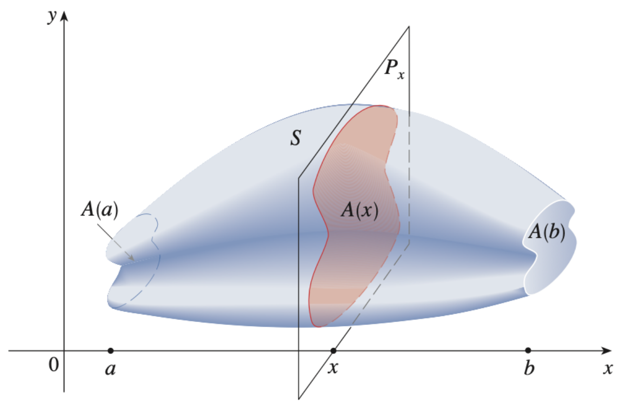
6.2 Volumes

Definition of Volume

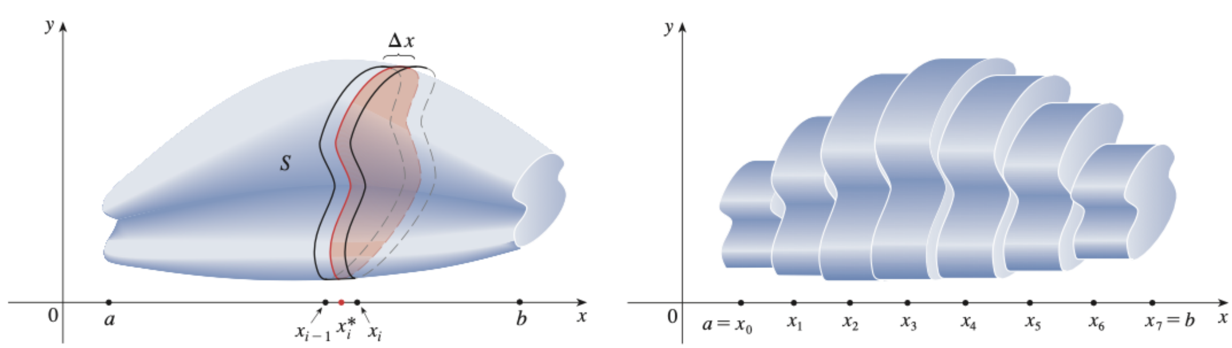
Question. How can we find the volume of a solid region S?



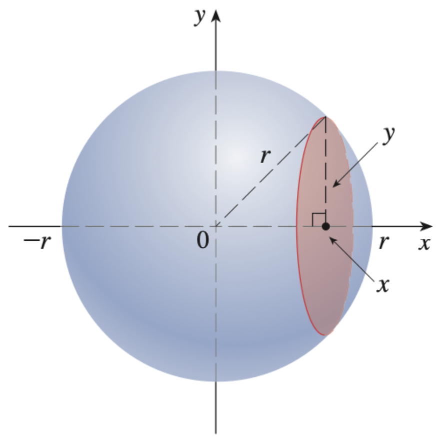
- Let $A(x)$ be the area of the cross-section in the plane P_x, which is the plane perpendicular to the x-axis, passing through x

  (think of slicing S with a knife)

- The area $A(x)$ will vary as a increases to b
- The total volume of S will be $\int_a^b A(x)\,dx$



Example. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.



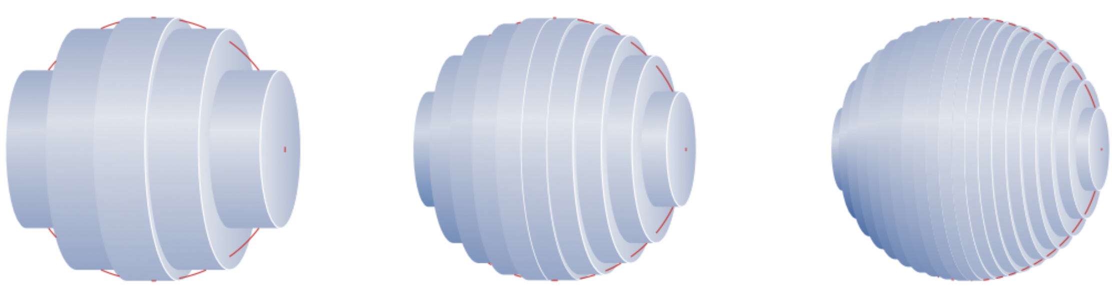
- Place the sphere so that the center is at the origin
- The plane P_x intersects the sphere in a circle of radius $y = \sqrt{r^2 - x^2}$
- The cross-sectional area is $A(x) = \pi y^2 = \pi(r^2 - x^2)$

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

$$= \pi\left[r^3 + r^3 - \frac{2r^3}{3}\right]$$

$$= \frac{4}{3}\pi r^3$$

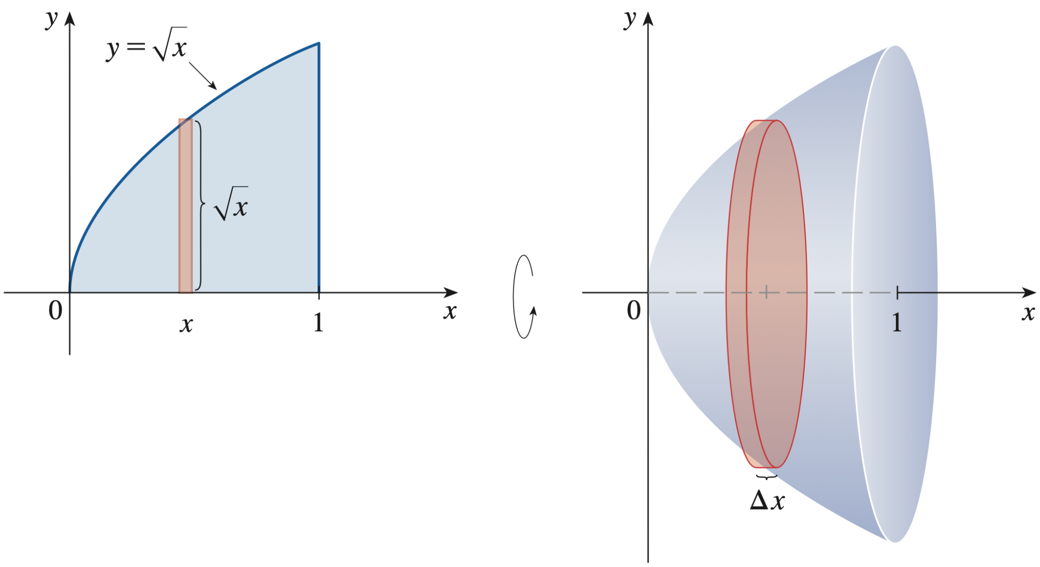


(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Approximating the volume of a sphere with radius 1

Volumes of Solids of Revolution

Example. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

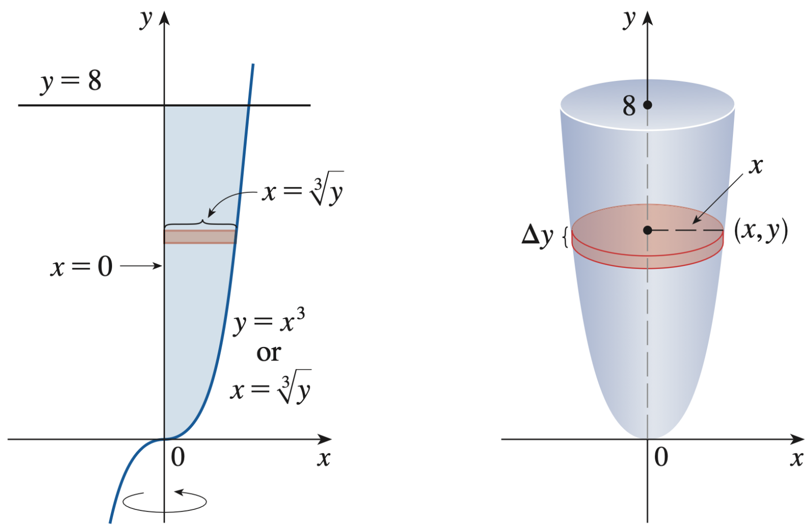


If we slice through the point x, we get a disk of radius $\sqrt{x}$

The area of the cross-section is $A(x) = \pi \underbrace{(\sqrt{x})}_{\text{radius}}^2 = \pi x$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x \, dx = \pi \left[\frac{x^2}{2} \right]_0^1 = \frac{\pi}{2}$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 8$, and $x = 0$ about the y-axis.



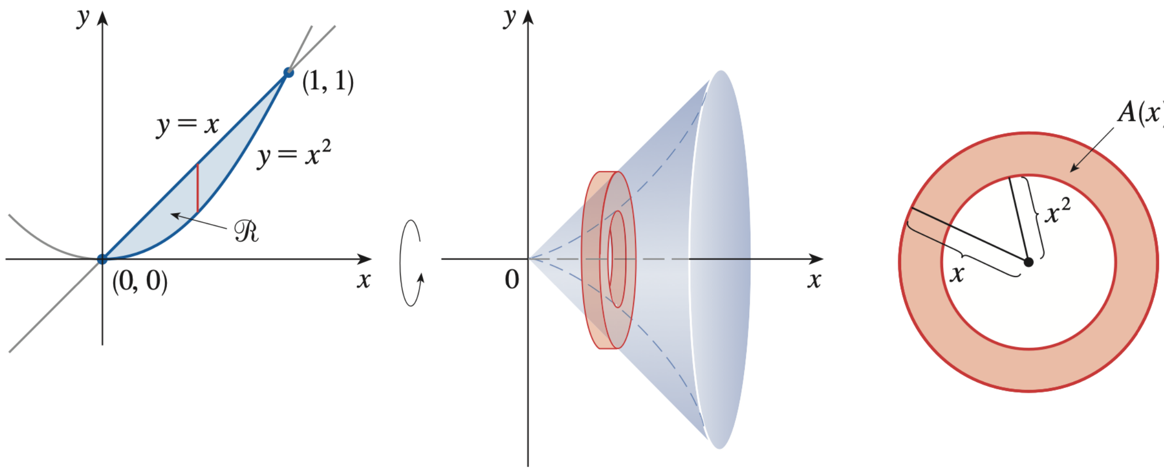
The region is being rotated about the y-axis ⇒ slice the solid perpendicular to the y-axis and integrate with respect to y.

At height y, the disk has radius $x = \sqrt[3]{y}$

The area of the disk is $A(y) = \pi\,(\underbrace{\sqrt[3]{y}}_{\text{radius}})^2 = \pi y^{2/3}$

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\left[\tfrac{3}{5} y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

Example. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



Here, a cross-section in the plane P_x is a washer

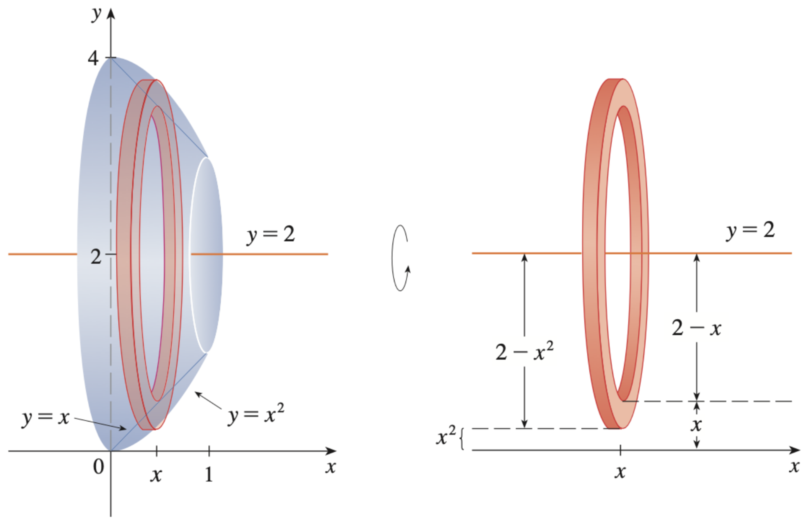
The cross-sectional area is

$$A(x) = \pi\underbrace{(x)}_{\text{outer radius}}{}^2 - \pi\underbrace{(x^2)}_{\text{inner radius}}{}^2 = \pi(x^2 - x^4)$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{2\pi}{15}$$

Example. Find the volume of the solid obtained by rotating the region R enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.



The cross-section is a washer

The cross-sectional area is $A(x) = \pi \underbrace{(2-x^2)}_{\text{outer radius}}{}^2 - \pi \underbrace{(2-x)}_{\text{inner radius}}{}^2$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left[(2-x^2)^2 - (2-x)^2\right] dx$$

$$= \pi \int_0^1 x^4 - 5x^2 + 4x \, dx$$

$$= \pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2} \right]_0^1$$

$$= \frac{8\pi}{15}$$

Summary

To calculate the volume of a solid of revolution, we use the defining formulas:

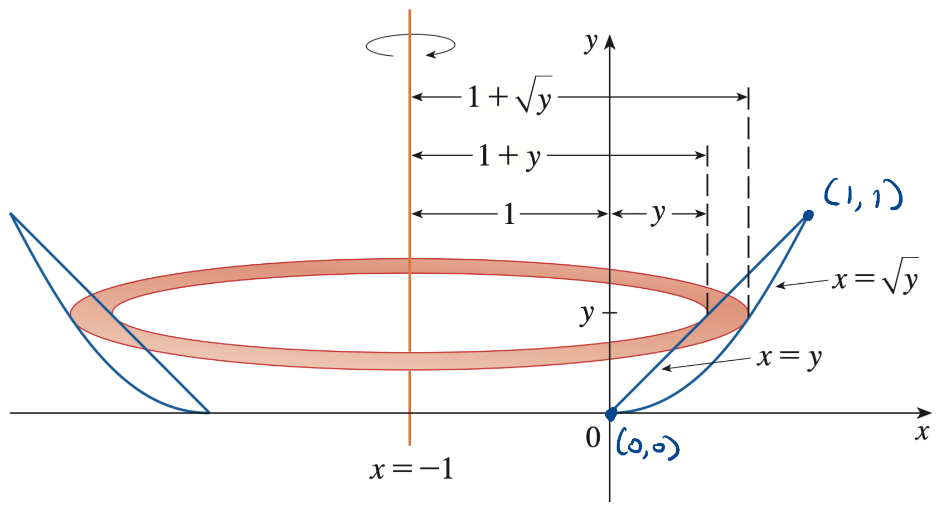
$$V = \int_a^b A(x)\,dx \quad \text{(rotation around the x-axis)}$$

$$V = \int_c^d A(y)\,dy \quad \text{(rotation around the y-axis)}$$

$A(x)$ or $A(y)$ represents the cross-sectional area, which is determined by the method used.

Cross-Section	**Process for Finding Area**
Disk	Cross-section is a solid disk. Determine the radius $r(x)$ or $r(y)$ based on the axis of rotation. Compute the area of the disk using: $$A = \pi \cdot [\text{radius}]^2$$
Washer	Cross-section is a washer (a disk with a hole). Find the inner radius r_{in} and outer radius r_{out} from a sketch or equation. Compute the area of the washer by subtracting the inner disk area from the outer disk area: $$A = \pi \cdot [r_{\text{out}}]^2 - \pi \cdot [r_{\text{in}}]^2$$

Example. Find the volume of the solid obtained by rotating the region R enclosed by the curves $y = x$ and $y = x^2$ about the line $x = -1$.



The cross-section is a washer

The cross-sectional area is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi(1+\sqrt{y})^2 - \pi(1+y)^2$$

$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 (1+\sqrt{y})^2 - (1+y)^2\,dy$$

$$= \pi\int_0^1 2\sqrt{y} - y - y^2\,dy$$

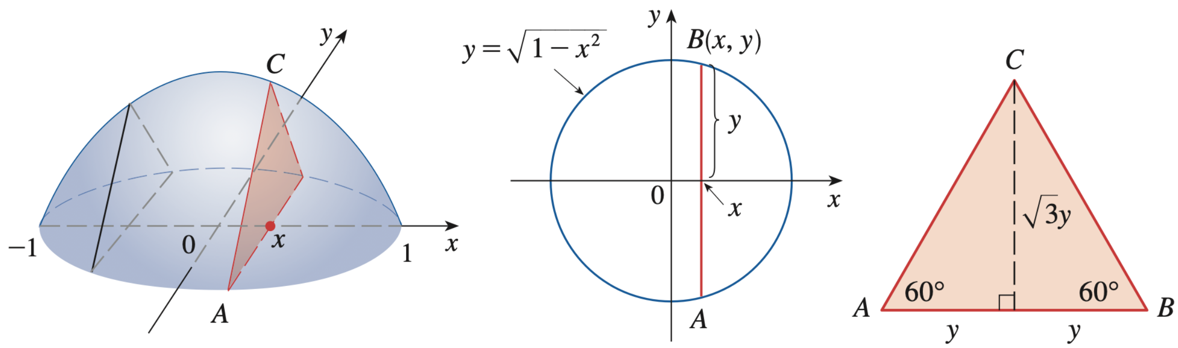
$$= \pi\left[\frac{4}{3}y^{3/2} - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{\pi}{2}$$

Finding Volume Using Cross-Sectional Area

We now find the volumes of solids that are not solids of revolution but whose cross-sections have areas that are readily computable.

Example. The figure below shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



(a) The solid (b) Its base (c) A cross-section

Goal: find $A(x)$ and compute $\int_{-1}^{1} A(x)\,dx$

From (a), $A(x)$ is the area of an equilateral triangle ABC.

We need the base and the height.

From (b), the base AB is $2y$. Because $x^2+y^2=1$, the formula for a circle

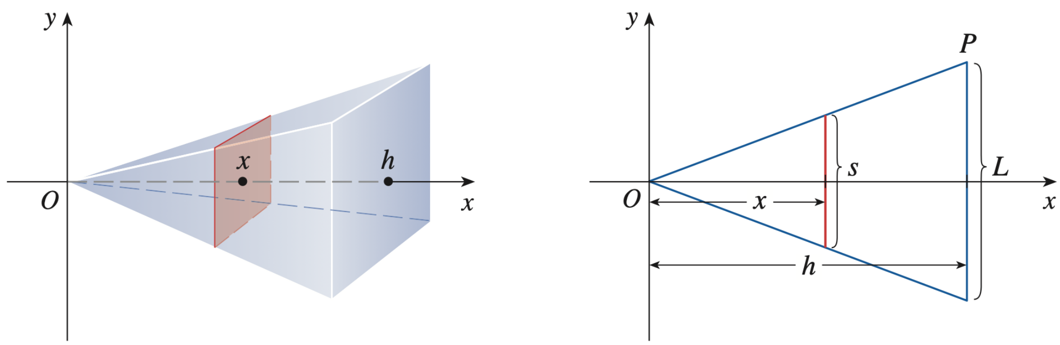
$2y = 2\sqrt{1-x^2}$

From (c), the height is $\sqrt{3}\,y = \sqrt{3}\cdot\sqrt{1-x^2}$

Hence, $A(x) = \frac{1}{2}\cdot \text{base}\cdot\text{height} = \frac{1}{2}\cdot 2\sqrt{1-x^2}\cdot\sqrt{3}\sqrt{1-x^2} = \sqrt{3}(1-x^2)$

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1}\sqrt{3}(1-x^2)\,dx = \sqrt{3}\left[x-\frac{x^3}{3}\right]_{-1}^{1} = \frac{4\sqrt{3}}{3}$$

Example. Find the volume of a pyramid whose base is square with side L and whose height is h.



Solution:

- Place the origin O at the vertex of the pyramid and the x-axis along its central axis
- Goal: find $A(x)$ and compute $\int_0^h A(x)\,dx$.
- $A(x)$ computes the area of a square with side s.
- From similar triangles,
$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$
and so $s = Lx/h$
- Therefore, $A(x) = s^2 = \frac{L^2}{h^2}x^2$
- The volume of the solid is
$$\begin{aligned} V &= \int_0^h A(x)\,dx \\ &= \int_0^h \frac{L^2}{h^2}x^2\,dx \\ &= \left[\frac{L^2}{h^2}\cdot\frac{x^3}{3}\right]_0^h \\ &= \frac{L^2h}{3} \end{aligned}$$

Example. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.