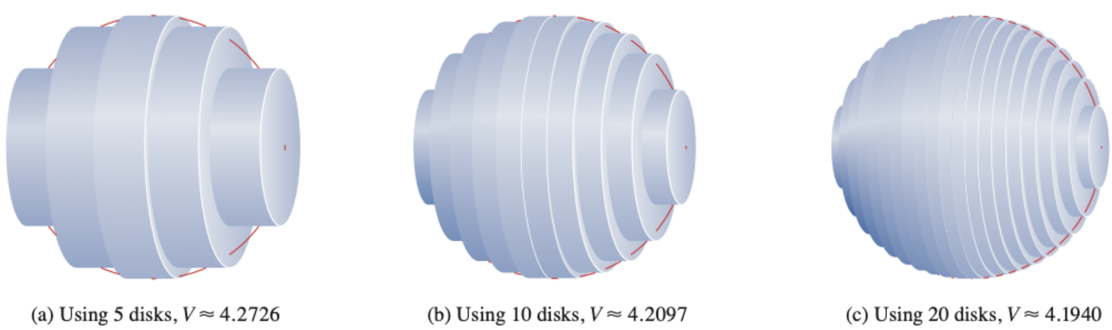
- The plane P_x intersects the sphere in a circle of radius $y = \sqrt{r^2 - x^2}$
- The cross-sectional area is $A(x) = \pi y^2 = \pi (r^2 - x^2)$

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi (r^2 - x^2)\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right) \right]$$

$$= \pi \left[r^3 + r^3 - \frac{r^3}{3} - \frac{r^3}{3} \right]$$

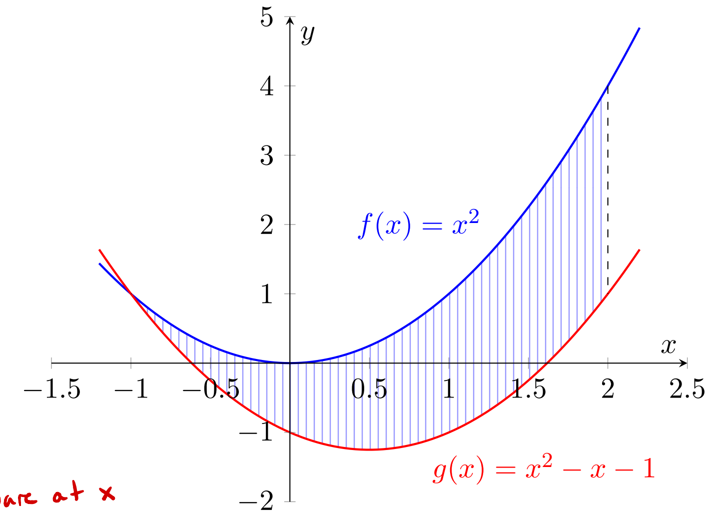
$$= \frac{4}{3}\pi r^3$$



Approximating the volume of a sphere with radius 1

Exam-style

Example. Let R be the region bounded by $f(x) = x^2$ and $g(x) = x^2 - x - 1$ on the interval $[-1, 2]$. A solid has base R, and cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.



Side of the square at x

① $A(x) = [s(x)]^2 = [f(x) - g(x)]^2 = [x^2 - (x^2 - x - 1)]^2$

$= [x + 1]^2$

② $V = \int_{-1}^{2} A(x)\, dx = \int_{-1}^{2} (x+1)^2\, dx$

Let $u = x + 1$
$du = dx$

$= \int_0^3 u^2\, du$

$= \left[\frac{1}{3} u^3 \right]_0^3$

$= 9$